HILBERT-MUMFORD CRITERION FOR NODAL CURVES

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Abstract. We prove by Hilbert-Mumford criterion that a slope stable polarized weighted pointed nodal curve is Chow asymptotic stable. This generalizes the result of Caporaso on stability of polarized nodal curves, and of Hassett on weighted pointed stable curves polarized by the weighted dualizing sheaves. It also solved a question raised by Mumford and Gieseker to prove the Chow asymptotic stability of stable nodal curves by Hilbert-Mumford criterion.

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1. INTRODUCTION AND SUMMARY OF MAIN RESULT

In the late seventies, Mumford [16] and Gieseker [7] constructed the coarse moduli space $\mathcal{M}_g$ of stable curves using Mumford's Geometric Invariant Theory (GIT). They proved the GIT stability of smooth curves by verifying Hilbert-Mumford stability criterion; for nodal curves, they proved the stability indirectly by using semistable replacement and using a numerical criterion to rule out curves with worse than nodal singularities. This construction has been very successful and is widely adopted subsequently, for instance, Caporaso’s proof of asymptotic stability of nodal curves [3].

In this paper, we will prove the Chow asymptotic stability of weighted pointed nodal curves by verifying Hilbert-Mumford criterion directly. As an application, we provide a GIT construction of the moduli of weighted pointed stable curves. An interesting consequence of this construction is that the GIT closure of the moduli of weighted pointed
smooth curves, using Chow asymptotic stability, is identical to Hassett’s coarse moduli of weighted pointed stable curves, while its universal family includes strictly semistable weighted pointed nodal curves.

Another application of our stability study is to show that a polarized nodal curve is $K$-stable (c.f. Section 7) if and only if the polarization is numerically proportional to the dualizing sheaf. This generalizes a theorem of Odaka that a stable nodal curve polarized with dualizing sheaf is $K$-stable.

The primary goal of this work is towards understanding the GIT compactification of moduli of canonically polarized varieties. The recent works on the relation between various notions of $K$-stabilities and the existence of constant scalar curvature Kähler (cscK) metrics suggest that some deep and interesting geometry is yet to be uncovered in this area. We hope this study will help us understand the stability of high dimensional singular varieties.

We briefly outline the results proved in this paper. In this paper, we work over a characteristic zero algebraically closed field $k$. A curve is a proper, reduced pure one dimensional scheme.

**Definition 1.1** (Hassett [10]). A weighted pointed nodal curve $(X, x, a)$ is a connected nodal curve $X$ coupled with with $n$ ordered (not necessarily distinct) weighted smooth points $x = (x_1, \cdots, x_n) \in X^n$ of weights $a = (a_1, \cdots, a_n) \in \mathbb{Q}^n$ such that the total weight at any point is no more than one (i.e. for any $p \in X$, $\sum_{x_i=p} a_i \leq 1$). It is polarized if it comes with a very ample line bundle $\mathcal{O}_X(1)$ of degree $d$.

In this paper, we will use $(X, \mathcal{O}_X(1), x, a)$ to denote a polarized weighted pointed nodal curve. As $\mathcal{O}_X(1)$ is very ample, we form its tautological embedding

\[
\iota : X \hookrightarrow \mathbb{P}(W), \quad W = H^0(\mathcal{O}_X(1))^\vee,
\]

and the Chow point

\[
\text{Chow}(X, x) = (\text{Chow}(X), x) \in \Xi := \text{Div}^{d, d}((\mathbb{P}W^\vee)^2) \times (\mathbb{P}W)^n,
\]

where $\text{Div}^{d, d}((\mathbb{P}W^\vee)^2)$ is the space of bi-degree $(d, d)$ hypersurfaces in $(\mathbb{P}W^\vee)^2$; $\text{Chow}(X) \in \text{Div}^{d, d}((\mathbb{P}W^\vee)^2)$ is the Chow point of $(X, \iota)$ consisting of the set of $(V_1, V_2) \in (\mathbb{P}W^\vee)^2$ such that $V_1 \cap V_2 \cap \iota(X) \neq \emptyset$.

The stability of the Chow point is tested by the positivity of the $a$-weight of any one parameter subgroup $\lambda : \mathcal{O}_m \to SL(W)$. (A one parameter subgroup, abbreviated to 1-PS, is always non-trivial.) Since $\text{Div}^{d, d}((\mathbb{P}W^\vee)^2)$ is a projective space, it has a canonical polarization $\mathcal{O}(1)$. We let

\[
\mathcal{O}_\Xi(1, a)
\]

be the $\mathcal{O}$-ample line bundle on $\Xi$ that has degree 1 on $\text{Div}^{d, d}((\mathbb{P}W^\vee)^2)$ and has degree $a_i$ on the $i$-th copy of $\mathbb{P}W$ in $(\mathbb{P}W)^n$. The group $SL(W)$ acts on $\Xi$, and an integral multiple of $\mathcal{O}_\Xi(1, a)$ is canonically linearized by $SL(W)$.

**Definition 1.2.** Given $(X, \mathcal{O}_X(1), x, a)$, and a 1-PS $\lambda$ of $SL(W)$, we let $\zeta = \lim_{t \to 0} \lambda(t) \cdot \text{Chow}(X, x) \in \Xi$, and define the $a$-$\lambda$-weight $\omega_a(\lambda)$ of $\text{Chow}(X, x) \in \Xi$ to be the weight of the $\lambda$-action on the fiber $\mathcal{O}_\Xi(1, a)|_{\zeta}$. We define the $\lambda$-weight $\omega(\lambda)$ of $\text{Chow}(X) \in \text{Div}^{d, d}((\mathbb{P}W^\vee)^2)$ similarly with $\text{Chow}(X, x)$ (resp. $\mathcal{O}_\Xi(1, a)$) replaced by $\text{Chow}(X)$ (resp. $\mathcal{O}(1)$).

**Definition 1.3.** We say that $(X, \mathcal{O}_X(1), x, a)$ is stable (resp. semistable) if for any 1-PS $\lambda$ of $SL(W)$, the $a$-$\lambda$-weight $\omega_a(\lambda)$ of $\text{Chow}(X, x)$ is positive (resp. non-negative).
To make an analogy with the slope stability of vector bundles, we introduce the notion of slope stable by testing on proper closed subcurves $Y \subseteq X$. First, with $\mathcal{O}_X(1)$ understood, for subcurve $Y \subseteq X$ we denote $\deg Y = \deg \mathcal{O}_X(1)|_Y$. For any proper subcurve $Y \subseteq X$, we define the number of linking nodes of $Y$ to be

$$\ell_Y = |L_Y|, \quad L_Y = Y \cap Y^c, \quad Y^c = X \setminus Y.$$ 

For simplicity, we abbreviate

$$a_Y = \sum_{i \in Y} a_i,$$

thus $a_X = \sum_{i=1}^n a_i$. We say that $(X, \mathcal{O}_X(1))$ is non-special if $h^1(\mathcal{O}_X(1)) = 0$. We call a subcurve $Y \subseteq X$ of $(X, \mathcal{O}_X(1), x)$ an exceptional component if $Y \cong \mathbb{P}^1$, $Y \cap x = \emptyset$, $\ell_Y = 2$ and $\deg_Y \mathcal{O}_X(1) = 1$.

**Definition 1.4.** We call $(X, \mathcal{O}_X(1), x, a)$ slope semistable if $(X, \mathcal{O}_X(1))$ is non-special, and for any proper subcurve $Y \subseteq X$ we have

$$\frac{\deg Y + \frac{\ell_Y}{2} + \frac{a_Y}{2}}{h^0(\mathcal{O}_Y(1))} \leq \frac{\deg X + \frac{a_X}{2}}{h^0(\mathcal{O}_X(1))}.$$ 

We say that it is stable if it is semistable and if the strict inequality (1.4) holds except when $Y^c$ is a disjoint union of exceptional components of $(X, \mathcal{O}_X(1), x)$.

In this paper, we will prove by verifying the Hilbert-Mumford criterion the following theorem. For the weight $a$ and $g(X) = g$, we denote

$$\chi_{a,g} := g - 1 + \frac{1}{2} a_X \quad \text{and} \quad \chi_{a,g}(X) := \chi_{a,g}(x).$$

**Theorem 1.5.** Given $g$ and $a$ such that $\chi_{a,g} > 0$, there is an $M = M(g, n, a)$ so that a genus $g$ polarized weighted pointed nodal curve $(X, \mathcal{O}_X(1), x, a)$ having $\deg X \geq M$ is (semi-)stable if and only if it is slope (semi-)stable.

By a straightforward extension of [3, Proposition 3.1], Theorem 1.5 can be reformulated (cf. Proposition 5.4) as follows

**Theorem 1.6.** Given $g$ and $a$ such that $\chi_{a,g} > 0$, there is an $M = M(g, n, a)$ so that a genus $g$ polarized weighted pointed nodal curve $(X, \mathcal{O}_X(1), x, a)$ having $\deg X \geq M$ is semistable if and only if for any proper subcurve $Y \subseteq X$, we have

$$\left| \left( \deg Y + \sum_{j \in Y} \frac{a_j}{2} \right) - \frac{\deg Y \omega_X(a \cdot x)}{\deg \omega_X(a \cdot x)} \left( \deg X + \sum_{j=1}^n \frac{a_j}{2} \right) \right| \leq \frac{\ell_Y}{2}.$$ 

It is stable if it is semistable (i.e. (1.6) holds) and the strict inequality holds except when $Y$ or $Y^c$ is a disjoint union of exceptional components of $(X, \mathcal{O}_X(1), x)$.

We remark that the constant $M$ in the theorem can be estimated effectively depending on $g, n$ and $a$. However, as our approach is unlikely to produce a near optimal bound $M$, we made no efforts in this paper to trace the dependence of $M$ on $g$ and $a$. It is certainly an interesting and important question to optimize $M$, and it is known (cf. [2]) that the optimal $M$ is $4(2g - 2) + \epsilon$ (with $0 < \epsilon \ll 1$) when $a = 0$.

The case $x = \emptyset$ is a theorem of Caporaso [3] on the stability of polarized nodal curves. The case of the asymptotic Hilbert stability of smooth weighted pointed curves is a theorem of David Swinarski [27] (see also [15]).

We now sketch the main ingredients of our proof. Our starting point is a theorem of Mumford that expresses the $a$-$\lambda$-weight of Chow $(X, x)$ in terms of the leading coefficient of
the Hilbert-Samuel polynomial of an ideal \( J \subset O_{X \times \mathbb{A}^1}(1) \) (cf. Prop. 2.1). Our observation is that this leading coefficient can be evaluated by the leading coefficient of the Hilbert-Samuel polynomial of the pull back \( \tilde{J} \) of \( J \) to the normalization \( \tilde{X} \) of \( X \). This transforms the evaluation of the \( a_\lambda \)-weight to the calculation of the areas of a class of Newton polygons associated to the pull back sheaf \( \tilde{J} \). We then obtain an effective bound of the areas of these Newton polygons, thus a bound of the \( a_\lambda \)-weight of Chow \((X, x)\). Since this bound is linear in the weights of \( \lambda \), we can apply linear programing to complete a proof of Theorem 1.5.

Our GIT construction of the moduli of weighted pointed stable curves goes as follows. We form the Hilbert scheme \( \mathcal{H} \) of pointed one-dimensional subschemes of \( \mathbb{P}^m \) of fixed degree. Let \( \psi : \mathcal{H} \to \mathcal{C} \) be the equivariant Hilbert-Chow morphism (map) to the Chow variety of pointed one-dimensional cycles in \( \mathbb{P}^m \) of the same degree. Applying our main theorem, we conclude that in case the degree is sufficiently large, the preimage under \( \psi \) of the set \( \mathcal{C}^{ss} \subset \mathcal{C} \) of GIT-semistable points is the set of semistable polarized weighted pointed nodal curves. Let \( \mathcal{K} \subset \mathcal{H} \) be the subset of canonically polarized weighted pointed smooth curves. We prove that the GIT-quotient of the closure \( \overline{\mathcal{K}} \) is isomorphic to Hassett’s moduli of weighted pointed stable curves. An interesting observation is that the complement \( \overline{\mathcal{K}} - \mathcal{K} \) contains polarized semistable but not canonically polarized weighted pointed nodal curves. Thus though GIT gives the same compactification as that of Hassett of the moduli of canonically polarized weighted pointed smooth curves, the geometric objects added to obtain the compactification in the mentioned two constructions are different. It is worth pursuing to see how this extends in the high dimensional case.

In the end, using a result of Stoppa and the fact that the Donaldson-Futaki invariants can be expressed as the limit of normalized Chow weights under a 1-PS, we apply our main theorem to prove that a polarized nodal curve \((X, O_X(1))\) is \( K \)-stable if and only if \( O_X(1) \) is numerically proportional to \( \omega_X \) (cf. Theorem 7.1). This implies that GIT compactification is same as the compactification of smooth curves using \( K \)-stability.

The paper is organized as follows. In Section two, we show that the weights can be evaluated via the leading coefficients of the Hilbert-Samuel polynomial of a sheaf on the normalization \( \tilde{X} \). In Section three, we reduce our study to a particular class of 1-PS: the staircase 1-PS. We will derive a sharp bound for each irreducible component in Section four. We complete the proof of our main theorems in Section five. The last two sections include the applications of our stability study to constructing moduli of weighted pointed nodal curves and to study the \( K \)-stability of polarized curves.

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List of notations

\[ \mathcal{I}(\lambda); \mathcal{J}(\lambda) \]
\[ \mathcal{e}(\mathcal{J}(\lambda)); \mathcal{e}(\mathcal{J}) \]
\[ (\mathcal{I} \mathcal{e} \mathcal{E} \mathcal{L})(\mathcal{I}) \]
\[ \omega(\lambda) \]
\[ v(\bar{s}_i, q) \]
\[ h(q) \]
\[ h_{\alpha} \]
\[ \Delta_{\alpha} \]
\[ \tilde{E}_{i} = \tilde{E}(\lambda)_{i} \]
\[ \Lambda_{\alpha}(\lambda); \Lambda(\lambda) \]
\[ \delta(\bar{s}_i, p) \]
\[ \text{inc}_{\alpha}(\bar{s}_i) \]
\[ \delta_{\alpha}(\bar{s}_i); \delta(\tilde{s}_i) \]
\[ w(\tilde{E}_{i}, p); w_{\alpha}(\tilde{E}_{i}) \]
\[ l_{\alpha} = l_{\alpha}(\lambda) \]
\[ L_{Y}; L_{\alpha}; \tilde{L}_{\alpha} \]
\[ N_{Y}; N_{\alpha}; \tilde{N}_{\alpha} \]
\[ \ell_{\alpha}; \ell_{\alpha, \beta}; \ell_{\alpha, \alpha} \]
\[ \mathcal{E}_{\alpha}^{\mathcal{e}}(\rho) \]
\[ W_{\mathcal{I}} = W_{\mathcal{I}}(\lambda) \]
\[ \omega(\alpha) \]
\[ \Phi : \mathcal{H} \to \mathcal{C} \]
\[ \delta(\mathcal{L}) \]

2. Chow stability, Chow weight and Newton polygon

In this section, we first recall some basic facts from [16] on stability of a polarized curve; we then localize the calculation of the weight of Chow(X) to a divisor on the normalization of X, and interpret the contribution from each point of the divisor as the area of a generalized Newton polytope.

Let \( (X, \mathcal{O}_{X}(1)) \) be a polarized connected nodal curve, together with its associated embedding \( i : X \to \mathbb{P}W \) (cf. (1.1)), and its Chow point Chow(X). We will reserve the symbol \( \lambda \) for a 1-PS of \( SL(W) \); for such \( \lambda \), we diagonalize its action by choosing

\[ s = \{s_{0}, \ldots , s_{m}\} \quad \text{a basis of} \quad W^{\vee} \]

so that under its dual bases the action \( \lambda \) is given by

\[ \lambda(t) := \text{diag}(\lambda^{p_{0}}; \ldots ; \lambda^{p_{m}}) \cdot t^{-\rho_{\text{ave}}}, \quad \rho_{0} \geq \rho_{1} \geq \cdots \geq \rho_{m} = 0, \]

and \( \rho_{\text{ave}} = \frac{1}{m+1}\sum p_{i} \). We will call \( s \) a diagonalizing basis of \( \lambda \).

In [16], Mumford introduced a subsheaf

\[ \mathcal{J}(\lambda) = (t^{p_{0}}s_{0}, \ldots , t^{p_{m}}s_{m}) \subset \mathcal{O}_{X \times \mathbb{A}^{1}}(1) \]

generated by sections in the parenthesis, where \( p_{X} : X \times \mathbb{A}^{1} \to X \) is the projection. Let \( e(\mathcal{J}(\lambda)) \) be the normalized leading coefficient (abbreviate to n.l.c.) of the Hilbert-Samuel
polynomial:

\[
\chi(O_{X\times\mathbb{A}^1}(k)/\mathcal{I}(\lambda)^k) = e(\mathcal{I}(\lambda)) \frac{k^2}{2} + \text{lower order terms.}
\]

**Proposition 2.1** (Mumford). The \(\lambda\)-weight of \(\text{Chow}(X)\) is

\[
\omega(\lambda) = \frac{2 \deg X}{m + 1} \sum_{i=0}^m \rho_i - e(\mathcal{I}(\lambda)).
\]

In the following, when the 1-PS \(\lambda\) and its diagonalizing basis \(s\) are understood, we will drop \(\lambda\) from \(\mathcal{I}(\lambda)\) and abbreviate \(\mathcal{I}(\lambda)\) to \(\mathcal{I}\). Our first step is to lift the calculation of \(e(\mathcal{I})\) \((= e(\mathcal{I}(\lambda)))\) to the normalization \(\pi : \hat{X} \to X\).

We let

\[
\tilde{s}_i = \pi^* s_i \in O_{\hat{X}}(1) := O_X(1) \otimes_{O_X} O_{\hat{X}},
\]

and let \(\tilde{\mathcal{I}}\) be the pull-back of \(\mathcal{I}\):

\[
\tilde{\mathcal{I}} = \langle t_0 \tilde{s}_0, \ldots, t_m \tilde{s}_m \rangle \subset O_{\hat{X} \times \mathbb{A}^1}(1) = O_{\hat{X}}(1) \otimes_{O_{\hat{X}}} O_{\hat{X} \times \mathbb{A}^1}.
\]

We define \(e(\tilde{\mathcal{I}}) = \text{n.l.c.} \chi(O_{\hat{X} \times \mathbb{A}^1}(k)/\tilde{\mathcal{I}})^k\). We have the following special case of [16, Lemma 5.6] which enables us to lift the evaluation of \(e(\mathcal{I})\) to \(\hat{X}\). As [16, Lemma 5.6] was not proved in [16], we give a proof here shortly.

**Proposition 2.2.** We have \(e(\mathcal{I}) = e(\tilde{\mathcal{I}})\).

Our next step is to localize the evaluation of \(e(\tilde{\mathcal{I}})\) to individual \(q \in \hat{X}\). Let \(z\) be a uniformizing parameter of \(\hat{X}\) at \(q\); let \(t\) be the standard coordinates of \(\mathbb{A}^1\). We denote by \(\hat{O}_{\hat{X},q}\) the formal completion of the local ring \(O_{\hat{X},q}\) at its maximal ideal. We fix an isomorphism of \(O_{\hat{X},q}\)-modules (the first isomorphism below):

\[
\varphi_q : O_{\hat{X}}(1) \otimes_{O_{\hat{X}}} \hat{O}_{\hat{X},q} \cong \hat{O}_{\hat{X},q} \cong k[z],
\]

where the second isomorphism is induced by the choice of \(z\).

**Definition 2.3.** Let \(\tilde{s}_i \in H^0(O_{\hat{X}}(1))\) be as in (2.5). We define

\[
v(\tilde{s}_i, q) = \text{the vanishing order of } \tilde{s}_i \text{ at } q;
\]

in case \(\tilde{s}_i \equiv 0 \text{ near } q\), we define \(v(\tilde{s}_i, q) = \infty\). We set

\[
h(q) = \max \{i \mid v(\tilde{s}_i, q) \neq \infty\} \text{ and } w(\tilde{\mathcal{I}}, q) = v(\tilde{s}_{h(q)}, q).
\]

The quantity \(w(\tilde{\mathcal{I}}, q)\) is the width of the polygon \(\Delta_q\) associated to \(\tilde{\mathcal{I}}\) (at \(q\)) to be defined later.

We now look at the image of \(\tilde{\mathcal{I}}\) under \(O_{\hat{X} \times \mathbb{A}^1}(1) \to \hat{O}_{\hat{X} \times \mathbb{A}^1}(q, 0)\). We let

\[
I_q = \langle z^{v(\tilde{s}_0, q)} t_0^p, \ldots, z^{v(\tilde{s}_{m-1}, q)} t_{m-1}^p, z^{v(\tilde{s}_m, q)} t_m^p \rangle \subset R = k[z, t],
\]

agreeing \(z^\infty = 0\). By construction, \(\varphi_q\) induces an isomorphism

\[
\left( O_{\hat{X} \times \mathbb{A}^1}(k)/\tilde{\mathcal{I}}^k \right) \otimes_{O_{\hat{X} \times \mathbb{A}^1}} \hat{O}_{\hat{X} \times \mathbb{A}^1}(q, 0) \cong R/I_q^k.
\]

Notice that the right hand side is not a finite module when \(h(q) < m\). Since for all \(i\) we have \(t_i \varphi_q(\tilde{s}_i) \in (t_i^{p_i}) \subset R\), the induced homomorphism \((t^{k p_i(q)})/I_q^k \to R/I_q^k\) is injective, and \((t^{k p_i(q)})/I_q^k\) is a finite module. We define

\[
e(\tilde{\mathcal{I}})_q = \text{n.l.c. dim}

\left( (t^{k p_i(q)})/I_q^k \right) + 2 p_{h(q)} \cdot w(\tilde{\mathcal{I}}, q).
\]
We have the following formula, independently obtained by Swinarski; a special case can be found in [24, p. 300].

**Lemma 2.4.** We have the summation formula \( e(\hat{j}) = \sum_{q \in \hat{X}} e(\hat{j})_q \).

**Proof of Proposition 2.2.** Let \( p_1, \ldots, p_l \) be the nodes of \( X \); let \( \xi = \pi \times 1_{\hat{A}} : \hat{X} \times \hat{A}^l \to X \times A^l \) be the projection. Tensoring the exact sequence

\[
0 \to \mathfrak{O}_{X \times A^l} \to \xi_* \mathfrak{O}_{\hat{X} \times \hat{A}^l} \to \oplus_{j=1}^l \mathfrak{O}_{p_j \times A^l} \to 0
\]

with \( \mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k \), we obtain an exact sequence

\[
\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k \overset{f_k}{\to} (\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k) \otimes_{\mathfrak{O}_{X \times A^l}} \xi_* \mathfrak{O}_{\hat{X} \times \hat{A}^l} \to \bigoplus_{\alpha=1}^r (\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k)|_{p_j \times A^l} \to 0.
\]

By projection formula, we have

\[
\xi_* (\mathfrak{O}_{\hat{X} \times \hat{A}^l}(k)/\mathfrak{I}^k) = (\xi^* (\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k)) = (\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k) \otimes_{\mathfrak{O}_{X \times A^l}} \xi_* \mathfrak{O}_{\hat{X} \times \hat{A}^l}.
\]

Thus

\[
e(\hat{j}) = \text{n.l.c.} (\chi (\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k)) = \text{n.l.c.} (\chi ((\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k) \otimes_{\mathfrak{O}_{X \times A^l}} \xi_* \mathfrak{O}_{\hat{X} \times \hat{A}^l}),
\]

which equals

\[
\text{n.l.c.} (\chi (\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k) - \dim \ker f_k + \sum_{i=1}^l \chi ((\mathfrak{O}_{X \times A^l}(k)|_{p_j \times A^l})).
\]

We claim that both

\[
(2.13) \quad \chi ((\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k) \otimes_{\mathfrak{O}_{X \times A^l}} \mathfrak{O}_{p_j \times A^l}) \quad \text{and} \quad \dim \ker f_k
\]

are linear in \( k \). This will prove the Proposition.

We begin with the first claim. We let \( q \) be one of the nodes of \( X \); let \( q^+ \) and \( q^- \) be the preimages \( \pi^{-1}(q) \subset \hat{X} \), and let \( x \) and \( y \) be uniformizing parameters of \( \hat{X} \) at \( q^+ \) and \( q^- \), respectively. Then after fixing an isomorphism \( \mathfrak{O}_X(1) \otimes_{\mathfrak{O}_X} \mathfrak{O}_{X,q} \cong \mathfrak{O}_{X,q} \) near \( q \) and denoting \( R = k[x,y] \/ (xy) \), we have isomorphism

\[
(2.14) \quad (\mathfrak{O}_{X \times A^l}(k)/\mathfrak{I}^k) \otimes_{\mathfrak{O}_{X \times A^l}} \mathfrak{O}_{q \times A^l} \cong (R[t]/I^k) \otimes_{R[t]} R[t]/(x,y),
\]

where \( I \subset R[t] \) is the ideal generated by \( t^{s_i} s_i, i = 0, \ldots, m \), and \( s_i \) are formal germs of \( s_i \) at \( q \) as elements in \( R \). Since for some \( i \) the value \( s_i(q) \neq 0 \), \( i_q = \max \{ i \mid s_j(q) \neq 0 \} \) is finite. Thus the right hand side of (2.14) is isomorphic to \( R[t]/(I^k, x,y) = \]}\kon[4]t^{-i_q}y{[4]}$$ which is linear in \( k \). This proves the first claim.

For the second claim, since the kernel of \( f_k \) consists of torsion elements supported on the union of \( p_1 \times A^l, \ldots, p_l \times A^l \). Hence to prove the claim, we only need to study the kernel of an analogous homomorphism

\[
\hat{f}_k : R[t]/I^k \to (R[t]/I^k) \otimes_{R[t]} (k[x] [t] \oplus k[y] / [t]),
\]

where \( I \) is as in the previous paragraph, and \( R[t] \to k[x][t] \oplus k[y][t] \) is the normalization homomorphism \( g(x,y,t) \to (g(x,0,t),g(0,y,t)) \). Since the domain and the target of \( \hat{f}_k \) are \( t \)-graded rings and \( \hat{f}_k \) is a homomorphism of graded rings, as vector spaces

\[
\ker \hat{f}_k = \bigoplus_{j \geq 1} \ker \left\{ \left( \hat{f}_k \right)_j : t^j R/(I^k \cap t^j R) \to (t^j R/(I^k \cap t^j R)) \otimes_R (k[x] \oplus k[y]) \right\}.
\]
Because $R = \mathbb{k}[x, y]/(xy)$, as $R$-modules, $t^j R/(I^k \cap t^j R)$ is isomorphic to $R/J$ for $J$ one of the ideals in the list:

$$R, \ (0), \ (x^e), \ (y^e), \ (x^e + y^e), \text{ where } e, e' \in \mathbb{N}.$$  

One checks that for $J$ of the first five kinds, $\ker(\tilde{f}_k)_j = 0$; for $J$ of the last kind, $\ker(\tilde{f}_k)_j \cong \mathbb{k}$. Thus we always have $\dim \ker(\tilde{f}_k)_j \leq 1$. On the other hand, since $s_{tq}(q) \neq 0$, $t^\rho_{sa} \in I$ and $t^{k\rho_{sa}} \in I^k$. Thus $\ker(\tilde{f}_k)_j = 0$ for $j \geq ki_q$. This proves that $\dim \ker f_k$ is at most linear in $k$. This proves the Proposition. $\square$

Because of this Proposition, we will work over the normalization $\tilde{X}$ of $X$ subsequently. To avoid possible confusion, we will reserve "\(\sim\)" to denote the associated objects lifted to $\tilde{X}$. For instance, we will denote by $X_1, \ldots, X_r$ the irreducible components of $X$, and denote by $\tilde{X}_1, \ldots, \tilde{X}_r$ their respective normalizations. For the sections $t^{\rho_i}s_i$ in $J$, $t^{\rho_i}\tilde{s}_i$ are their lifts in $J = J \otimes_{\mathcal{O}_{\tilde{X}\times \mathbb{A}^l}} \mathcal{O}_{\tilde{X}\times \mathbb{A}^l}$. For consistency, we reserve subindices $i$ for the sections $s_i$, and reserve the Greek $\alpha$ for the indices of the irreducible components $\{X_\alpha\}_{1 \leq \alpha \leq r}$.

**Proof of Lemma 2.4.** Letting $\tilde{J}_\alpha = J|_{\tilde{X}_\alpha \times \mathbb{A}^l} \subset \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(1)$, then

$$e(\tilde{J}) = \sum_{\alpha=1}^r \text{n.l.c. } \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{J}_\alpha^k) = \sum_{\alpha=1}^r e(\tilde{J}_\alpha).$$

For $q \in \tilde{X}_\alpha$, we denote $e(\tilde{J}_\alpha)_q = e(\tilde{J})_q$. Thus to prove the Lemma, we only need to show that

$$e(\tilde{J}_\alpha) = \sum_{q \in X_\alpha} e(\tilde{J}_\alpha)_q.$$

To proceed, we first note that $h(q)$ (cf.(2.9)) is locally constant on $\tilde{X}$, hence constant on each individual component $\tilde{X}_\alpha \subset X$. We let $h_\alpha = h(q)$ for a $q \in \tilde{X}_\alpha$. Then we have

$$h_\alpha = \max_i \{i \mid \tilde{s}_j|_{\tilde{X}_\alpha} = 0, \text{ for } j \geq i + 1\}.$$  

We claim $t^{\rho_{sa}}$ divides $t^{\rho_i} \tilde{s}_i$ for all $i$. Indeed, the case $i > h_\alpha$ follows from $\tilde{s}_i|_{\tilde{X}_\alpha} \equiv 0$; the case $i \leq h_\alpha$ follows from $\rho_i \geq \rho_{sa}$. We let $\bar{\rho}_i = \rho_i - \rho_{sa}$, and introduce ideal

$$\tilde{R}_\alpha = (t^{\bar{\rho}_0} \tilde{s}_0, t^{\bar{\rho}_1} \tilde{s}_1, \ldots, t^{\bar{\rho}_h} \tilde{s}_h) \subset \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(1).$$

This way, $\tilde{J}_\alpha = t^{\rho_{sa}} \tilde{R}_\alpha \subset t^{\rho_{sa}} \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(1)$.

We let $(t^{k\rho_{sa}}) = t^{k\rho_{sa}} \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)$; it belongs to the exact sequence

$$0 \rightarrow (t^{k\rho_{sa}})/\tilde{J}_\alpha^k \rightarrow \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{J}_\alpha^k \rightarrow \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/(t^{k\rho_{sa}}) \rightarrow 0.$$  

Since $(t^{k\rho_{sa}})/\tilde{J}_\alpha^k = t^{k\rho_{sa}} (\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{R}_\alpha^k)$ and $\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{R}_\alpha^k$ is finite, we have

$$\chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{J}_\alpha^k) = \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{R}_\alpha^k) + \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/(t^{k\rho_{sa}})).$$

Taking the n.l.c. of individual term, and using

$$\chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/(t^{k\rho_{sa}})) = \rho_{sa} \cdot \chi(\mathcal{O}_{\tilde{X}_\alpha}(k)) = \rho_{sa} \cdot \deg X_\alpha + O(k),$$

we obtain

$$e(\tilde{J}_\alpha) = \text{n.l.c. } \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{J}_\alpha^k) = \text{n.l.c. } \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{R}_\alpha^k) + 2\rho_{sa} \cdot \deg \tilde{X}_\alpha.$$  

Next let $\{q_1, \ldots, q_r\}$ be the support of $(\tilde{s}_h) = 0 \cap \tilde{X}_\alpha$. Following the convention in (2.11), we have an isomorphism

$$\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^l}(k)/\tilde{R}_\alpha^k \cong \bigoplus_{\alpha=1}^r t^{k\rho_{sa}} R/(I_{q_\alpha} \cap t^{\rho_{sa}} R)^k,$$
induced by restricting to germs at \( q_a \) after multiplying by \( t^{k \rho_{h_a}} \). Adding that \( \deg \tilde{X}_\alpha = \dim \mathcal{O}_{\tilde{X},\alpha}(1)/(\tilde{h}_\alpha) = \sum_{a=1}^l w(\tilde{\lambda}, q_a) \), (2.17) gives us

\[
 e(\tilde{\lambda}) = \sum_{a=1}^l \left( \text{n.l.c. dim}(t^{k \rho_{h_a}} R/(I_{q_a} \cap t^{\rho_{h_a}} R)^k) + 2 \rho_{h_a} \cdot w(\tilde{\lambda}, q_a) \right) = \sum_{q \in \tilde{X}_\alpha} e(\tilde{\lambda})_q.
\]

This proves the Lemma. \( \square \)

**Example 2.5.** Let \( \lambda \) be a 1-PS with diagonalizing basis \( \{ s_i \} \) and weights \( \rho_0 = 1 > \rho_1 = \cdots = \rho_m = 0 \). Suppose \( (s_1 = \cdots = s_m = 0) \) is a reduced point \( q \in X \). Then \( e(\lambda) = 1 \) and \( \omega(\lambda) = \frac{2 \deg X_m + 1}{m + 1} \) (resp. \( e(\lambda) = 2 \) and \( \omega(\lambda) = \frac{2 \deg X_m}{m + 1} - 2 \) when \( q \) is a smooth point (resp. nodal point) of \( X \).

We give a useful geometric interpretation of the quantity \( e(\lambda)_q \). Let \( I \subset \mathbb{k}[z_1, z_2] \) be a monomial ideal and let \( \Gamma \) be the set of exponents of monomials in \( I \); namely, \( I \) is the linear span of the monomials \( \{ x^\gamma | \gamma \in \Gamma \} \), thus

\[
 \Gamma \subset (\mathbb{N} \cup \{ 0 \})^2 \subset \mathbb{R}^2_0 := (\mathbb{R} \geq 0)^2 \subset \mathbb{R}^2.
\]

We then form the closed convex hull \( \text{Conv}(\mathbb{R}^2_0 + \Gamma) \subset \mathbb{R}^2 \) of \( \mathbb{R}^2_0 + \Gamma \), and let \( \bar{\Gamma} = \text{Conv}(\mathbb{R}^2_0 + \Gamma) \cap \mathbb{N}^2 \); then the integral closure \( \bar{I} \) of \( I \) is the ideal generated by \( \{ x^\gamma | \gamma \in \Gamma \} \) (cf. [5, Exercise. 4.23, p. 141])

We let \( \Delta(I) \) be the Newton polygon of \( I \):

\[
 \Delta(I) = \mathbb{R}^2_0 - \text{Conv}(\mathbb{R}^2_0 + \Gamma) \subset \mathbb{R}^2.
\]

**Lemma 2.6.** Let \( |\Delta(I)| \) be the area of the \( \Delta(I) \). Then

\[
 \dim \mathbb{k}[z_1, z_2]/I^k = |\Delta(I)| \cdot k^2 + O(k).
\]

**Proof.** Since \( \bar{I} \) is the integral closure of \( I \), by Briancon-Skoda theorem [13, Thm 9.6.26], \( I^k \subset \bar{I}^k \subset I^{k-1} \) for \( k \) sufficiently large. Since \( \dim I^{k-1}/I^k \) is bounded from above by a linear function in \( k \), \( \dim \mathbb{k}[z_1, z_2]/I^k = \dim \mathbb{k}[z_1, z_2]/\bar{I}^k + O(k) \).

Further, \( \dim \mathbb{k}[z_1, z_2]/I^k \) is precisely the number of lattice points in \( k \Delta(\bar{I}) = k \Delta(I) \).

From the work of Kantor and Khovanskii [12, 4], the number of lattice points inside the polygon is given by \( |\Delta(I)| \cdot k^2 + O(1) \). This proves the Lemma. \( \square \)

We now come back to the 1-PS \( \lambda \) and its diagonalizing basis \( s = \{ s_i \} \).

**Definition 2.7.** For any \( q \in \tilde{X} \), we define

\[
 \Gamma_q = \{(v(\tilde{s}_i, q), \rho_i) \mid i = 0, \cdots, m; v(\tilde{s}_i, \rho_i) < \infty \} \subset (\mathbb{N} \cup \{ 0 \})^2;
\]

we define the Newton polygon (of \( \tilde{\lambda} = \tilde{\lambda}(\lambda) \)) at \( q \) to be

\[
 \Delta_q(\lambda) := (\mathbb{R}^2_0 - \text{Conv}(\mathbb{R}^2_0 + \Gamma_q)) \cap ([0, w(\tilde{\lambda}, q)] \times \mathbb{R}_0^2).
\]

We will abbreviate \( \Delta_q(\lambda) \) to \( \Delta_q \) when the choice of the basis \( s \) is understood. Let \( |\Delta_q| \) be the area of \( \Delta_q \). We state a formula useful for estimating the quantity \( e(\lambda)_q = e(\tilde{\lambda})_q \).

**Corollary 2.8.** We have \( e(\tilde{\lambda})_q = 2|\Delta_q| \); hence, \( e(\tilde{\lambda}) = 2 \sum_{q \in \tilde{X}} |\Delta_q| \).

**Proof.** Since \( \Delta_q \) is the union of \( \Delta_q \cap [0, w(\tilde{\lambda}, q)] \times [\rho_{h_q}, \infty) \) with \( [0, w(\tilde{\lambda}, q)] \times [0, \rho_{h_q}) \), by (2.4), (2.12) and Lemma 2.6,

\[
 e(\tilde{\lambda})_q = 2 \cdot |\Delta_q \cap [0, w(\tilde{\lambda}, q)] \times [\rho_{h_q}, \infty)| + 2 \cdot \rho_{h_q} \cdot w(\tilde{\lambda}, q) = 2 |\Delta_q|.
\]

The second identity follows from Lemma 2.4. \( \square \)
3. Staircase One-parameter subgroups

We begin with some conventions attached to a fixed 1-PS \( \lambda \) and its diagonalizing basis \( \{ s_0, \ldots, s_m \} \). For simplicity, we denote
\[
\emptyset = \{0,1,\ldots, m\}.
\]
For each \( i \in \emptyset \), we introduce subsheaves
\[
(3.1) \quad \mathcal{E}_i = \mathcal{E}(\lambda)_i := (s_i, s_{i+1}, \ldots, s_m) \subset \mathcal{O}_X(1);
\]
they form a decreasing sequence of subsheaves. Similarly, we introduce \( \mathcal{O}_X \)-submodules
\[
(3.2) \quad \tilde{\mathcal{E}}_i = \tilde{\mathcal{E}}(\lambda)_i := (\tilde{s}_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_m) \subset \mathcal{O}_{\tilde{X}}(1).
\]

**Definition 3.1.** We call \( i \in \emptyset \) a base index if \( i = h_\alpha \) (cf. (2.16)) for some irreducible component \( X_\alpha \). For each \( X_\alpha \), we define \( \Lambda_\alpha(\lambda) = \{ q \in X_\alpha \mid s_{h_\alpha}(q) = 0 \} \); define \( \Lambda(\lambda) = \bigcup_{\alpha=1}^n \Lambda_\alpha(\lambda) \); define \( \tilde{\Lambda}_\alpha(\lambda) = \{ p \in \tilde{X}_\alpha \mid \tilde{s}_{h_\alpha}(p) = 0 \} \), and define \( \tilde{\Lambda}(\lambda) = \bigcup_{\alpha=1}^n \tilde{\Lambda}_\alpha(\lambda) \).

In the following, for any sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{F} \) and \( p \in \tilde{X} \), we denote \( \mathcal{F}_p := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\tilde{X},p} \), the localization of \( \mathcal{F} \) at \( p \). We remark that for any \( p \in \tilde{X}_\alpha \), \( \tilde{h}(p) = h_\alpha \) is the largest index \( i \) so that \( (\tilde{\mathcal{E}}_i)_p \neq 0 \).

**Definition 3.2.** For a closed point \( p \in \tilde{X}_\alpha \subset \tilde{X} \), we define
\[
(3.3) \quad \delta(\tilde{s}_i, p) = \text{length}(\tilde{\mathcal{E}}_i/\tilde{\mathcal{E}}_{i+1})_p \quad \text{when } i \leq h_\alpha - 1 = h(p) - 1; \quad \delta(\tilde{s}_i, p) = 0 \quad \text{otherwise}.
\]
We define the increments of \( \tilde{s}_i \) along \( \tilde{X}_\alpha \) and \( \tilde{X} \) be the 0-cycles
\[
\text{inc}_\alpha(\tilde{s}_i) = \sum_{p \in \tilde{X}_\alpha} \delta(\tilde{s}_i, p) \quad \text{and} \quad \text{inc}(\tilde{s}_i) = \sum_{\alpha} \text{inc}_\alpha(\tilde{s}_i);
\]
we define their degrees be \( \delta_\alpha(\tilde{s}_i) = \sum_{p \in \tilde{X}_\alpha} \delta(\tilde{s}_i, p) \) and \( \delta(\tilde{s}_i) = \sum_{\alpha} \delta_\alpha(\tilde{s}_i) \). We define the width of \( \tilde{\mathcal{E}}_i \) at \( p \in \tilde{X}_\alpha \) and at \( \tilde{X}_\alpha \) for \( i \leq h_\alpha \) be
\[
(3.4) \quad w(\tilde{\mathcal{E}}_i, p) := \text{length}(\mathcal{O}_{\tilde{X}}(1)/\tilde{\mathcal{E}}_i)_p \quad \text{and} \quad w_\alpha(\tilde{\mathcal{E}}_i) := \sum_{p \in \tilde{X}_\alpha} w(\tilde{\mathcal{E}}_i, p).
\]

We remark that for \( p \in \tilde{X}_\alpha \), \( i + 1 \leq h(p) \) is equivalent to \( (\tilde{\mathcal{E}}_{i+1})_p \neq 0 \).

**Definition 3.3.** For any irreducible component \( X_\alpha \subset X \) we introduce
\[
(3.5) \quad \ll_\alpha = \ll_\alpha(\lambda) = \{ i \in \emptyset \mid \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha \neq \emptyset \ or \ i = h_\alpha \};
\]
for \( m_\alpha + 1 = |\ll_\alpha| \), the order of \( \ll_\alpha \), we introduce a re-indexing map
\[
(3.6) \quad \text{ind}_\alpha : \ll_\alpha \longrightarrow [0, m_\alpha] \cap \mathbb{Z}, \quad \text{order preserving and bijective}.
\]
Similarly, for \( p \in \tilde{X} \), we introduce
\[
(3.7) \quad \ll_p = \{ i \in \emptyset \mid p \in \text{inc}(\tilde{s}_i) \}.
\]
For \( m_p + 1 = |\ll_p| \); we define similarly
\[
\text{ind}_p : \ll_p \longrightarrow [0, m_p] \cap \mathbb{Z}, \quad \text{order preserving and bijective}.
\]

**Definition 3.4.** We say a 1-PS \( \lambda \) is a semi-staircase at index \( i \) if
\[
\mathcal{E}_i \supseteq \mathcal{E}_{i+1} \supseteq \cdots \supseteq \mathcal{E}_m.
\]
We say \( \lambda \) is a semi-staircase when \( \lambda \) is a semi-staircase after index 1.

**Proposition 3.5.** Given a 1-PS \( \lambda \), there is a semi-staircase 1-PS \( \lambda' \) with \( \rho_i' = \rho_i \) for all \( i \) so that \( \omega(\lambda) \geq \omega(\lambda') \).
Proof. Suppose λ is a semi-staircase at index i but not at i − 1, then
\[(3.5)\]  
\[\mathcal{E}_0 \supseteq \cdots \supseteq \mathcal{E}_{i-2} \supseteq \mathcal{E}_{i-1} = \mathcal{E}_i \supseteq \mathcal{E}_{i+1} \supseteq \cdots \supseteq \mathcal{E}_m.\]

Therefore, there is a point \(p \in X\) such that if we denote by \(\hat{s}_j \in \mathfrak{O}_{X,p}(1)\) the formal germ of \(s_j\) at \(p\), then as \(\mathfrak{O}_{X,p}\)-modules
\[(3.6)\]  
\[\hat{\mathfrak{O}}_{X,p}(1) \supset (\hat{s}_{i-1}, \ldots, \hat{s}_m) = (\hat{s}_i, \ldots, \hat{s}_m) \supset (\hat{s}_{i+1}, \ldots, \hat{s}_m).\]

By the middle equality, we can find \(\hat{c}_j \in \mathfrak{O}_{X,p}\) such that \(\hat{s}_{i-1} = \sum_{j=1}^m \hat{c}_j \hat{s}_j\). We now construct a new basis \(s'\). Let \(c = \hat{c}_i(p) \in \k\). We define
\[(3.7)\]  
\[s'_j = s_j \text{ for } j \neq i, i-1; \quad s'_i = s_{i-1} - cs_i; \quad s'_{i-1} = s_i.\]

Clearly, \(s' = \{s'_i\}\) is a basis of \(H^0(\mathfrak{O}_X(1))\). Let \(\mathcal{E}'_j\) be the \(\mathcal{E}_j\) in (3.1) with \(s_i\) replaced by \(s'_i\). For \(j \neq i\), because the linear spans of \(\{s_j, \ldots, s_m\}\) and of \(\{s'_j, \ldots, s'_m\}\) are identical, we have \(\mathcal{E}_j = \mathcal{E}'_j\). For \(i\), we claim that \(\mathcal{E}'_i \subseteq \mathcal{E}_i\). The inclusion \(\mathcal{E}'_i \subset \mathcal{E}_{i-1} = \mathcal{E}_i\). For the inequality, we claim that
\[\hat{s}_{i-1} - cs_i, \hat{s}_{i+1}, \ldots, \hat{s}_m \neq (\hat{s}_i, \hat{s}_{i+1}, \ldots, \hat{s}_m).\]

Suppose instead the equality holds, then there are constants \(a_j \in \k\) such that
\[\hat{s}_i = \sum_{j=i+1}^m a_j \hat{s}_j = (a_i(\hat{s}_{i-1} - \hat{c}_i \hat{s}_i) + \sum_{j=i+1}^m a_j \hat{s}_j) + a_i(\hat{c}_i - c) \hat{s}_i.\]

Combined with \(\hat{s}_{i-1} = \sum_{j=i+1}^m \hat{c}_j \hat{s}_j\), we conclude that \(\hat{s}_i \in (\hat{s}_{i+1}, \ldots, \hat{s}_m) + \hat{s}_i \mathfrak{m}\), where \(\mathfrak{m} \subset \hat{\mathfrak{O}}_{X,p}\) is the maximal ideal. By Nakayama Lemma, \(\hat{s}_i \in (\hat{s}_{i+1}, \ldots, \hat{s}_m)\), contradicting to (3.6). This proves the claim.

Finally, we claim that if we define \(\lambda'\) be the 1-PS with diagonalizing basis \(s'\) and associated weights \(\rho'_i = \rho_i\), then \(\omega(\lambda') \leq \omega(\lambda)\). By Mumford’s formula (cf. Prop. 2.1),
this is equivalent to \( e(\mathcal{I}(\lambda')) \geq e(\mathcal{I}(\lambda)) \). By our construction, \( \mathcal{E}_i' \subseteq \mathcal{E}_i \) for all \( i \in I \); hence since \( \rho_{i-1} \geq \rho_i \), \( \mathcal{I}(\lambda') \subset \mathcal{I}(\lambda) \). Thus \( \mathcal{O}_{X \times \mathbb{A}}(k)/\mathcal{I}(\lambda')^k \) surjects onto \( \mathcal{O}_{X \times \mathbb{A}}(k)/\mathcal{I}(\lambda)^k \). This proves \( e(\mathcal{I}(\lambda')) \geq e(\mathcal{I}(\lambda)) \).

In conclusion, for any \( \lambda \) that is not a semi-staircase (cf. black part in Figure 1), we have constructed a new \( \lambda' \) whose associated filtration of subsheaves \( \mathcal{E}_i' \) satisfying \( \mathcal{E}_i' = \mathcal{E}_i \) for \( j \neq i, i-1 \), and

\[
\mathcal{E}_0' \supseteq \cdots \supseteq \mathcal{E}_{i-1}' \supseteq \mathcal{E}_i' \supseteq \mathcal{E}_{i+1}' \supseteq \cdots \supseteq \mathcal{E}_m'.
\]

If \( \mathcal{E}_i' = \mathcal{E}_{i+1}' \) (cf. blue part in Figure 1), we repeat this process at \( i+1 \). Since we always have \( \mathcal{E}_{m-1} \supseteq \mathcal{E}_m \), after finitely many steps, we obtain a \( \lambda' \) that is a semi-staircase at \( i-1 \). An induction on \( i \) proves the Proposition.

**Definition 3.6.** We say a semi-staircase 1-PS \( \lambda \) is a staircase if for any \( p \in \hat{\Lambda} \), \( v(\hat{s}_i, p) \leq v(\hat{s}_{i+1}, p) \) for all \( i \) (cf. Definition 2.3).

**Proposition 3.7.** Given a 1-PS \( \lambda \), there is a staircase 1-PS \( \lambda' \) with \( \rho'_i = \rho_i \) for all \( i \) so that \( \omega(\lambda) \geq \omega(\lambda') \).

**Figure 2.** The figure shows how vanishing order \( v(s_1, p) \) varies under the general perturbation of the section \( s_1 \) when one creates a staircase from a semi-staircase.

**Proof.** By Proposition 2.1, the \( \lambda \)-weight \( \omega(\lambda) \) (of Chow \( (X) \)) depends only the sheaf \( \mathcal{I}(\lambda) \) and the weights \( \{\rho_i\} \). Thus, for any 1-PS \( \lambda' \) with \( \mathcal{I}(\lambda) = \mathcal{I}(\lambda') \) and having identical weights \( \{\rho'_i\} \) as those of \( \lambda \), we have \( \omega(\lambda) = \omega(\lambda') \).

Given any 1-PS, we let \( \lambda \) be the corresponding semi-staircase constructed in Proposition 3.5. Let \( \hat{\Lambda} \) be the associated objects of \( \lambda \). Since \( \hat{\Lambda} \) is a finite set, if we replace \( s_i \) by \( s'_i = s_i + \sum_{j > i} c_{ij} s_j \) for a general choice of \( c_{ij} \in \mathbb{k} \), the new 1-PS with the same \( \{\rho_i\} \) but new basis \( \{s'_i\} \) will be the desired staircase 1-PS.

**Lemma 3.8.** Suppose \( \lambda \) is a staircase 1-PS, then for any \( p \in \hat{X}_\alpha \) and \( i \leq h_\alpha \), \( w(\hat{s}_i, p) = v(\hat{s}_i, p) \), and \( \delta(\hat{s}_{i-1}, p) = v(\hat{s}_i, p) - v(\hat{s}_{i-1}, p) \).

**Proof.** As \( i \leq h_\alpha \), both \( \hat{s}_i \) and \( \hat{s}_{i-1} \) are restrict non-trivially to \( X_\alpha \). The identity is a direct consequence of the definition of staircase 1-PS.
As we will see, if $\lambda$ is a staircase 1-PS then for most of $i$, $\delta(\tilde{s}_i) = 1$. For those $i$ with $\delta(\tilde{s}_i) > 1$, we will give a detailed characterization (cf. Prop. 3.9). To this purpose, for any subscheme $Y \subset X$, we let $N_Y = X_{\text{node}} \cap Y$ be the set of nodes of $X$ in $Y$. We let (recall $L_Y := Y \cap Y^0$ cf. (1.3))

$$\tilde{N}_Y := \pi^{-1}(N_Y) \cap \tilde{Y} \quad \text{and} \quad \tilde{L}_Y := \pi^{-1}(L_Y) \cap \tilde{Y} \subset \tilde{N}_Y.$$  

As $\alpha$ is reserved for the index of the components $X_\alpha$, we abbreviate

$$N_\alpha := N_{X_\alpha}, \quad \tilde{N}_\alpha := \tilde{N}_{X_\alpha}, \quad L_\alpha := L_{X_\alpha}, \quad \tilde{L}_\alpha := \tilde{L}_{X_\alpha}, \quad \ell_\alpha := |L_\alpha|.$$

**Proposition 3.9.** Suppose $\lambda$ is a staircase 1-PS. Let $i \in I_\alpha$ be a non-base index (cf. Definition 3.1) and $p \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$. Suppose $\delta(\tilde{s}_i) \geq 2$, and that either $\deg X_\alpha = 1$ or

$$w_\alpha(\tilde{E}_i) + 1 \leq \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha,$$

then $q = \pi(p) \in X$ is a node of $X$, $\text{ind}_p(i) = 0$ and $\delta(\tilde{s}_i, p) = 1$. In this case, let $\{p, p'\} = -\tilde{q}$ and let $\tilde{X}_\beta$ be a component satisfying $p' \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\beta$ (possibly $\tilde{X}_\alpha = \tilde{X}_\beta$), assuming $\deg X_\beta > 1$ and $w_\beta(\tilde{E}_i) + 1 \leq \deg X_\beta - 2g(X_\beta) - \ell_\beta$, then $\text{inc}(\tilde{s}_i) = p + p'$.  

*Proof.* We adopt the following convention. Since $\tilde{X}_\alpha$ is smooth, we can view a zero-subscheme of $X_\alpha$ as a divisor as well. This way, the union of two effective divisors is the union as zero subschemes, and the sum is as sum of divisors. For example, $(\sum n_p p) \cup (\sum n'_p p) = \max\{n_p, n'_p\}p$ and $(\sum n_p p) + (\sum n'_p p) = \sum (n_p + n'_p)p$.

We will prove each part of the statement by repeatedly applying the following strategy. Suppose $i$ satisfies (3.10) and $\delta(\tilde{s}_i) \geq 2$, we will construct a section $\zeta \in H^0(\mathcal{O}_X(1))$ so that the $O_X$-modules $\mathcal{F}_j = (\zeta, s_j, \ldots, s_m)$ fits into a strict filtration

$$\mathcal{F}_0 \supseteq \cdots \supseteq \mathcal{F}_i \supseteq \mathcal{F}_{i+1} \supseteq \mathcal{E}_{i+1} \supseteq \cdots \supseteq \mathcal{E}_m \neq 0.$$  

Since $\mathcal{E}_j$ and $\mathcal{F}_j$ are generated by global sections of $H^0(\mathcal{O}_X(1))$, this implies $h^0(\mathcal{O}_X(1)) \geq m + 2$, a contradiction.

We first assume $\deg X_\alpha > 1$. Then $w_i(\tilde{E}_i)$ satisfies (3.10). We recall an easy consequence of a vanishing result. Let $B \subset \tilde{X}_\alpha$ be a closed zero-subscheme satisfying

$$\deg B \leq \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha + 1;$$

let $\tilde{N}_\alpha$ be as defined in (3.9). We claim that the $\gamma$ in the exact sequence

$$H^0(\mathcal{O}_{\tilde{X}_\alpha}(1)) \xrightarrow{\gamma} H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1)) \rightarrow H^1(\mathcal{O}_{\tilde{X}_\alpha}(1)(-\tilde{N}_\alpha \cup B))$$  

is surjective. Indeed, this follows from $\deg \tilde{N}_\alpha = 2g(X_\alpha) - 2g(\tilde{X}_\alpha) + \ell_\alpha$ and (3.12), which gives $\deg \mathcal{O}_{\tilde{X}_\alpha}(1)(-\tilde{N}_\alpha \cup B) \geq 2g(\tilde{X}_\alpha) - 1$, thus the last term in (3.13) vanishes.

The section $\zeta$ mentioned before (3.11) will be chosen by picking an appropriate $B$ and $v \in H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1))$ so that any element $\tilde{\zeta}_\alpha \in \gamma^{-1}(v)$ descends to a section in $H^0(\mathcal{O}_{\tilde{X}_\alpha}(1))$ which glues with $s_{i+1}|_{\tilde{X}_\alpha}^0$ to form the desired section $\zeta$.

We let

$$\tilde{Z}_{\alpha,j} := (\tilde{s}_j = \cdots = \tilde{s}_m = 0) \cap \tilde{X}_\alpha \subset \tilde{X}_\alpha.$$  

Since $p \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$, $\delta_\alpha(\tilde{s}_i) \geq 1$. In case $\delta_\alpha(\tilde{s}_i) = 1$, we choose $B = \tilde{Z}_{\alpha,i} + p$, which is a subscheme of $\tilde{Z}_{\alpha,i+1}$. In case $\delta_\alpha(\tilde{s}_i) \geq 2$ and $\delta(\tilde{s}_i, p) = 1$, then there exists a $p' \neq p \in \tilde{X}_\alpha$ such that $p + p' \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$, (which is equivalent to $\tilde{Z}_{\alpha,i} + p + p' \subset \tilde{Z}_{\alpha,i+1}$). In case $\delta(\tilde{s}_i, p) \geq 2$, we choose $p' = p$. Combined, we let $B = \tilde{Z}_{\alpha,i} + p + p'$.

We then let

$$v_1 = \tilde{s}_{i+1}|_{\tilde{N}_\alpha} \in H^0(\mathcal{O}_{\tilde{N}_\alpha}(1)) \quad \text{and} \quad v_2 \neq 0 \in H^0(\mathcal{O}_B(1)) \quad \text{s.t.} \quad v_2|_{B-p} = 0.$$  


We claim that when \( p \notin \tilde{N}_\alpha \), or \( \text{ind}_p(i) \geq 1 \), or \( \delta(\tilde{s}_i, p) \geq 2 \), then both \( v_1|_{\tilde{N}_\alpha \cap B} \) and \( v_2|_{\tilde{N}_\alpha \cap B} \) are zero.

Indeed, since \( \tilde{N}_\alpha \cap B \subset \tilde{Z}_{\alpha,i+1} \) and \( \tilde{s}_{i+1}|_{\tilde{Z}_{\alpha,i+1}} = 0 \), we have \( v_1|_{\tilde{N}_\alpha \cap B} = \tilde{s}_{i+1}|_{\tilde{N}_\alpha \cap B} = 0 \). For \( v_2 \), we prove case by case. Suppose \( p \notin \tilde{N}_\alpha \), then \( \tilde{N}_\alpha \cap B = \tilde{N}_\alpha \cap (B - p) \); therefore since \( v_2|_{B - p} = 0 \), \( v_2|_{\tilde{N}_\alpha \cap B} = 0 \). Now suppose \( p \in \tilde{N}_\alpha \). Since \( v_2|_{B - p} = 0 \), \( v_2(p) = 0 \) for all \( \tilde{p} \in (\tilde{N}_\alpha \cap B) - \{p\} \). It remains to show that \( v_2(p) = 0 \). We write \( B = \sum_{k=0}^{l} n_k p_k \), \( p_k \) distinct, as an effective divisor. Since \( p \in B \), we can arrange \( p_0 = p \). In case \( \text{ind}_p(i) \geq 1 \), we have \( n_0 \geq 2 \); in case \( \delta(\tilde{s}_i, p) \geq 2 \), then \( p' = p \) we still have \( n_0 \geq 2 \). Thus \( p \in B - p \) and \( v_2(p) = 0 \). This proves that \( v_1 \) and \( v_2 \) have identical images in \( H^0(\mathcal{O}_{\tilde{N}_\alpha \cap B}(1)) \).

Consequently, \( (v_1, v_2) \) lifts to a section \( v \in H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1)) \) using the exact sequence

\[
H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1)) \rightarrow H^0(\mathcal{O}_{\tilde{N}_\alpha}(1)) \oplus H^0(\mathcal{O}_B(1)) \rightarrow H^0(\mathcal{O}_{\tilde{N}_\alpha \cap B}(1)) .
\]

Since \( \deg B \leq w_\alpha(\tilde{E}_i) + 2 + i \) satisfies (3.10), by the assumption that \( \deg X_\alpha > 1 \), \( \deg B \) satisfies the inequality (3.12). Therefore, the \( \gamma \) in (3.13) is surjective. We let \( \tilde{\zeta}_\alpha \in \gamma^{-1}(v) \subset H^0(\mathcal{O}_{\tilde{N}_\alpha}(1)) \) be any lift. Because it is a lift of \( v \), \( \tilde{\zeta}_\alpha|_{\tilde{N}_\alpha} = \tilde{s}_{i+1}|_{\tilde{N}_\alpha} \). This implies that \( \tilde{\zeta}_\alpha \) descends to a section \( \zeta_\alpha \in H^0(\mathcal{O}_{X_\alpha}(1)) \), and the descent \( \zeta_\alpha \) glues with \( s_{i+1}|_{X_\alpha} \) to form a new section \( \zeta \in H^0(\mathcal{O}_{X}(1)) \).

We now prove the first part of the Proposition. We let \( Z_{\alpha,j} \subset X_\alpha \) be the subscheme \( Z_{\alpha,j} = (s_j = \cdots = s_m = 0) \cap X_\alpha \). We decompose \( Z_{\alpha,j} \) into disjoint union \( Z_{\alpha,j} = R_j \cup R'_j \) so that \( R_j \) is supported at \( q = \pi(p) \) and \( R'_j \) is disjoint from \( q \). We let \( \tilde{Z}_\alpha = (\zeta = s_{i+1} = \cdots = s_m = 0) \cap X_\alpha \) and decompose \( \tilde{Z}_\alpha = \tilde{R} \cup \tilde{R}' \) accordingly.

Suppose \( q \) is a smooth point of \( X \). Then \( R_j \) and \( \tilde{R} \) are divisors, and can be written as \( R_j = n_j q \) and \( \tilde{R} = \tilde{n} q \). In case \( \delta_\alpha(\tilde{s}_i) = 1 \), the choice of \( B \) ensures that \( n_i = \tilde{n} = n_{i+1} - 1 \) and \( R'_i \subset \tilde{R} \subset \tilde{R}_i \subset \tilde{R}_i+1 \). Thus

\[
(s_i, \cdots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supset (s_i, s_{i+1}, \cdots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supset (s_i, s_{i+1}, \cdots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} .
\]

Further, since \( \delta(\tilde{s}_i) \geq 2 \) and \( \zeta|_{X_\alpha} = s_{i+1}|_{X_\alpha} \), we have

\[
(s_i, \cdots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supset (s_i, s_{i+1}, \cdots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supset (s_i, s_{i+1}, \cdots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} .
\]

Thus we have

\[
(3.15) \quad \mathcal{E}_i \supseteq \mathcal{F}_{i+1} \supseteq \mathcal{E}_{i+1} .
\]

In case \( \delta_\alpha(\tilde{s}_i) = 2 \), the choice of \( B \) ensures that \( R_i \subset \tilde{R} \subset R_{i+1} \). Thus

\[
(s_i, \cdots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supset (s_i, s_{i+1}, \cdots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} .
\]

This implies (3.15) as well. In summary, by the argument at the beginning of the proof, (3.15) leads to a contradiction which proves that \( q \) must be a node of \( X \).

It remains to study when \( q \) is a node of \( X \). A careful case by case study shows that when either \( \text{ind}_p(i) \geq 1 \) or \( \delta(\tilde{s}_i, p) \geq 2 \), then \( Z_{\alpha,i} \subset \tilde{Z}_\alpha \subset Z_{\alpha,i+1} \). Thus (3.15) holds, which leads to a contradiction. This proves that when \( q \) is a node, \( \text{ind}_p(i) = 0 \) and \( \delta(\tilde{s}_i, p) = 1 \).

We complete the proof of the first part by looking at the case \( \deg X_\alpha = 1 \). In this case \( \text{ind}_p(i) = 0 \) and \( \delta(\tilde{s}_i, p) = 1 \), since otherwise \( \deg X_\alpha = 1 \) implies that \( i = h_\alpha \), contradicting the assumption that \( i \) is not a base index. We next show that \( p \in L_\alpha \). But this is parallel to the proof of the case \( \deg X_\alpha > 1 \) by letting \( B = p \) because \( \delta_\alpha(\tilde{s}_i) = 1 \). This completes the proof of the first part.

We now prove the second part. Let \( \pi^{-1}(q) = \{p, p'\} \) with \( p' \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\beta \) so that the assumption on \( X_\beta \) holds. Then by the first part of the Proposition, we have \( \text{ind}_p(i) = \text{ind}_{p'}(i) = 0 \); hence \( s_i(q) \neq 0 \). Thus for \( Z_j = (s_j = \cdots = s_m = 0) \subset X \), we have \( p \notin Z_i \) and
\( Z_{i+1} = p \cup S \), where \( S \) is a zero-subscheme disjoint from \( p \). Since \( Z_i \subset Z_{i+1} \) and \( p \notin Z_i \), we have \( Z_i \subset S \). In case \( Z_i = S \), then the second part of the Proposition holds. Suppose \( Z_i \subset S \), then repeating the proof of the first part of the Proposition, we can find a section 
\( \zeta \in H^0(\mathcal{O}_X(1)) \) so that \( p \notin (\zeta = 0) \) and \( S \subset (\zeta = 0) \). This way, we will have (3.15) again, which leads to a contradiction. This proves the second part of the Proposition. \( \square \)

The Proposition above motivates the following

**Definition 3.10.** For \( \deg X_\alpha > 1 \), we define the primary indices of \( X_\alpha \) to be

\[
\mathbb{I}^\text{pri}_\alpha = \{ i \in \mathbb{I}_\alpha \mid w_\alpha(\tilde{\mathcal{E}}_{i+1}) \leq \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha - 1 \};
\]

for \( \deg X_\alpha = 1 \), we define \( \mathbb{I}^\text{pri}_\alpha = \text{ind}^{-1}_\alpha(0) \subset \mathbb{I}_\alpha \). We say \( i \in \mathbb{I}_\alpha \) is primary at \( p \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha \), if \( i \in \mathbb{I}^\text{pri}_\alpha \); otherwise we say it is secondary. We define \( \tilde{\mathbb{J}}_\alpha := \max \{ i \mid i \in \mathbb{I}^\text{pri}_\alpha \} \).

Note that in the proof above, the assumption \( \delta(\tilde{s}_i) \geq 2 \) is used only to show that (3.11) is strict. If \( i = h_\alpha \) for some \( \alpha \), then length(\( \mathcal{E}_i/\mathcal{E}_{i+1} \)) = \( \infty \). This time we choose \( \zeta \) so that \( \mathcal{E}_i/\mathcal{E}_{i+1} \) is finite. Since \( \mathcal{E}_i/\mathcal{E}_{i+1} \) is strict. If \( p, p' \in \mathcal{S} \), we have

Corollary 3.12. Denoting \( w^\text{pri}_\alpha := w_\alpha(\tilde{\mathcal{E}}_{\tilde{j}_\alpha+1}) \), suppose \( X_\alpha \subset X \), then

\[
0 \leq \deg X_\alpha - w^\text{pri}_\alpha \leq 2g(X_\alpha) + \ell_\alpha + 1.
\]

**Proof.** The first inequality is trivial. We now prove the second one. If \( \deg X_\alpha = 1 \) we obtain \( \deg X_\alpha - w^\text{pri}_\alpha = 0 \), from which the second inequality follows. So from now on we assume \( \deg X_\alpha > 1 \). We let \( i \in \mathbb{I}_\alpha \) be the index succeeding \( \tilde{j}_\alpha \); namely, \( \tilde{i} \) is the smallest index \( i > \tilde{j}_\alpha \) so that \( \delta_\alpha(\tilde{s}_i) \geq 1 \). In particular, this implies that

\[
\delta_\alpha(\tilde{s}_{\tilde{j}_\alpha}) = \cdots = \delta_\alpha(\tilde{s}_{\tilde{i}-1}) = 0.
\]

Since \( \tilde{i} \notin \mathbb{I}^\text{pri}_\alpha \),

\[
w^\text{pri}_\alpha = w_\alpha(\tilde{\mathcal{E}}_{\tilde{j}_\alpha+1}) = w_\alpha(\tilde{\mathcal{E}}_{\tilde{i}+1}) - \delta_\alpha(\tilde{s}_{\tilde{i}}) > \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha - 1 - \delta_\alpha(\tilde{s}_{\tilde{i}}).
\]

Thus when \( \delta_\alpha(\tilde{s}_{\tilde{i}}) \leq 2 \), the second inequality follows from \( \ell_\alpha \geq 1 \) (since \( X_\alpha \subset X \)).

Suppose \( \delta_\alpha(\tilde{s}_{\tilde{i}}) > 2 \). By our assumption \( \tilde{i} \) is the index in \( \mathbb{I}_\alpha \) immediately succeeding \( \tilde{j}_\alpha \), thus we have \( w_\alpha(\tilde{\mathcal{E}}_{\tilde{i}}) = w_\alpha(\tilde{\mathcal{E}}_{\tilde{j}_\alpha+1}) \) because of (3.17). By Definition 3.10, \( w_\alpha(\tilde{\mathcal{E}}_{\tilde{i}}) \) satisfies (3.10). So we can apply Proposition 3.9 to the index \( \tilde{i} \) to conclude that every \( p \in \text{inc}(\tilde{s}_{\tilde{i}}) \cap \tilde{X}_\alpha \) lies in \( \tilde{N}_\alpha \) and has \( \delta(\tilde{s}_{\tilde{i}}, p) = 1 \).

We claim that \( \text{inc}(\tilde{s}_{\tilde{i}}) \cap \tilde{X}_\alpha \subset \tilde{L}_\alpha \). Indeed, let \( p \in \text{inc}(\tilde{s}_{\tilde{i}}) \cap (\tilde{N}_\alpha \setminus \tilde{L}_\alpha) \), then the second part of Proposition 3.9 implies that \( \text{inc}(\tilde{s}_{\tilde{i}}) = p + p' \) and \( \delta(\tilde{s}_{\tilde{i}}, p) = 2 \), contradicting the assumption \( \delta_\alpha(\tilde{s}_{\tilde{i}}) > 2 \). This proves that \( \text{inc}(\tilde{s}_{\tilde{i}}) \cap \tilde{X}_\alpha \subset \tilde{L}_\alpha \). Adding that \( \delta(\tilde{s}_{\tilde{i}}, p) = 1 \) for \( p \in \text{inc}(\tilde{s}_{\tilde{i}}) \cap \tilde{X}_\alpha \), we conclude that \( \delta_\alpha(\tilde{s}_{\tilde{i}}) \leq \ell_\alpha \). These and (3.18) prove the second inequality in (3.16). \( \square \)
4. Main estimate for irreducible curves

Throughout this section, we fix a staircase 1-PS $\lambda$, and an irreducible $X_\alpha$. We will derive an estimate of $e(\tilde{J}_\alpha(\lambda))$ for the $X_\alpha \subset X$.

We let $g_\alpha$ be the genus of $X_\alpha$; we define the set of special points
\begin{equation}
\tilde{S}_\alpha = (\pi^{-1}(X) \cap X_\alpha) \cup \tilde{N}_\alpha \subset \tilde{X}_\alpha ,
\end{equation}
where $X = (x_1, \ldots, x_n) \subset X$ is the set of weighted points. We continue to denote $\bar{\rho}_i = \rho_i - \rho_n$. For each $p \in \tilde{\Lambda}_\alpha$, we define the initial index
\begin{equation}
i_0(p) := \min \{ i \mid i \in \downarrow_p \} .
\end{equation}

Given a fixed $\epsilon > 0$, we define
\begin{equation}
E^\epsilon_\alpha(p) := \left( 2 + \frac{2\epsilon}{\deg X_\alpha} \right) \sum_{i \in \tilde{\mu}_p} \delta(\tilde{s}_i) \bar{\rho}_i - \left( 1 + \frac{2\epsilon}{\deg X_\alpha} \right) \sum_{q \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \bar{\rho}_{i_0(q)} + 2 \deg X_\alpha \cdot \rho_n
\end{equation}
for $\deg X_\alpha > 1$; and for $\deg X_\alpha = 1$, we define
\begin{equation}
E^\epsilon_\alpha(p) := \delta(\tilde{s}_{i_0}) \bar{\rho}_{i_0} + 2 \cdot \rho_n; \quad i_0 = \text{ind}_{\alpha}^{-1}(0).
\end{equation}

It is clear that in both cases $E^\epsilon_\alpha(p)$ are linear in $\rho \in \mathbb{R}_+^{m+1}$. Our main result of this section is the following

**Theorem 4.1.** For any $1 \geq \epsilon > 0$ there is a constant $M_1 = M_1(g_\alpha, \ell_\alpha, n, \epsilon)$, which is a rational function of $g_\alpha, \ell_\alpha, n$ and $\epsilon$, such that either when $\deg X_\alpha \geq M_1$, or when $\deg X_\alpha = 1$, we have
\[ e(\tilde{J}_\alpha(\lambda)) \leq E^\epsilon_\alpha(p) . \]

Note that the theorem implies that we can bound $e(\tilde{J}(\lambda))$ in terms of the primary $\rho_i$’s only, with an additional margin related to the markings and nodes. This extra margin will be crucial to study the stability of curves with nodes and markings.

We begin with a useful bound on the area of $\Delta_p$.

**Lemma 4.2.** Let 1-PS $\lambda$ be a staircase. Then for any $p \in \tilde{\Lambda}_\alpha$, we have
\begin{equation}
|\Delta_p| - \rho_n \cdot w(\tilde{\epsilon}_{\alpha}, p) \leq \left| \sum_{i \in \downarrow_p} \delta(\tilde{s}_i) \bar{\rho}_i - \frac{\bar{\rho}_{i_0}(p)}{2} \right| .
\end{equation}

**Proof.** Let $0 \leq l \leq k \leq h_\alpha$; let $i_{\text{min}} := \min(\downarrow_p \cap [l, k])$ and $i_{\text{max}} := \max(\downarrow_p \cap [l, k])$, we prove
\begin{equation}
|\Delta_p \cap ([w(\tilde{\epsilon}_l, p), w(\tilde{\epsilon}_k, p) \times \mathbb{R}])| - \rho_n \cdot (w(\tilde{\epsilon}_k, p) - w(\tilde{\epsilon}_l, p)) \leq \sum_{i \in \downarrow_p \cap [l, k]} \delta(\tilde{s}_i) \bar{\rho}_i - \frac{(\bar{\rho}_{i_{\text{min}}}(p) + \bar{\rho}_{i_{\text{max}}}(p))}{2} .
\end{equation}

Note that by letting $l = 0$ and $k = h_\alpha$, we obtain the Lemma.

We prove (4.6). As it is invariant when varying $\rho_n$, without loss of generality we assume $\rho_n = 0$; hence $\bar{\rho}_i = \rho_i$. Let $\Gamma_p := \{(w(\tilde{\epsilon}_i, p), \rho_i) \mid 0 \leq i \leq m, w(\tilde{\epsilon}_i, p) \neq \infty \}$; it follows from Definition 2.7 and 3.6 that
\begin{equation}
\Delta_p = (\mathbb{R}_+^2 - \text{Conv}(\mathbb{R}_+^2 + \Gamma_p)) \cap ([0, w(\tilde{\epsilon}, p)] \times \mathbb{R}) .
\end{equation}

Fixing an indexing
\begin{equation}
\downarrow_p = \{i_0(p), \cdots, i_d(p)\} \subset \emptyset, \quad i_j(p) \text{ increasing and } d + 1 = |\downarrow_p| ,
\end{equation}

we let \( T \) be the continuous piecewise linear function on \([0, w(\tilde{\gamma}, p)]\) defined by linearly interpolating the points
\[
\{(0, \rho_{\alpha}), \ldots, (w(\tilde{E}_{i_k}, p), \rho_{i_k}), \ldots, (w(\tilde{E}_{i_d}, p), \rho_{n_0})\} \subset \mathbb{R}^2,
\]
and let \( \Delta_{\tilde{T}} \) be the polygon bounded on two sides by \( x = 0 \) and \( x = w(\tilde{E}_k, p) \), from below by \( y = 0 \) and from above by the graph of \( y = T \). By the convexity of \( \Delta_p \), we have
\[
\Delta_p \cap ([w(\tilde{E}_l, p), w(\tilde{E}_k, p)] \times \mathbb{R}) \subset \Delta_{\tilde{T}} \cap ([w(\tilde{E}_l, p), w(\tilde{E}_k, p)] \times \mathbb{R}) \subset \mathbb{R}^2.
\]
By Lemma 3.8, \( w(\tilde{E}_i, p) = \sum_{j=0}^{i-1} \delta(\tilde{\gamma}_j, p) \); hence
\[
|\Delta_p \cap ([w(\tilde{E}_l, p), w(\tilde{E}_k, p)] \times \mathbb{R})| \leq |\Delta_{\tilde{T}} \cap ([w(\tilde{E}_l, p), w(\tilde{E}_k, p)] \times \mathbb{R})|
= \sum_{i \in I_p, i \in [l, k]} \delta(\tilde{\gamma}_i, p) \rho_i - \frac{1}{2}(\rho_{\min}(p) + \rho_{\max}(p)).
\]
This proves (4.6), and the Lemma. \( \square \)

The idea of the proof of the theorem is as follows: When \( |\tilde{\Lambda}_\alpha| \) (cf. Definition 3.1) is large, applying Lemma 4.2, we gain a sizable multiple of \( \frac{1}{2}\rho_{\alpha}(p) \)'s (cf. (4.5) and Figure 5) in the estimate of \( \Delta_p \); these extra gains will take care of the contributions from non-primary \( \rho_i \)'s. When \( |\tilde{\Lambda}_\alpha| \) is small, one large \( \Delta_p \) (cf. Figure 4) is sufficient to cancel the contribution from the non-primary \( \rho_i \)'s.

We need a few more notions. For any \( p \in \tilde{\Lambda}_\alpha \), we let \( I_p^{\text{pri}} := I^{\text{pri}} \cap I_p \), and define
\[
(4.9) \quad \tilde{\gamma}_p := \max\{i \in I_p^{\text{pri}}\}, \quad w^{\text{pri}}(p) := w(\tilde{E}_{\tilde{\gamma}_p+1}, p), \quad \text{and} \quad w(p) := w(\tilde{\gamma}, p) \text{ (cf. (2.9))}.
\]
Note that \( w(p) \) is the base-width of the Newton polygon \( \Delta_p \). Using \( \tilde{\gamma}_p \), we truncate the Newton polygon \( \Delta_p \) by intersecting it with the strip \([0, w^{\text{pri}}(p)] \times \mathbb{R} \):
\[
\Delta_p^{\text{pri}} := \Delta_p \cap [0, w^{\text{pri}}(p)] \times \mathbb{R}.
\]
Our next Lemma says that if one \( \Delta_p \) is big enough, the contribution from the non-primary \( \rho_i \)'s can be absorbed by the difference between \( E^p_{\alpha}(\rho) \) and \( e(\tilde{\gamma}_p(\rho)) \). Recall that \( w^{\text{pri}}_{\alpha} \) is defined in Corollary 3.12.

![Figure 3. The shape for a typical Newton polygon and its area.](image-url)
Lemma 4.3. For any \(1 > \epsilon > 0\), there is an \(M = M(g_\alpha, \ell_\alpha, \epsilon)\) such that whenever \(w(p) \geq M\) (cf. (4.9)),
\[
|\Delta_p^{\text{pri}}| + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\bar{\rho}_j \leq \left(1 + \frac{\epsilon}{w(p)}\right) \sum_{i \in I_p^{\text{pri}}} \delta(\bar{s}_i, p)\bar{\rho}_i + \rho_{h_\alpha} \cdot w_\alpha^{\text{pri}}(p) - \left(\frac{1}{2} + \frac{\epsilon}{w(p)}\right)\bar{\rho}_i w(p),
\]
and if \(|\Delta_p^{\text{pri}}| \leq 2\epsilon\), we let \(\rho_{h_\alpha} = 0\); hence \(\bar{\rho}_i = \rho_i\). Our proof is based on studying the proximity of \(\partial^+ \Delta_p\) (\(\partial^+ \Delta_p\) is the boundary component of \(\Delta_p\) lying in the (open) first quadrant) with the lattice points \((w(\bar{\xi}_i, p), \rho_i)\) (cf. (3.2)). In case they differ slightly, then the term \(\frac{\epsilon}{w(p)} \sum_{i \in I_p^{\text{pri}}} \delta(\bar{s}_i, p)\bar{\rho}_i\) is sufficient to absorb the term \(2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{h_\alpha}\) in the inequality (note \(\bar{\rho}_i = \rho_i\) by assumption). Otherwise, the difference between \(\sum_{i \in I_p^{\text{pri}} \cap \{\xi, \bar{\xi}\}} \delta(\bar{s}_i, p)\bar{\rho}_i\) (for some \(c\) that will be specified below) and \(|\Delta_p^{\text{pri}}|\) is sufficient to imply the desired estimate.

We assume \(M > 4\), then \(w(p) - \sqrt{w(p)} \geq 2\) whenever \(w(p) \geq M\). We introduce \(c = \max\{i \in I_p^{\text{pri}} \mid (w(\bar{\xi}_i, p), \frac{\rho_i}{2}) \in \Delta_p \subset \mathbb{R}^2\}\), and let \(w^c(p) := w(\bar{\xi}_c, p)\) and \(\Delta_p^{\leq c} = \Delta_p \cap [0, w^c(p)] \times \mathbb{R}\).

![Figure 4](image-url) A big Newton polygon at point \(p\).

We divide our study into two cases. The first is when \(w(p) - w^c(p) \leq \sqrt{w(p)}\), which implies \(w^c(p) - 1 \geq w(p) - \sqrt{w(p)} - 1 \geq (w(p) - \sqrt{w(p)})/2\). We let \(\Theta\) be the trapezoid that is bounded on two sides by \(x = 1\) and \(x = w^c(p)\), from below by \(y = 0\) and from above by the line passing through \((w(p), 0)\) and \((w^c(p), \rho_c/2)\). Since the length of its two vertical edges are \(\rho_c/2\) and \(\frac{w(p) - 1}{w(p) - w^c(p)} \cdot \frac{\rho_c}{2}\), a simple estimate gives
\[
|\Theta| \geq \left(\frac{\sqrt{w(p)}}{2} + 1\right) \cdot \frac{w(p) - \sqrt{w(p)}}{2} \cdot \frac{\rho_c}{4} \geq \frac{w(p)^{3/2}}{32} \cdot \rho_c.
\]
Since the piecewise linear \(\partial^+ \Delta_p\) is convex, \(\Theta\) lies inside \(\Delta_p\), hence
\[
|\Delta_p| - \frac{\rho_i w(p)}{2} > |\Theta| > \frac{w(p)^{3/2}}{32} \cdot \rho_c.
\]
By the definition of $\Delta_p^\text{pri}$, the difference between the base-width of $\Delta_p^\text{pri}$ and of $\Delta_p$ is bounded by $w(p) - w^\text{pri}(p)$; therefore by Lemma 4.2 we have

$$|\Delta_p^\text{pri}| + 2(\deg X_\alpha - w_\alpha^\text{pri})\rho_{j_n} \geq |\Delta_p^\text{pri}| + (w(p) - w^\text{pri}(p))\rho_{j_n} \geq |\Delta_p| \geq \frac{w(p)^{3/2}}{32} \rho_c + \frac{\rho_{i_0}(p)}{2}.$$  

Since $\rho_{j_n} \leq \rho_c$, this implies

$$|\Delta_p^\text{pri}| - \frac{\rho_{i_0}(p)}{2} > \left( \frac{w(p)^{3/2}}{32} - 2(\deg X_\alpha - w_\alpha^\text{pri}) \right) \rho_c.$$  

(4.10)

We now choose $M$ so that $M^{3/2} \geq 2^8(g_\alpha + \ell_\alpha + 1)$. By Corollary 3.12, we have $\deg X_\alpha - w_\alpha^\text{pri} \leq 2(g_\alpha + \ell_\alpha + 1)$. Therefore when $w(p) \geq M$, we have

$$2(\deg X_\alpha - w_\alpha^\text{pri}) \leq 4(g_\alpha + \ell_\alpha + 1) \leq \frac{w(p)^{3/2}}{64}.$$  

Plugging this into (4.10), we obtain $\rho_c \leq \frac{2^6}{w(p)^{3/2}} (|\Delta_p^\text{pri}| - \frac{\rho_{i_0}(p)}{2}).$ Hence

$$2(\deg X_\alpha - w_\alpha^\text{pri})\rho_{j_n} \leq 2(\deg X_\alpha - w_\alpha^\text{pri})\rho_c \leq \frac{2^6(\deg X_\alpha - w_\alpha^\text{pri})}{w(p)^{3/2}} (|\Delta_p| - \frac{\rho_{i_0}(p)}{2}).$$

So if we assume further $M \geq 2^{14}(g_\alpha + \ell_\alpha + 1)^2/\epsilon^2$, then whenever $w(p) \geq M$ we have $2^6(\deg X_\alpha - w_\alpha^\text{pri})w(p)^{-3/2} \leq \epsilon/w(p)$, thus

$$|\Delta_p^\text{pri}| - \frac{1}{2}\rho_{i_0}(p) + 2(\deg X_\alpha - w_\alpha^\text{pri})\rho_{j_n} \leq \left(1 + \frac{\epsilon}{w(p)}\right)\left(|\Delta_p^\text{pri}| - \frac{\rho_{i_0}(p)}{2}\right) \leq \left(1 + \frac{\epsilon}{w(p)}\right)\left(\sum_{i \in I_p^\text{pri}} \delta(\tilde{s}_i, p)\rho_i - \rho_{i_0}(p)\right),$$

where the last inequality follows from Lemma 4.2. Thus this case is settled.

The other case is when $w(p) - w^\text{c}(p) > \sqrt{w(p)}$. By the definition of $c$, for $j \in J := \{p \cap (c, \bar{c}), (w(\tilde{c}_j, p), \rho_j/2) \notin \Delta_p\}$. Since $\partial^+ \Delta_p$ is convex, by Lemma 4.2, we have

$$\sum_{i \in J} \delta(\tilde{s}_i, p)\rho_i - |\Delta_p^\text{pri} \setminus \Delta_p^\text{c}| \geq \sum_{i \in J} \delta(\tilde{s}_i, p)\rho_i/2.$$  

Since $\deg X_\alpha - w_\alpha^\text{pri} \geq w(p) - w^\text{pri}(p)$ and $w(p) - w^\text{c}(p) > \sqrt{w(p)}$ by our assumption, we have

$$\sum_{i \in J} \delta(\tilde{s}_i, p) = w(p) - w^\text{c}(p) - (w(p) - w^\text{pri}(p)) > \sqrt{w(p)} - (\deg X_\alpha - w_\alpha^\text{pri}),$$

We choose

$$M \geq 10^2(g_\alpha + \ell_\alpha + 1)^2 \geq 5^2(\deg X_\alpha - w_\alpha^\text{pri})^2$$

and require $w(p) \geq M$, then $\sum_{i \in J} \delta(\tilde{s}_i, p) \geq 4(\deg X_\alpha - w_\alpha^\text{pri})$. This implies

$$\sum_{i \in J} \delta(\tilde{s}_i, p)\rho_i - |\Delta_p^\text{pri} \setminus \Delta_p^\text{c}| \geq \sum_{i \in J} \delta(\tilde{s}_i, p)\rho_i/2 \geq 2(\deg X_\alpha - w_\alpha^\text{pri})\rho_{j_n},$$

where the last inequality follows from Lemma 4.2. Thus this case is settled.
and combined with Lemma 4.2, we obtain
\[
|\Delta^\text{pri}_p| + 2(\deg X_\alpha - w^\text{pri}_\alpha)\rho_{j_\alpha} \\
\leq |\Delta^\text{ce}_p| + |\Delta^\text{pri}_p \setminus \Delta^\text{ce}_p| + 2(\deg X_\alpha - w^\text{pri}_\alpha)\rho_{j_\alpha} \\
\leq |\Delta^\text{ce}_p| + \sum_{i \in J} \delta(\hat{s}_i, p)\rho_i + \sum_{i \in J} \delta(\tilde{s}_i, p)\rho_i + 2(\deg X_\alpha - w^\text{pri}_\alpha)\rho_{j_\alpha} \\
\leq |\Delta^\text{ce}_p| + \sum_{i \in J} \delta(\hat{s}_i, p)\rho_i + \sum_{i \in J} \delta(\tilde{s}_i, p)\rho_i - \frac{\rho_{i_0}(p)}{2} \\
< \left(1 + \frac{\epsilon}{w(p)}\right)\left(\sum_{i \in J^\text{pri}} \delta(\hat{s}_i, p)\rho_i - \frac{\rho_{i_0}(p)}{2}\right) + \frac{\rho_{i_0}(p)}{2}.
\]

In the end, since \(\epsilon < 1\) we choose \(M := 2^{14}(g_\alpha + \ell_\alpha + 1)^2/\epsilon^2\). Then for \(w(p) > M\), (4.3) holds. This proves the Lemma.

\[\]

**Figure 5.** Newton polygons supported at many points gain us a lot of \(\frac{\rho_{i_0}(p)}{2}\)'s.

**Proof of Theorem 4.1.** First, by the same reason we can assume \(\rho_{h_\alpha} = 0\) and \(\rho_i = \rho_i\). Also, when \(\deg X_\alpha = 1\), then the statement is a direct consequence of Lemma 4.2. So from now on we assume that \(\deg X_\alpha \geq M_1 \geq 2\). Let \(1 > \epsilon > 0\) be any constant. Since \(\epsilon < 1\), we have \(\frac{\epsilon}{\deg X_\alpha} \leq 1/2\). We define \(\sigma\) to be the number of Newton polytopes supported on \(\tilde{X}_\alpha\). We divide our study into two cases.

The first case is when \(\sigma > 10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|\). Since Corollary 3.12 implies
\[
|\{p \in \tilde{X}_\alpha \cap \tilde{S}_\alpha | i_0(p) > \alpha\}| \leq \sum_{i \in \Lambda_\alpha \setminus \tilde{S}_\alpha} \delta_\alpha(\tilde{s}_i) \leq (\deg X_\alpha - w^\text{pri}_\alpha) \leq 2(g_\alpha + \ell_\alpha + 1),
\]
the number of \(p \in \tilde{X}_\alpha \setminus \tilde{S}_\alpha\) satisfying \(\rho_{i_0}(p) \geq \rho_{j_\alpha}\) is at least \(8(\tilde{g}_\alpha + \ell_\alpha + 1)\). By Lemma 4.2, for each \(p \in \tilde{X}_\alpha\), we gain an extra \(\rho_{i_0}(p)/2\) on the right hand side in the estimate \(\Delta_p\) in terms of \(\{\rho_i\}_{i=0}^\infty\). This implies
\[
\sum_{i \in \Lambda_\alpha \setminus \tilde{S}_\alpha} \delta_\alpha(\tilde{s}_i)\rho_i \leq (\deg X_\alpha - w^\text{pri}_\alpha)\rho_{j_\alpha} \leq 2(g_\alpha + \ell_\alpha + 1)\rho_{j_\alpha} \leq \frac{1}{4} \sum_{p \in \tilde{X}_\alpha \setminus \tilde{S}_\alpha} \rho_{i_0}(p)\).
\]
So we obtain, via using Lemma 4.2 and summing over \( p \in \tilde{\Lambda}_\alpha \),

\[
\sum_{p \in \tilde{\Lambda}_\alpha} |\Delta_p| \leq \sum_{i \in \mathbb{P}^n} \delta_\alpha(\tilde{s}_i) \rho_i + \sum_{i \in \mathbb{P}^n} \delta_\alpha(\tilde{s}_i) \rho_i - \frac{1}{2} \sum_{p \in \tilde{\Lambda}_\alpha} \rho_{i_0(p)} \\
= \left( \sum_{i \in \mathbb{P}^n} \delta_\alpha(\tilde{s}_i) \rho_i - \frac{\epsilon}{\deg X_\alpha} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} - \frac{1}{2} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} \right) + \\
+ \left( \sum_{i \in \mathbb{P}^n \setminus \mathbb{P}^n} \delta_\alpha(\tilde{s}_i) \rho_i + \frac{\epsilon}{\deg X_\alpha} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} - \frac{1}{2} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} \right).
\]

Using (4.11) and

\[
\frac{\epsilon}{\deg X_\alpha} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} \leq \sum_{i \in \mathbb{P}^n \setminus \mathbb{P}^n} \delta_\alpha(\tilde{s}_i) \rho_i \leq \frac{1}{4} \sum_{p \in \tilde{\Lambda}_\alpha \setminus \tilde{S}_\alpha} \rho_{i_0(p)},
\]

the sum in the line of (4.12) is non-positive. Therefore, for any \( 0 < \epsilon < 1 \) we have

\[
\sum_{p \in \tilde{\Lambda}_\alpha} |\Delta_p| \leq \sum_{i \in \mathbb{P}^n} \delta_\alpha(\tilde{s}_i) \rho_i - \frac{\epsilon}{\deg X_\alpha} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} - \frac{1}{2} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} \\
\leq \left( 1 + \frac{\epsilon}{\deg X_\alpha} \right) \sum_{i \in \mathbb{P}^n} \delta_\alpha(\tilde{s}_i) \rho_i - \left( \frac{1}{2} + \frac{\epsilon}{\deg X_\alpha} \right) \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} = \frac{E_\alpha^\epsilon(\rho)}{2},
\]

since

\[
\frac{\epsilon}{\deg X_\alpha} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} \leq \frac{\epsilon}{\deg X_\alpha} \sum_{i \in \mathbb{P}^n} \delta_\alpha(\tilde{s}_i) \rho_i.
\]

This verifies the Theorem in this case.

The other case is when \( \sigma \leq 10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha| \). By the pigeonhole principle, there exists at least one \( p_0 \in \tilde{\Lambda}_\alpha \) such that

\[
w(\tilde{j}, p_0) \geq \frac{\deg X_\alpha}{\sigma} \geq \frac{\deg X_\alpha}{10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|}.
\]

By Corollary 2.8, we have

\[
\frac{\epsilon}{\deg X_\alpha} \langle \tilde{\beta}(\lambda) \rangle = \sum_{p \in \tilde{\Lambda}_\alpha} |\Delta_p|.
\]

Our assumption \( \epsilon \leq 1 \), \( 1/\deg X \leq 1/2 \) and Corollary 3.12 imply

\[
\left( \frac{1}{2} + \frac{\epsilon}{\deg X_\alpha} \right) \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \rho_{i_0(p)} \leq \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \rho_{i_0(p)} \leq (\deg X_\alpha - w_\alpha^{\text{pri}}) \rho_{\alpha}.
\]
So we obtain
\[\frac{\epsilon_{X_\alpha}(\mathcal{J}(\lambda))}{2} - \sum_{p \in \hat{S}_\alpha \cap \hat{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} = |\Delta_{p_0}^{\text{pri}}| + |\Delta_{p_0} \setminus \Delta_{p_0}^{\text{pri}}| + \sum_{p_0 \neq p \in \hat{\Lambda}_\alpha} (|\Delta_{p}^{\text{pri}}| + |\Delta_{p} \setminus \Delta_{p}^{\text{pri}}|) - \sum_{p \in \hat{S}_\alpha \cap \hat{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2}.\]

By Lemma 4.2 and the first inequality of (4.11), we have
\[|\Delta_{p_0} \setminus \Delta_{p_0}^{\text{pri}}| + \sum_{p_0 \neq p \in \hat{\Lambda}_\alpha} |\Delta_{p} \setminus \Delta_{p}^{\text{pri}}| = \sum_{p \in \hat{\Lambda}_\alpha} |\Delta_{p} \setminus \Delta_{p}^{\text{pri}}| \leq (\deg X_\alpha - w_{\alpha}^{\text{pri}}) \rho_{j_\alpha}.
\]

So
\[\frac{\epsilon_{X_\alpha}(\mathcal{J}(\lambda))}{2} - \sum_{p \in \hat{S}_\alpha \cap \hat{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \leq |\Delta_{p_0}^{\text{pri}}| + (\deg X_\alpha - w_{\alpha}^{\text{pri}}) \rho_{j_\alpha} + \sum_{p_0 \neq p \in \hat{\Lambda}_\alpha} |\Delta_{p}^{\text{pri}}| - \sum_{p \in \hat{S}_\alpha \cap \hat{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2}\]

\[\leq |\Delta_{p_0}^{\text{pri}}| + 2(\deg X_\alpha - w_{\alpha}^{\text{pri}}) \rho_{j_\alpha} - \frac{\rho_{i_0(p_0)}}{2} |\{p_0\} \cap \hat{S}_\alpha| + \sum_{p_0 \neq p \in \hat{\Lambda}_\alpha} |\Delta_{p}^{\text{pri}}| - \sum_{p \in \hat{S}_\alpha \cap \hat{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2}\]

\[= |\Delta_{p_0}^{\text{pri}}| + 2(\deg X_\alpha - w_{\alpha}^{\text{pri}}) \rho_{j_\alpha} - \frac{\rho_{i_0(p_0)}}{2} |\{p_0\} \cap \hat{S}_\alpha| + \sum_{p_0 \neq p \in \hat{\Lambda}_\alpha} |\Delta_{p}^{\text{pri}}| - \sum_{p \in \hat{S}_\alpha \cap \hat{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \leq \frac{\epsilon}{10(g_\alpha + \ell_\alpha + 1) + |\hat{S}_\alpha|}.
\]

By (4.13) we obtain
\[\frac{\epsilon_0}{w(\mathcal{J}, p_0)} \leq \frac{\epsilon}{w(\mathcal{J}, p_0)(10(g_\alpha + \ell_\alpha + 1) + |\hat{S}_\alpha|)} \leq \frac{\epsilon}{\deg X_\alpha}.
\]

If we let \(M = M(g_\alpha, \ell_\alpha, \epsilon_0)\) be the constant fixed in Lemma 4.3 for \(\epsilon = \epsilon_0\) and choose
\[M_1(g_\alpha, \ell_\alpha, n, \epsilon) := (11(g_\alpha + \ell_\alpha + 1) + n)M \geq (10(g_\alpha + \ell_\alpha + 1) + |\hat{S}_\alpha|)M,
\]
then \(\deg X_\alpha \geq M_1\) implies \(w(\mathcal{J}, p_0) > M\). In particular, we have \(i_0(p_0) \in \mathcal{I}_{a_0}^{\text{pri}}\). The whole term after (4.15) is equal to
\[= |\Delta_{p_0}^{\text{pri}}| + 2(\deg X_\alpha - w_{\alpha}^{\text{pri}}) \rho_{j_\alpha} - \frac{\rho_{i_0(p_0)}}{2} |\{p_0\} \cap \hat{S}_\alpha| + \sum_{p_0 \neq p \in \hat{\Lambda}_\alpha} |\Delta_{p}^{\text{pri}}| - \sum_{p \in \hat{S}_\alpha \cap \hat{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \leq \frac{\epsilon}{\deg X_\alpha}.
\]

Applying Lemma 4.2 to the term \(|\Delta_{p_0}^{\text{pri}}| + 2(\deg X_\alpha - w_{\alpha}^{\text{pri}}) \rho_{j_\alpha}\), Lemma 4.3 to the term \(\sum_{p_0 \neq p \in \hat{\Lambda}_\alpha} |\Delta_{p}^{\text{pri}}|\), and using (4.16), we
Lemma 5.1. Let \((X, \mathcal{O}_X(1), x, a)\) be a polarized weighted pointed nodal curve. Suppose \((X, \mathcal{O}_X(1))\) is non-special. Then it satisfies (1.6) for all subcurves \(Y \subseteq X\) if and only if it satisfies (1.4) for all subcurves \(Y \subseteq X\).

Proof. Let \(Y \subseteq X\) be a subcurve. Since \((X, \mathcal{O}_X(1))\) is non-special, we have vanishing \(h^1(\mathcal{O}_Y(1)) = h^1(\mathcal{O}_{Y_C}(1)) = 0\). Following the proof of [3, Proposition 3.1], we see that (1.4) holds for \(Y \subseteq X\) is equivalent to

\[
\left(\deg Y + \frac{a_Y}{2}\right) - \frac{\deg Y}{\deg X} \omega_X(a \cdot x) \left(\deg X + \frac{a_X}{2}\right) \geq -\frac{\ell_Y}{2},
\]

and (1.4) holding for \(Y_C \subseteq X\) is equivalent to

\[
\left(\deg Y + \frac{a_Y}{2}\right) - \frac{\deg Y}{\deg X} \omega_X(a \cdot x) \left(\deg X + \frac{a_X}{2}\right) \leq \frac{\ell_Y}{2}.
\]

So (1.4) holding for any subcurve \(Y \subseteq X\) implies (1.6) holding for any subcurve \(Y \subseteq X\).

The other direction is trivial, since (1.6) is equivalent to both (5.1) and (5.2). This proves the Lemma.

Lemma 5.2. Given \(g, n\) and \(a \in \mathbb{Q}_+^n\) satisfying \(\chi_{a, g} > 0\) (cf. (1.5)), there are positive constants \(M_2 = M_2(g, n, a)\) and \(C = C(g, n, a)\) such that for any genus \(g\) polarized weighted pointed nodal curve \((X, \mathcal{O}_X(1), x, a)\) satisfying (1.6) and \(\deg X > M_2\), any connected subcurve \(Y \subset X\) either has \(\deg Y \geq C \deg X\) and \(\deg Y \omega_X > 0\), or is an exceptional component.
Proof. Suppose \( \deg_Y \omega_X = 2g(Y) - 2 + \ell_Y \geq 1 \), since \( a_i \geq 0 \), (1.6) implies
\[
\frac{\deg Y + \frac{a_Y}{2}}{g(Y) - 1 + \frac{a_Y}{2} + \frac{\ell_Y}{2}} \leq \frac{\deg X + \frac{a_X}{2}}{g(Y) - 1 + \frac{a_X}{2} + \frac{\ell_Y}{2}} \geq \deg X + \frac{a_X}{2} - 3.
\]
This inequality implies
\[
\deg Y \geq \left( \frac{\deg X}{2\chi_{a,g}} - 6 \right) - \frac{n}{2}.
\]
Therefore, by choosing \( C = 1/4\chi_{a,g} \) and \( M_2 \geq 4\chi_{a,g}(6 + n/2) \), we obtain
\[
\deg Y \geq (4\chi_{a,g})^{-1} \deg X = C \cdot \deg X
\]
provided \( \deg X > M_2 \).

Now suppose \( \deg_Y \omega_X = 2g(Y) - 2 + \ell_Y \leq 0 \). Since \( X \) is connected, \( \ell_Y \geq 1 \) and \( g(Y) \geq 0 \). Thus \( g(Y) = 0 \) and \( \ell_Y \leq 2 \). In this case, (1.6) becomes
\[
(5.3) \quad \left| \deg Y + \frac{a_Y}{2} - \frac{\deg X + \frac{a_X}{2}}{g(Y) - 1 + \frac{a_X}{2} + \frac{\ell_Y}{2}} \cdot (-1 + \frac{a_Y}{2} + \frac{\ell_Y}{2}) \right| \leq 1.
\]
Let \( A := -1 + \frac{a_Y}{2} + \frac{\ell_Y}{2} \). In case \( A \leq 0 \), then we have \( \deg Y = 1, a_Y = 0 \) and \( \ell_Y = 2 \). Because \( O_X(1) \) is ample, \( Y \) must be irreducible, thus isomorphic to \( \mathbb{P}^1 \). Thus \( Y \subsetneq X \) is an exceptional component.

In case \( A > 0 \). We let
\[
(5.4) \quad A_0 = \min_{\ell \in \{1, \ldots, n\}, k \geq 0} \left\{ \sum_{i \in I} a_i/2 + k/2 \mid \sum_{i \in I} a_i/2 + k/2 > 0 \right\},
\]
which is positive by the finiteness of \( \{a_i\} \). Then \( A \geq A_0 \) and
\[
\deg Y > \frac{\deg X + \frac{a_X}{2}}{\chi_{a,g}} \cdot 2A - 1 - \frac{a_Y}{2} > \frac{\deg X}{\chi_{a,g}} A_0,
\]
when \( \deg X \geq M_2 \geq \frac{\chi_{a,g}(1-a_Y/2)}{A_0} \). Combined, we have proved the lemma by choosing
\[
M_2 := \max \{ \frac{\chi_{a,g}(1-a_Y/2)}{A_0}, 4\chi_{a,g}(6 + n/2) \} \quad \text{and} \quad C = \min \{ 1/4\chi_{a,g}, A_0/\chi_{a,g} \}. \quad \square
\]

Corollary 5.3. Let the situation and the constant \( C \) be as in Lemma 5.2. Then for any genus \( g \) polarized weighted pointed nodal curve \((X, O_X(1), x, a)\) (with \( O_X(1) \) only assumed to be ample) satisfying \( \deg X > M_3 = M_3(g, n, a) := \max \{ M_2, (9g + n)/C \} \) and the inequality (1.6), we have that \( O_X(1) \) is very ample, \( h^1(O_X(1)) = 0 \), and the number of nodes of \( X \) is bounded from above by \( 6(g + n) \).

Proof. First, we notice that \( h^1(O_X(1)) = 0 \) if \( h^1(O_Y(1)(-L_Y)) = 0 \) for any irreducible component \( Y \subset X \). In case \( Y \cong \mathbb{P}^1 \) with \( \ell_Y = 2 \), this follows from the fact that \( O_X(1) \) is ample. Otherwise, \( Y \) is not an exceptional component, and then by the previous Lemma, \( \deg O_Y(1) \geq C \deg X \geq CM_3 \). As \( X \) has genus \( g \), we have \( g(Y) \leq g \) and \( \ell_Y \leq g + n + 1 \). Therefore, by our choice of \( M_3 \) we obtain
\[
(5.5) \quad \deg Y - \ell_Y \geq 8g(Y) - 1
\]
from which we deduce that \( h^1(O_Y(1)(-L_Y)) = 0 \) for all irreducible components of \( X \) and \( O_X(1)|_Y \) is very ample. This proves the non-speciality of \((X, O_X(1))\).

By (1.6), we conclude that any chain of exceptional components consists of a single component. Thus the number of nodes of \( X \) is no more than twice the number of the
nodes of the stabilization (cf. (6.6)) of \((X, x)\) which is less than \(3g - 3 + n < 3(g + n)\)\(^1\), and the stated bound follows.

Finally, we prove that \(O_X(1)\) is very ample. First, one notices that for each irreducible component \(Y \subset X\), the inequality (5.5) implies the very ampleness of \(O_X(1)|_Y\) and the exact sequence

\[
H^0(O_X(1)|_Y(-L_Y)) \to H^0(O_X(1)|_Y) \to H^0(O_X(1)|_{L_Y}) \to H^1(O_X(1)|_Y(-L_Y)) = 0
\]

from which we conclude that for any non-exceptional irreducible component \(X_\alpha \subset X\), there is a section \(s \in H^0(O_X(1)|_{X_\alpha})\) taking any given boundary value on \(L_\alpha\).

For any two points \(p_1, p_2 \in X\), \(\alpha = 1, 2\), we claim that there is a section \(s \in H^0(O_X(1))\) such that \(s(p_1) = 0\) and \(s(p_2) \neq 0\). Without loss of generality, we assume \(p_1\) lies in the irreducible component \(X_\alpha\), \(\alpha = 1, 2\). If \(X_1 = X_2\) then our claim follows from the very ampleness of \(O_X(1)|_{X_1}\). From now on, we assume \(X_1 \neq X_2\). We first consider the case where both \(X_1\) and \(X_2\) are exceptional, which is the most involved case. Since \(X_1\) is an exceptional component, there is a section \(0 \neq s_1 \in H^0(O_X(1)|_{X_1})\) satisfying \(s_1(p_1) = 0\). Since \(X_2\) is also exceptional, we have \(X_1 \cap X_2 = \emptyset\) by (1.6), which allows us to choose a section \(s_2 \in H^0(O_X(1)|_{X_2})\) with \(s_2(p_2) \neq 0\). To construct the global section \(s \in H^0(O_X(1))\), we first let \(s = s_1 + s_2\) on \(X_1\) and \(X_2\), respectively; we let it be the zero section on the exceptional components of \(X\) different from \(X_1\) and \(X_2\). We next extend it to nonexceptional components, one at a time.

Suppose we have extended it to a section \(s_\beta\) on a component \(X_\beta \subset X\), we then apply (5.6) to construct a section \(s_{\beta+1} \in H^0(O_X(1)|_{X_{\beta+1}})\) satisfying the boundary value prescribed by previous stage. By continuing this process, we obtain the section \(s\) we want.

The other cases are similar, and will be left to the readers. Because of the claim, we deduce that the complete linear system \(W^r = H^0(O_X(1))\) provides an embedding of \(X \subset PW\). This completes the proof. \(\square\)

As a consequence, we have the following

**Proposition 5.4.** Given \(g, n\) and \(a \in \mathbb{Q}^+_0\) satisfying \(\chi_{a, g} > 0\), then for any polarized weighted pointed nodal curve \((X, O_X(1), x, a)\) \((\text{with } O_X(1) \text{ only assumed to be ample})\) of \(\deg X \geq M_1\) (the constant in Corollary 5.3), the following two are equivalent:

1. \(O_X(1)\) is very ample and \((X, O_X(1), x, a)\) is slope-semistable (resp. slope-stable);
2. \((X, O_X(1), x, a)\) satisfies (1.6) for all subcurves \(Y \subset X\) (resp. and the strict (1.6) holds except when \(Y\) or \(Y^c\) is a disjoint union of exceptional components of \((X, O_X(1), x)\)).

**Proof.** By the Definition of slope semistability (Definition 1.4), (1) implies that \((X, O_X(1))\) is non-special. On the other hand, by Corollary 5.3 \((X, O_X(1))\) is non-special and \(O_X(1)\) is very ample if it satisfies (1.6) and \(\deg X \geq M_1\). Hence in both cases we have \(h^1(X, O_X(1)) = 0\). Applying Lemma 5.1, we conclude that in cases (1) and (2), (1.4) is equivalent to (1.6). This proves the equivalence of the (non-resp. cases of) (1) and (2).

We now prove the case of slope-stable. Suppose (1) holds for \((X, O_X(1), x, a)\) and the later is slope-stable, but for a subcurve \(Y \subset X\) (1.6) is an equality; then either (5.2) or (5.1) is an equality. It follows from the proof of Lemma 5.1 that (1.4) becomes an equality

\(^1\)One can see this by induction. Notice that adding one marked point will introduce at most one node when \(\mathbb{P}^3\) with 3 marked points is bubbled off. On the other hand, increasing the genus by 1 will increase the nodes at most by 3 when a nodal rational curve is connected to the main component through a \(\mathbb{P}^3\) with 3 marked points.
for either $Y$ or $Y^c$. By the slope-stable assumption, either $Y$ or $Y^c$ is a disjoint union of exceptional components. This proves one direction for the “resp.” case. The other case is similar and we leave the proof to the readers. 

In order to prove Theorem 1.5 for the stable case, we also need the following

**Lemma 5.5.** Given $g$, $n$ and $a \in \mathbb{Q}^+_n$ satisfying $\chi_{a,g} > 0$, there is a constant $M_4 = M_4(g,n,a)$ such that for any slope semistable polarized weighted pointed nodal curve $(X, \mathcal{O}_X(1), x,a)$ of $\deg X \geq M_4$, a subcurve $Y \subset X$ satisfies $h^0(\mathcal{O}_X(1)|_Y) = h^0(\mathcal{O}_X(1))$ if and only if $Y^c$ is a disjoint union of exceptional components. In particular, if we assume further that $(X, \mathcal{O}_X(1), x,a)$ is slope stable, then $h^0(\mathcal{O}_X(1)|_Y) < h^0(\mathcal{O}_X(1))$ implies that (1.4) is a strict inequality.

**Proof.** For any subcurve $Y \subset X$, let $W_Y = \{ v \in W \mid s(v) = 0, \forall s \in H^0(\mathcal{O}_X(1) \otimes J_Y) \} \subset W$ denote the linear subspace spanned by $Y$. By our slope semistability assumption, the embedding $X \subset \mathbb{P}W$ is given by a complete non-special linear system, hence $\dim W_Y = h^0(\mathcal{O}_X(1)|_Y)$. So to prove the Lemma all we need to show is that for $\deg X$ sufficiently large, $\dim W_Y = \dim W$ if and only if $Y^c$ is a disjoint union of exceptional components.

To achieve that, we notice that for any component $X_\alpha \subset Y^c$, we have

$$\dim W_Y \cup X_\alpha = \dim W_Y + \dim W_{X_\alpha} - \dim W_Y \cap W_{X_\alpha}. \quad (5.7)$$

We claim that there is an $M_4 = M_4(g,n,a)$ such that whenever $\deg X \geq M_4$, we have $\dim W_Y \cap W_{X_\alpha} = |X_\alpha \cap Y|$. This is trivially true when $X_\alpha$ is exceptional. If $X_\alpha$ is non-exceptional and $\deg X \geq M_2$ (the constant in Lemma 5.2), we have $\deg X_\alpha \geq C \deg X$ by the semistability assumption and Lemma 5.2. So as long as $\deg X \geq \max\{M', M_2\}$ with $M'$ satisfying

$$CM' > 2g - 2 + \text{number of nodes in } X \geq 2g - 2 + |X_\alpha \cap Y|,$$

where $C$ is given in Lemma 5.2, by the vanishing theorem we have the surjectivity of the following restriction maps

$$H^0(\mathcal{O}_X(1)|_Y) \to H^0(\mathcal{O}_X(1)|_{X_\alpha \cap Y}) \text{ and } H^0(\mathcal{O}_X(1)|_{X_\alpha}) \to H^0(\mathcal{O}_X(1)|_{X_\alpha \cap Y}),$$

from which we deduce the exact sequence

$$0 \to H^0(\mathcal{O}_X(1)|_{X_\alpha \cap Y}) \to H^0(\mathcal{O}_X(1)|_{X_\alpha}) \oplus H^0(\mathcal{O}_X(1)|_{X_\alpha}) \to H^0(\mathcal{O}_X(1)|_{X_\alpha \cap Y}) \to 0.$$ 

This together with the non-specialness assumption and (5.7) imply $\dim W_Y \cap W_{X_\alpha} = |X_\alpha \cap Y|$. On the other hand, by Corollary 5.3 the number of nodes in $X$ is bounded by $6(g + n)$ provided $\deg X \geq M_3$. So our claim follows if we choose $M_4 \geq \max\{(2g - 2 + 6(g + n))/C, M_3\}$.

Now let us define $M_4 = M_4(g,n,a) := \max\{8(g + n + 1)/C, M_3\}$ and assume $\deg X > M_4$. Then for any $X_\alpha \subset Y^c$ non-exceptional we have

$$\dim W_{X_\alpha} - |X_\alpha \cap Y| \geq \deg X_\alpha + 1 - g(X_\alpha) - |X_\alpha \cap Y| \geq CM_4 + 1 - g - 6(g + n) > 1.$$ 

Plug the above inequality and $\dim W_Y \cap W_{X_\alpha} = |X_\alpha \cap Y|$ into (5.7); we obtain

$$\dim W_Y \cup X_\alpha = \dim W_Y + \dim W_{X_\alpha} - \dim W_Y \cap W_{X_\alpha} \geq \dim W_Y + 1,$$

from which we deduce that $W_Y = W$ if and only if $X_\alpha \subset Y^c$ is exceptional and $|X_\alpha \cap Y| = 2$. This proves the Lemma. 

\[\square\]
Let $s$ be a diagonalizing basis of $\lambda$:
\[
\lambda(t) := \text{diag}(t^{\rho_0}, \ldots, t^{\rho_m}) \cdot t^{-\rho_{\text{ave}}}, \quad \text{with} \quad \rho_0 \geq \rho_1 \geq \cdots \geq \rho_m = 0.
\]
The $a$-$\lambda$-weight of Chow($X, x$) is the sum of the contributions from $\text{Div}^{d,d}([PW^\vee]^2)$ and $(FW)^n$. By Proposition 2.1, the contribution from $\text{Div}^{d,d}([PW^\vee]^2)$ is $\omega(\lambda)$.

For the contribution from $(FW)^n$, we introduce subspaces
\[
(W_i =) \ W_i(\lambda) := \{ v \in W \mid s_i(v) = \cdots = s_m(v) = 0 \} \subset W = H^0(\mathcal{O}_X(1))^\vee.
\]
They form a strictly increasing filtration of $W$. Also, for any closed subscheme $\Sigma \subset X$, we denote by
\[
W_\Sigma := \{ v \in W \mid s(v) = 0 \text{ for all } s \in H^0(\mathcal{O}_X(1) \otimes J_\Sigma) \} \subset W
\]
the linear subspace spanned by $\Sigma \subset X$. For instance, for a marked point $x_i$, $W_{x_i}$ is the line in $W$ spanned by $x_i \in FW$.

By [14, Prop 4.3], the $a$-$\lambda$-weight of $x = (x_1, \ldots, x_n) \in (FW)^n$ is
\[
\mu_a(\lambda) := \sum_{j=1}^n a_j \left( \frac{\sum_{i=0}^m \rho_i}{m+1} + \sum_{i=0}^{m-1} (\rho_{i+1} - \rho_i) \dim(W_{x_j} \cap W_{i+1}(\lambda)) \right).
\]
($\mu_a(\lambda)$ implicitly depends on $\rho_i$, which we fix for the moment.) Therefore, the $a$-$\lambda$-weight $\omega_a(\lambda)$ of Chow($X, x$) $\in \Xi$ is
\[
\omega_a(\lambda) = \omega(\lambda) + \mu_a(\lambda).
\]

We now argue that for the staircase $\lambda'$ constructed from $\lambda$ by applying Proposition 3.5, we have
\[
\omega_a(\lambda) \geq \omega_a(\lambda').
\]
Indeed, since $\omega(\lambda) \geq \omega(\lambda')$, if suffices to show that $\mu_a(\lambda) \geq \mu_a(\lambda')$. To see that, we first notice that
\[
\dim(W_{x_j} \cap W_{i+1}(\lambda)) = \#(x_i \cap \text{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda)_i)).
\]
(Here $\mathcal{E}(\lambda)_i = (s_i, s_{i+1}, \ldots, s_m) \subset \mathcal{O}_X(1)$.) On the other hand, by the proof of Proposition 3.5, we conclude
\[
\text{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda)_i) \subset \text{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda')_i).
\]
This together with (5.13) proves
\[
\dim(W_{x_j} \cap W_{i+1}(\lambda)) \leq \dim(W_{x_j} \cap W_{i+1}(\lambda')).
\]
The inequality $\mu_a(\lambda) \geq \mu_a(\lambda')$ then follows from the facts $\rho_i \geq \rho_{i+1}$ and $\rho_i = \rho_i'$. Therefore, to prove Theorem 1.5, it suffices to show that $\omega_a(\lambda) > 0$ for all staircase 1-PS's $\lambda$. From now on we assume $\lambda$ is a staircase. For simplicity, we denote $W_i = W_i(\lambda)$.

To state the estimate of this section, we define $E^e_\alpha(\lambda, \rho) := \sum_{\alpha=1}^r E^e_\alpha(\rho)$. Since $\lambda$ is a staircase 1-PS, $\bigcup_{\alpha=1}^r \mathbb{I}_\alpha = \{0, \ldots, m\}$, where $\mathbb{I}_\alpha$ is the index set of the component $X_\alpha$ defined in (3.3). This allows us to define the shifted weights $\hat{\rho}_i$ by
\[
\hat{\rho}_i := \min_{\alpha} \{ \rho_i - \rho_{\mathbb{I}_\alpha} \mid i \in \mathbb{I}_\alpha \} \geq 0.
\]
We caution that $\hat{\rho}_i$ are only defined for staircase 1-PS, and possibly are non-monotone.
Sublemma 5.7. with the understanding that for any closed subset

\[ \sum_{a \geq \bar{r}} E^*_{\alpha}(\lambda, \rho) = 2 \deg X_{\alpha} \]

where \( \bar{S}_{\text{reg}} := \bigcup_{m=1}^{\infty}(\pi^{-1}(x) \cap \tilde{X}_a \cap \tilde{\Lambda}) \) and \( C > 0 \) is the constant given in Lemma 5.2.

Proof. By the definition of \( E^*_a(\rho) \) (cf. (4.3)), \( E^*_a(\lambda, \rho) = \sum_{\alpha=1} E^*_a(\rho) \) is linear in \( \rho = (\rho_i) \).

By linear programming, (5.15) holds on

\[ \mathbb{R}_{m+1}^+ := \{ (\rho_0, \cdots, \rho_m) \in \mathbb{R}^{m+1} | \rho_0 \geq \rho_1 \geq \cdots \geq \rho_m = 0 \} \]

if and only if it holds on every edge of \( \mathbb{R}_{m+1}^+ \); these edges are spanned by the vectors

\[ \rho = (1, \cdots, 1, 0, \cdots, 0), \quad 0 < m_0 < m. \]

We now fix a \( 0 < m_0 < m \). By possibly reindexing the irreducible components of \( X \), we can assume that for an \( \bar{r} \leq r \), \( h_1 \leq \cdots \leq h_r \leq m_0 \leq h_{r+1} \leq \cdots \leq h_r \). In other words,

\[ \rho_{h_1} = \cdots = \rho_{h_r} = 1, \quad \rho_{h_{r+1}} = \cdots = \rho_{h_r} = 0. \]

We let \( Y := \bigcup_{\alpha \geq \bar{r}} X_{\alpha} \); thus its complement \( Y^c := \bigcup_{\alpha < \bar{r}} X_{\alpha} \).

We claim that \( Y^c \) is the maximal subcurve of \( X \) contained in the linear subspace \( \mathbb{P} W_{m_0} \) (cf. (5.8)). By definition, for any \( \alpha, h_{\alpha} \) is the largest index \( 0 < i \leq m \) of which \( s_{i|X_{\alpha}} \neq 0 \).

On the other hand, because \( \mathbb{P} W_{m_0} = \{ s_{m_0} = \cdots = s_m = 0 \} \), \( \bigcup_{\alpha \geq \bar{r}} X_{\alpha} \subset \mathbb{P} W_{m_0} \) if and only if \( s_{i|X_{\alpha}} = 0 \) for all \( i \geq m_0 \), which is equivalent to \( h_{\alpha} < m_0 \). This proves the claim.

Let \( X_{\alpha} \) be a component in \( Y^c \). Since \( \rho_{h_{\alpha}} = 1 \), \( \rho_i = 1 \) for \( i \in \bar{I}_{\alpha} \). Using the explicit expression of \( E^*_a(\lambda, \rho)(\text{cf. (4.3)}) \), we obtain \( E^*_a(\lambda, \rho) = 2 \deg X_{\alpha} \).

Thus

\[ \sum_{\alpha \geq \bar{r}} E^*_a(\lambda, \rho) = \sum_{\alpha \geq \bar{r}} 2 \deg X_{\alpha} = 2 \deg Y^c. \]

We next look at \( Y \). Following (1.3) and (3.8), \( \tilde{L}_Y := \pi^{-1}(Y \cap Y^c) \cap \tilde{Y} \). We claim that \( \tilde{L}_Y \subset \tilde{\Lambda}_Y := \bigcup_{\alpha \geq \bar{r}} \tilde{\Lambda}_{\alpha} \). Indeed, for any \( \alpha > \bar{r} \), there is an \( i \geq m_0 \) so that \( s_{i|X_{\alpha}} \neq 0 \). However for any \( \beta \leq \bar{r}, i \geq m_0 \) implies \( s_{i|X_{\beta}} = 0 \). Thus \( s_{i|X_{\alpha} \cap X_{\beta}} = 0 \), and consequently, \( \pi^{-1}(X_{\alpha} \cap X_{\beta}) \cap \tilde{X}_{\alpha} \subset \tilde{\Lambda}_{\alpha} \). Summing over all \( \alpha > \bar{r} \) and \( \beta \leq \bar{r} \), we obtain \( \tilde{L}_Y \subset \tilde{\Lambda}_Y \).

As a consequence,

\[ \sum_{p \in \tilde{L}_Y} \rho_{i_0}(p) = \ell_Y. \]

To simplify the notation, in the remaining part of this section, we will abbreviate

\[ \sum_{p \in \Sigma} \rho_{i_0}(p) := \sum_{p \in \Sigma \cap \tilde{\Lambda}} \rho_{i_0}(p), \]

with the understanding that for any closed subset \( \Sigma \subset \tilde{X} \), \( \sum_{p \in \Sigma} \) only sums over \( p \in \Sigma \cap \tilde{\Lambda} \).

Sublemma 5.7. Let the notation be as before. Then

\[ \sum_{\alpha > \bar{r}} E^*_a(\lambda, \rho) = \frac{\ell_Y}{2} \leq \left( 1 + \frac{C^{-1} \epsilon}{\deg X} \right) \left( \sum_{\alpha > \bar{r}} \delta_{i_0}(s_{i_0}) \rho_i - \sum_{p \in \tilde{L}_Y} \rho_{i_0}(p) - \sum_{p \in \tilde{X} \setminus \tilde{L}_Y} \frac{\rho_{i_0}(p)}{2} \right) - \sum_{p \in \tilde{Y} \cap Y} \frac{\rho_{i_0}(p)}{2} \]
Lemma 5.8. For $1 \leq k \leq m_0$, we have

$$\sum_{\alpha > \bar{t} \atop i \in \mathcal{I}_a \cap [0,k)} \delta_\alpha(\tilde{s}_i) \rho_{i_\alpha} - \sum_{p \in \mathcal{L}_Y \setminus \mathcal{X}_a} \frac{\rho_{i_\alpha}(p)}{2} - \sum_{p \in \mathcal{N}_Y \setminus \mathcal{L}_Y} \frac{\rho_{i_\alpha}(p)}{2} \leq \dim W_Y \cap W_k - \dim W_{Y \cap Y^0} \cap W_k.$$
where $W_{Y \cap Y^c}$ is the linear subspace in $W$ spanned by $Y \cap Y^c$.

**Proof.** We prove the Lemma by induction on $k$. When $k = 0$, then both sides of the inequality are zero, and the inequality follows. Suppose the Lemma holds for a $0 \leq k < m_0$. Then the Lemma holds for $k + 1$ if for the expressions

$$A_{k,1} := \sum_{\alpha > \bar{r}_k, \ k \in \mathbb{I}_p^{pri}} \delta_{\alpha}(\tilde{s}_k)\rho_k, \quad A_{k,2} := \sum_{p \in L_Y, \ i_0(p) = k} \rho_{i_0(p)}, \quad A_{k,3} := \sum_{p \in N_Y \setminus L_Y, \ i_0(p) = k} \frac{\rho_{i_0(p)}}{2}$$

and

$$B_{k,1} := \dim W_Y \cap W_{k+1} - \dim W_Y \cap W_k, \quad B_{k,2} := \dim W_{Y \cap Y^c} \cap W_{k+1} - \dim W_{Y \cap Y^c} \cap W_k,$$

the following inequality holds

$$(5.20) \quad A_{k,1} - A_{k,2} - A_{k,3} \leq B_{k,1} - B_{k,2}.$$ 

To study the left hand side of (5.20), we introduce the set

$$(5.21) \quad R_k = \{ p \in \hat{Y} \mid k \in \mathbb{I}_p^{pri}\}.$$ 

By Proposition 3.9 and 3.11, $R_k$ can take three possibilities according to

$$(5.22) \quad \sum_{\alpha > \bar{r}_k, \ k \in \mathbb{I}_p^{pri}} \delta_{\alpha}(\tilde{s}_k)$$

taking values 0, 1 or $\geq 2$. Notice that if $A_{k,1} = 0$, then $A_{k,1} - A_{k,2} - A_{k,3} \leq 0$. The Lemma holds trivially in this case since the right hand side of (5.20) is non-negative. So from now on, we will assume that $A_{k,1} \geq 1$, in particular, (5.22) is positive.

We first observe that since $\dim W_{k+1} - \dim W_k = 1$, both $B_{k,1}$ and $B_{k,2}$ can only take values 0 or 1. We now investigate the case when $B_{k,2} = 1$.

**Claim 5.9.** Suppose (5.22) is positive and $B_{k,2} = 1$. Then there is a $p \in R_k$ (cf. (5.21)) such that $i_0(p) = k$ and

$$(5.23) \quad q = \pi(p) \in Y \cap Y^c \cap (\mathbb{P}W_{k+1} - \mathbb{P}W_k).$$

**Proof.** Suppose (5.22) is positive then there is a $p \in \text{inc} (\tilde{s}_k) \cap \tilde{X}_\alpha$ with $\alpha > \bar{r}$ and $k \in \mathbb{I}_p^{pri}$. Let $Z_k = \mathbb{P}W_k \cap X$ be defined after (5.9), and $W_{z_k, q}$ be defined in (5.9). Then $W_{k+1} = W_{z_k, q}$, since $\dim W_{k+1} = \dim W_k + 1$. Suppose $q = \pi(p) \not\in Y \cap Y^c$ and $k \in \mathbb{I}_p^{pri}$ then by applying the argument parallel to Proposition 3.9 and 3.11, we deduce

$$(5.24) \quad W_{z_k, q} + W_{Y \cap Y^c} \supseteq W_{z_k} + W_{Y \cap Y^c}.$$ 

On the other hand, $B_{k,2} = 1$ implies that

$$\dim(W_k + W_{Y \cap Y^c}) = \dim W_k + \dim W_{Y \cap Y^c} - \dim W_k \cap W_{Y \cap Y^c}$$

$$= \dim W_{k+1} + \dim W_{Y \cap Y^c} - \dim W_{k+1} \cap W_{Y \cap Y^c}$$

$$= \dim(W_{k+1} + W_{Y \cap Y^c}),$$

which means $W_k + W_{Y \cap Y^c} = W_{k+1} + W_{Y \cap Y^c}$ contradicting (5.24). So we must have $q \in Y \cap Y^c$.

By definition, $q \in \mathbb{P}W_{k+1}$ (cf. (5.8)) implies that $s_i(q) = 0$ for $i \geq k + 1$; $q \not\in \mathbb{P}W_k$ implies that not all $s_i(q), \ k \leq i \leq m$, are zero. Combined, we have $s_k(q) \neq 0$. This implies $i_0(q) = k$. As an easy consequence, this shows that $B_{k,2} = 1$ forces $W_Y \cap W_{k+1} \neq W_Y \cap W_k$, and hence $B_{k,1} = 1$. In particular, the right hand side of (5.20) is non-negative. This proves the Claim. \qed
We complete our proof of Lemma (5.8). When (5.22) takes value 1, then \( R_k \) consists of a single point, say \( p \in \tilde{Y} \). In case \( \pi(p) \in Y \) is a smooth point of \( X \), \( A_{k,1} = 1 \) and \( A_{k,2} = A_{k,3} = 0 \). We claim that \( B_{k,1} = 1 \) and \( B_{k,2} = 0 \). Indeed, if \( B_{k,1} = 0 \), then \( \mathcal{F}_Y \cap \mathcal{F}_k = \mathcal{F}_Y \cap \mathcal{F}_k \), which is the same as \( Y \cap (s_k = \cdots = s_m = 0) = Y \cap (s_{k+1} = \cdots = s_m = 0) \) as subschemes of \( Y \). But this contradicts to \( \sum_{\alpha \in G_k} \delta_\alpha(s_k) = 1 \). Thus \( B_{k,1} = 1 \). On the other hand, if \( B_{k,2} = 1 \), then Claim 5.9 shows that \( R_k \cap \tilde{Y} \) contains an element in \( \tilde{L}_Y \), contradicting our assumption that \( R_k = \{ p \} \) lies over a smooth point of \( X \).

In case \( p \in \tilde{L}_Y \), then the previous paragraph shows that \( A_{k,1} = B_{k,1} = 1, A_{k,3} = 0 \). For the values of \( A_{k,2} \) and \( B_{k,2} \), when \( i_0(p) = k \), then both \( A_{k,2} = B_{k,2} = 1 \); when \( i_0(p) \neq k \), then both \( A_{k,2} = B_{k,2} = 0 \). Therefore, (5.20) holds.

The last case is when \( p \in \tilde{N}_Y - \tilde{L}_Y \). In this case, since the point \( p' \) in \( \tilde{Y} \cap \pi = \pi(p) \) other than \( p \) is not contained in \( R_k \), either \( i_0(p) \neq k \) or \( i_0(p) = i_0(p') = k \) and \( k \notin {p}' \). In both cases, \( A_{k,1} = B_{k,1} = 1, A_{k,3} = B_{k,2} = 0 \); the inequality (5.20) holds.

Lastly, when (5.22) is bigger than 1, by Proposition 3.9 and 3.11, either \( R_k = \{ p_-, p_+ \} \) such that \( \pi(p_-) = \pi(p_+) \) is a node of \( Y \), i.e. \( p_\pm \in \tilde{N}_Y \), and \( i_0(p_-) = i_0(p_+) = k \), or \( R_k = \{ p_1, \cdots, p_l \} \) so that \( i_0(p_i) = k \) and \( \{ \pi(p_i) \}_{1 \leq i \leq s} \) are distinct nodes of \( X \). In case \( R_k = \{ p_-, p_+ \} \), since \( p_1 \in \tilde{N}_Y \cap \tilde{L}_Y \), \( A_{k,1} = 1, A_{k,2} = 2, A_{k,3} = B_{k,3} = 0 \); the inequality (5.20) holds.

The other case is when \( R_k = \{ p_1, \cdots, p_l \} \). By reindexing, we may assume \( p_1, \cdots, p_l \) are in \( \tilde{N}_Y \cap \tilde{L}_Y \) and \( p_{l+1}, \cdots, p_l \) are in \( \tilde{L}_Y \). We let \( p_i' \) be such that \( \pi^{-1}(\pi(p_i)) = \{ p_i, p_i' \} \) for \( i \leq l_1 \). Then \( i_0(p_i') = k \) as well, but \( k \notin {p_i}' \) because of Proposition 3.9 and 3.11. This in particular implies that the interior linking nodes \( \tilde{N}_Y \cap \tilde{L}_Y \) contribute once in \( A_{k,1} \) but twice in \( A_{k,3} \); namely, only \( \rho_i(p_i) \) appears in \( A_{k,1} \), but both \( \rho_i(p_i) \) and \( \rho_i(p_i') \) appear in \( A_{k,3} \). Therefore, \( A_{k,1} = l; A_{k,2} = l - l_1 \), and \( A_{k,3} = 2l_1 / l_1 \). Hence the left hand side of (5.20) is 0. This proves (5.20) in this case; hence for all cases. This proves the Lemma.

We continue our proof of Proposition 5.6. We apply Lemma 5.7 and Lemma 5.8 with \( k = m_0 \). Noticing \( \rho_{i_0}(p) = 0 \) for \( i_0(p) > m_0 \), we obtain

\[
\frac{E_Y(\lambda, \rho)}{2} - \ell_Y = \frac{\sum_{\alpha = \pi_{\Delta}} E'_\alpha(\lambda, \rho)}{2} - \frac{\ell_Y}{2} \leq \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) \left( \dim W_Y \cap W_{m_0} - \ell_Y \right) - \frac{1}{2} \left( \sum_{\pi(p) \in \mathcal{S}^* \cap \pi(\Delta) \cap Y} \rho_{i_0}(p) \right).
\]

Here we used that for all \( p' \in \pi(...) X \cap \pi(...) Y \cap Y, \rho_{i_0}(p') = 0 \). And the last inequality holds since by the definition of \( \mathcal{S}_{\text{reg}} \) and \( \rho_i \) (cf. (5.14)), we have \( \sum_{q \in \mathcal{S}_{\text{reg}}} \rho_{i_0}(q) \leq \sum_{\pi(p) \in \mathcal{S}^* \cap \pi(\Delta) \cap Y} \rho_{i_0}(p) \).
Using $\deg X - g = m$ (hence $2/(m + 1) \geq 1/\deg X$), and $E_Y(\lambda, \rho) = 2\deg Y^\xi$, we obtain
\[
\frac{E_X(\lambda, \rho)}{2} = \left( \deg Y^\xi + \frac{\ell_Y}{2} \right) + \left( \frac{E_Y(\lambda, \rho)}{2} - \frac{\ell_Y}{2} \right) \\
\leq \left( \deg Y^\xi + \frac{\ell_Y}{2} \right) + \left( 1 + \frac{2C^{-1}\epsilon}{m + 1} \right) (m_0 + 1 - \dim W_Y^\xi) - \frac{1}{2} \sum_{p \in S_{\text{reg}}} \rho_{i_0(p)}.
\]

Here the last inequality follows from
\[
\dim W_{m_0} \geq \dim(W_{m_0} \cap W_Y + W_{m_0} \cap W_Y^\xi) = \dim W_{m_0} \cap W_Y + \dim W_{m_0} \cap W_Y^\xi - \dim W_{m_0} \cap W_Y \cap W_Y^\xi \\
= \dim W_{m_0} \cap W_Y + \dim W_Y^\xi - \ell_Y.
\]

Now we consider the right hand side of (5.15) for $\rho$ chosen as in (5.16), which gives
\[
\sum_{i=0}^{m} \hat{\rho}_i = m_0 + 1 - \dim W_Y^\xi.
\]
Indeed, from our choice of $\rho$ and the definition of $\hat{\rho}$ (cf. (5.14)), for any $0 \leq i \leq m$, $\hat{\rho}_i = 1$ or 0, and it is $0$ if and only if either $i > m_0$ or there is an $X_\alpha$ with $i \in \Lambda_\alpha$ (cf. (3.3)) such that $\rho_{\hat{\alpha}} = 1$, that is, $i \in \{1\}^m \cup X_\alpha \subset Y^\xi$. This proves $\sum_{i=0}^{m} \hat{\rho}_i = m_0 + 1 - ||Y^\xi||$.

Our claim will follow once we prove $||Y^\xi|| = \dim W_Y^\xi$; but this follows from the

**Criterion:** $i \in \{1\}^m$ if and only if $\dim W_{i+1} \cap W_{X_\alpha} - \dim W_i \cap W_{X_\alpha} = 1$ for some $X_\alpha \subset Y^\xi$.

To justify this criterion, we notice that $\dim W_{i+1} \cap W_{X_\alpha} = \dim W_i \cap W_{X_\alpha}$ for all $X_\alpha \subset Y^\xi$ is equivalent to $Y^\xi \cap \{ s_1 = \cdots = s_m = 0 \} = Y^\xi \cap \{ s_{i+1} = \cdots = s_m = 0 \}$ as subschemes of $Y^\xi$; that is, $\dim(s_\alpha) \cap Y^\xi = 0$. Since $\lambda$ is a staircase
\[
i \notin \{1\}^m \quad \text{for all } X_\alpha \subset Y^\xi \quad \text{if and only if} \quad \dim(s_\alpha) \cap Y^\xi = 0 \quad \text{(cf. (3.3))}.
\]

This proves the criterion.

With those in hand, we obtain
\[
\frac{E_X(\lambda, \rho)}{2} = \left( \deg Y^\xi + \frac{\ell_Y}{2} \right) + \left( \frac{E_Y(\lambda, \rho)}{2} - \frac{\ell_Y}{2} \right) \\
\leq \left( \deg Y^\xi + \frac{\ell_Y}{2} \right) + \left( 1 + \frac{2C^{-1}\epsilon}{m + 1} \right) (m_0 + 1 - \dim W_Y^\xi) - \sum_{q \in S_{\text{reg}}} \hat{\rho}_{i_0(q)} \\
\leq m_0 + 1 - \sum_{q \in S_{\text{reg}}} \hat{\rho}_{i_0(q)} + \left( \deg Y^\xi + \frac{\ell_Y}{2} - \dim W_Y^\xi \right) + \frac{2C^{-1}\epsilon}{m + 1} (m_0 + 1 - \dim W_Y^\xi) \\
= \sum_{i=0}^{m} \rho_i - \sum_{q \in S_{\text{reg}}} \hat{\rho}_{i_0(q)} + \sum_{\alpha=1}^{r} \left( \deg X_\alpha + \frac{\ell_{\alpha}}{2} - m_\alpha - 1 \right) \cdot \rho_{\hat{\alpha}} + \frac{2C^{-1}\epsilon}{m + 1} \sum_{i=0}^{m} \rho_i.
\]

So the proof of Proposition is completed. \qed
Let
\[
(5.26) \quad \hat{\omega}(\lambda) = \hat{\omega}(\lambda, \rho) := \frac{2 \deg X}{m + 1} \sum_{i=0}^{m} \rho_i - E^X_\lambda(\lambda, \rho),
\]
we now state and prove the main result of this section.

**Theorem 5.10.** Given \(g, n\) and \(a \in \mathbb{Q}^n_+\) such that \(\chi_{a,g} > 0\), we let \(C\) be the constant given in Lemma 5.2. Suppose \(1 > \epsilon > 0\) such that \((2C^{-1} + 1)\epsilon < \chi_{a,g}\). Then there exists a constant \(M_5 = M_5(g, n, a, \epsilon)\) such that for any slope stable (resp. semistable) weighted pointed nodal curve \((X, \mathcal{O}_X(1), x, a)\) of genus \(g\) and \(\deg X > M_5\), and for any staircase 1-PS \(\lambda\), we have
\[
(5.27) \quad \omega_a(\lambda) = \omega(\lambda) + \mu_a(\lambda) \geq \hat{\omega}(\lambda) + \mu_a(\lambda) > (\text{resp.} \geq) \frac{2\epsilon}{m + 1} \sum_{i=0}^{m} \rho_i.
\]

**Proof.** We will give a proof of the stable case, from which the semistable case follows easily.

First, let us justify the first inequality. Given \((2C^{-1} + 1)\epsilon < \chi_{a,g}\), we define \(M_5 = M_5(g, n, a, \epsilon) := \max\{M_4(g, n, a), M_1(g, 6(g + n), n, \epsilon)/C\} > 0\), where \(C\) is the constant introduced in Lemma 5.2. Then by slope stability assumption and Lemma 5.2, whenever \(\deg X > M_5\), either \(X_\alpha\) is exceptional or
\[
\deg X_\alpha > CM_5 > M_1(g, 6(g + n), n, \epsilon) > M_1(g_\alpha, \ell_\alpha, n, \epsilon),
\]
where the last inequality follows from that \(g \geq g_\alpha, 6(g + n) \geq \ell_\alpha\) (cf. Corollary 5.3) and the definition of \(M_1\) in the proof of Theorem 4.1. Hence the assumption of Theorem 4.1 is satisfied. Applying Theorem 4.1 to \(\hat{\omega}(\lambda)\) and using (2.15), we obtain \(\omega(\lambda) > \hat{\omega}(\lambda)\). Thus the first inequality is proved.

By Proposition 5.6, it suffices to prove
\[
(5.28) \quad \sum_{i=0}^{m} \rho_i - \sum_{q \in S_{\text{reg}}} \frac{\hat{\mu}_a(q)}{2} + \sum_{\alpha=1}^{r} \left(\deg X_\alpha + \frac{\ell_\alpha}{2} - m_\alpha - 1\right) \cdot \rho_{\alpha} + \frac{(2C^{-1} + 1)\epsilon}{m + 1} \sum_{i=0}^{m} \rho_i < \frac{\deg X}{m + 1} \sum_{i=0}^{m} \rho_i + \frac{\mu_a(\lambda, \rho)}{2}.
\]

By linear programming, we only need to prove the above estimate for \(\rho\) of the form (5.16). We will break the verification into several inequalities. First, we have (defining \(a_X = \sum_{j=1}^{n} a_j\))
\[
(5.29) \quad \mu_a(\lambda, \rho) = \frac{m_0 + 1}{m + 1} a_X - \sum_{x_j \in Y \cap \mathcal{A} \cap W_{m_0}} a_j - \sum_{x_j \in Y \cap \mathcal{B} \cap W_{m_0}} a_j.
\]
Here \(x_j\) runs through all marked points of the curve. We claim that
\[
(5.30) \quad \sum_{q \in S_{\text{reg}}} \frac{\hat{\mu}_a(q)}{2} = \left| x \cap \pi(\hat{\lambda}) \cap Y \cap \mathcal{P}W_{m_0} \right| \geq \sum_{x_j \in Y \cap \mathcal{P}W_{m_0}} a_j.
\]
To this purpose, we first show that
\[
(5.31) \quad x \cap \pi(\hat{\lambda}) \cap Y \cap \mathcal{P}W_{m_0} = x \cap Y \cap \mathcal{P}W_{m_0}.
\]
Indeed, for any \(x_j\) in \(x\) that lies in \(Y \cap \mathcal{P}W_{m_0}\), \(s_k(x_j) = 0\) for \(k \geq m_0\). On the other hand, let \(x_j \in X_\alpha \subset Y\); since \(Y^6\) is the largest subcurve of \(X\) contained in \(\mathcal{P}W_{m_0}\), for some
\[ k \geq m_0, \ s_k|_{X_0} \neq 0. \] Combined with \( s_k(x_j) = 0 \), we conclude \( x_j \in \pi(\bar{A}) \) (cf. Definition 3.1). In particular \( x \cap Y \cap \mathbb{F} W_{m_0} \subset \pi(\bar{A}) \). This proves (5.31).

Applying (5.31), and using that for any colliding subset \( \{x_1, \ldots, x_i\} \) (i.e. \( x_i = \cdots = x_{i} \)) necessarily \( a_i + \cdots + a_i \leq 1 \), we obtain

\[ \frac{a_j}{2} \text{ dim } W_{m_0} \text{ for } x_j \in Y \cap \mathbb{F} W_{m_0} \]

hence (5.30).

By putting (5.29) and (5.30) together, we obtain

\[ \frac{\sum_{q \in S_{reg}} \hat{\rho}_{a(q)} - \mu_a(\lambda, \rho)}{2} \leq m_0 + 1 \frac{aX}{2} + \sum_{x_j \in Y \cap \mathbb{F} W_{m_0}} a_j \frac{1}{2}. \]

On the other hand, for \( \rho \) of the form in (5.16), we have

\[ \sum_{i=0}^{m} \rho_i + \sum_{a=1}^{r} \left( \deg X_a + \frac{\ell_a}{2} - m_a - 1 \right) \cdot \rho_a + \frac{(2C^{-1} + 1)\epsilon}{m + 1} \sum_{i=0}^{m} \hat{\rho}_i \]

\[ = m_0 + 1 + \left( \deg Y_c + \frac{\ell_c}{2} - \dim W_{Y_c} \right) + \frac{(2C^{-1} + 1)\epsilon}{m + 1} \left( m_0 + 1 - \dim W_{Y_c} \right). \]

Plugging (5.34) and (5.33) into (5.28), we obtain

\[ \frac{\mu_a(\lambda, \rho)}{2} + E_a(\lambda, \rho) + \frac{(2C^{-1} + 1)\epsilon}{m + 1} \sum_{i=0}^{m} \hat{\rho}_i \]

\[ \leq m_0 + 1 + \left( \deg Y_c + \frac{\ell_c}{2} + \sum_{x_j \in Y_c \cap \mathbb{F} W_{m_0}} \frac{a_j}{2} - \dim W_{Y_c} \right) - \frac{m_0 + 1}{2} \frac{aX}{m + 1} + \frac{(2C^{-1} + 1)\epsilon}{m + 1} \left( m_0 + 1 - \dim W_{Y_c} \right) \]

\[ = \frac{\deg Y_c + \frac{\ell_c}{2} + \sum_{x_j \in Y_c \cap \mathbb{F} W_{m_0}} \frac{a_j}{2}}{\dim W_{Y_c}} \dim W_{Y_c} - \frac{m_0 + 1}{2} \frac{aX}{m + 1} + \left( 1 + \frac{(2C^{-1} + 1)\epsilon}{m + 1} \right) \left( m_0 + 1 - \dim W_{Y_c} \right). \]

Since \( \dim W_{Y_c} \leq m_0 < m \), we have

\[ \frac{\deg Y_c + \frac{\ell_c}{2} + \sum_{x_j \in Y_c \cap \mathbb{F} W_{m_0}} \frac{a_j}{2}}{\dim W_{Y_c}} < \frac{\deg X + \frac{aX}{2}}{m + 1} \]

by our stability assumption and Lemma 5.5. Hence we have

LHS of (5.35) \[ \frac{\deg X + \frac{aX}{m + 1}}{\dim W_{Y_c}} \left( 1 + \frac{(2C^{-1} + 1)\epsilon}{m + 1} \right) \left( m_0 + 1 - \dim W_{Y_c} \right) - \frac{m_0 + 1}{2} \frac{aX}{m + 1} \]

\[ \leq \frac{\deg X + \frac{aX}{m + 1}}{\dim W_{Y_c}} \left( \dim W_{Y_c} + m_0 + 1 - \dim W_{Y_c} \right) - \frac{m_0 + 1}{2} \frac{aX}{m + 1} \]

\[ = \frac{\deg X}{m + 1} \left( m_0 + 1 \right) = \frac{\deg X}{m + 1} \sum_{i=0}^{m} \rho_i. \]
where we have used the assumption \((2C^{-1} + 1)\epsilon < \chi_{a,g}\) to conclude
\[
\frac{\deg X + \frac{a_X}{m+1}}{m+1} > 1 + \frac{(2C^{-1} + 1)\epsilon}{m+1}
\]
in the second inequality. This completes the proof. \(\square\)

**Proof of Theorem 1.5.** We first prove that slope stable implies Chow stable. By our assumption \(\chi_{a,g} > 0\), we may choose an \(0 < \epsilon < 1\) such that \((2C^{-1}+1)\epsilon < \chi_{a,g}\). Fixing such \(\epsilon\), we let \(M_5 = M_5(g,n,a,\epsilon)\) be the constant given in Theorem 5.10. Then by Theorem 5.10, whenever \(\deg X > M_5\) and \((X, \mathcal{O}_X(1), x, a)\) is slope stable we have \(\omega_a(\lambda >= \frac{2\epsilon}{m+1} \sum_{i=0}^m \rho_i \geq 0; \) hence \((X, \mathcal{O}_X(1), x, a)\) is stable by Definition 1.3. The proof for semistable case is the similar. This proves the sufficient part.

We now prove the other direction: Chow stable implies slope stable. Let \(Y \subset X\) be any proper subcurve; let \(W_Y \subset W\) be the linear subspace spanned by \(Y\), and let \(m_0 + 1 = \dim W_Y\). We choose a 1-PS \(\lambda = \text{diag}[t^{p_0}, \cdots, t^{p_m}] \cdot t^{-\rho_{ave}}\) such that the corresponding filtration \(\{W_i\}_{i=0}^m\) satisfies \(W_{m_0+1} = W_Y\); we choose the weights \(\{\rho_i\}\) as in (5.16). Then
\[
\mu_a = a_X \left(\frac{m_0 + 1}{m+1}\right) - \sum_{x_j \in PW_Y} a_j.
\]
Thus by the proof of [16, Proposition 5.5, page 60], \(e(\lambda)/2 \geq \deg Y + \ell_Y/2\); hence
\[
0 \leq \frac{\omega_a(\lambda) - \ell_Y}{2} = \frac{\sum_{i=0}^m \rho_i \cdot \deg X - \frac{e(\lambda)}{2} + \mu_a(\lambda)}{2} \leq \frac{m_0 + 1}{m+1} \deg X - \left(\deg Y + \frac{\ell_Y}{2}\right) + \frac{m_0 + 1}{m+1} \frac{a_X}{2} - \sum_{x_j \in PW_Y} \frac{a_j}{2}
\]
\[
= (m_0 + 1) \left(\frac{\deg X + \frac{a_X}{m+1}}{m+1} - \frac{\deg Y + \frac{\ell_Y}{2} + \frac{\mu_a(\lambda)}{2}}{m_0 + 1}\right).
\]
which is equivalent to the slope semistability (cf. Definition 1.4) provided that \((X, \mathcal{O}_X(1))\) being non-special, which will be proved in Proposition 6.2.

Finally, if we assume further \((X, \mathcal{O}_X(1), x, a)\) is stable, then (5.36) becomes equality only if \(m_0 = m\). Since \(M_5 \geq M_4\) by our choice, by Lemma 5.5 \(m_0 = m\) only when \(Y^G\) is a disjoint union of exceptional components provided \(\deg X > M_5\). By choosing \(M(g,n,a) := \max\{M_5, M_6\}\) with \(M_6\) being determined in Proposition 6.2, we complete the proof of the theorem. \(\square\)

**Proof of Theorem 1.6.** By our choice that \(M \geq M_5 \geq M_4\), our claim follows from Proposition 5.4. \(\square\)

6. Re-construction of the Moduli of Weighted Pointed Curves

In this section, we use GIT quotient of Hilbert scheme to construct the moduli of weighted pointed stable curves, first introduced and constructed by Hassett [10] using a different method. First, following Caporaso [3, section 3.3], we introduce the following

**Definition 6.1.** A weighted pointed quasi-stable curve is a weighted pointed nodal curve \((X, x, a)\) such that
1. \(\omega_X(a \cdot x)\) is numerically non-negative;
2. the total degree \(2\chi_{a,g}(X) = \deg \omega_X(a \cdot x)\) is positive;
(3) any connected subcurve $E \subset X$ satisfying $\deg \omega_X(a \cdot x)|_E = 0$ must have $E \cap x = \emptyset$ and $E \cong \mathbb{P}^1$, called an exceptional component.

We say $(X, x, a)$ is weighted pointed stable if it is weighted pointed quasistable and does not contain exceptional components.

6.1. As a GIT quotient. We fix integers $n$, $g$ and weights $a \in \mathbb{Q}_+^n$ satisfying $\chi_{a,g}(X) > 0$; for a large integer $k$ such that $k \cdot a_i \in \mathbb{Z}$ for all $i$, we let $d = (|a| + 2g - 2) \cdot k$, and form

\[ P(t) = d \cdot t + 1 - g \in \mathbb{Z}[t], \quad m + 1 = P(1). \]

We denote by $\text{Hilb}^P_{P^m}$ the Hilbert scheme of subschemes of $P^m$ of Hilbert polynomial $P$; we define $\mathcal{H}$ be the fine moduli scheme of flat families of $(X, x, a)$, where

\[ [t : X \to P^m] \in \text{Hilb}^P_{P^m} \quad \text{and} \quad x = (x_1, \ldots, x_n) \in X^n. \]

Using that Hilbert schemes are projective, we see that $\mathcal{H}$ exists and is projective. We denote by

\[ (\pi_H, \varphi) : X \longrightarrow \mathcal{H} \times P^m, \quad \pi_i : \mathcal{H} \to X, \quad i = 1, \ldots, n, \]

the universal family of $\mathcal{H}$.

We introduce a parallel space for the Chow variety. We let $\text{Chow}^d_{P^m}$ be the Chow variety of degree $d$ dimension one effective cycles in $P^m$. For any such cycle $Z$, we denote by $\text{Chow}^d(Z) \in \text{Div}^d([P^m])$ its associated Chow point (cf. Section 1). We define

\[ C := \{(Z, x) \in \text{Chow}^d_{P^m} \times (P^m)^n | x = (x_1, \ldots, x_n) \in (\supp Z)^n\}. \]

By Chow Theorem, $C$ is projective. Using the Chow coordinate, we obtain an injective morphism

\[ C \longrightarrow \text{Div}^d([P^m]) \times (P^m)^n. \]

As before (cf. Section 1), we endow it with the ample $\mathbb{Q}$-line bundle $\mathcal{O}_C(1, a)$, which is canonically linearized by the diagonal action of $G := SL(m + 1)$ on $C$. We let $C^{ss} \subset C$ be the (open) set of semistable points with respect to the $G$ linearization on $\mathcal{O}_C(1, a)$.

For any one-dimensional subscheme $X \subset P^m$, we denote by $[X]$ its associated one-dimensional cycle. By sending $(X, \iota, x) \in \mathcal{H}$ to $([X], x) \in C$, we obtain the $G$-equivariant Hilbert-Chow morphism (cf. [14, Section 5.4])

\[ \Phi : \mathcal{H} \longrightarrow C. \]

To characterize the members in $\Phi^{-1}(C^{ss})$, we need the following

Proposition 6.2. For $g$, $n$ and $a \in \mathbb{Q}_+^n$ satisfying $\chi_{a,g}(X) > 0$, there is an integer $M_0 = M_0(g, n, a)$ so that for $d > M_0$, a connected one-dimensional closed subscheme $X \subset P^m$ satisfies $(X, x, a) \in \Phi^{-1}(C^{ss})$ if and only if the associated data $(X, x, \iota \circ O_{P^m}(1), a)$ is a slope semistable polarized weighted pointed nodal curve.

The proof is a slight modification of the one given in [16, Proposition 3.1] by incorporating the weighted points. To do that, we need the following

Lemma 6.3. Let $\lambda = \text{diag}[t, 1, \ldots, 1]$, $X' = \text{diag}[t^4, t^2, t, 1, \ldots, 1]$ and $x_0 := [1, 0, \ldots, 0] \in P^m$. Then the $a$-$\lambda$-weight (resp. $a$-$\lambda'$-weight) of $x = (x_1, \ldots, x_n) \in (P^m)^n$ is

\[ \mu_a(\lambda) = -\frac{m \cdot \sum_{x_j = x_0} a_j}{m + 1} \left( \text{resp. } \mu_a(\lambda') = -\frac{(4m - 3) \cdot \sum_{x_j = x_0} a_j}{m + 1} \right). \]

In particular, $\mu_a(\lambda), \mu_a(\lambda') \leq 0$ as long as $m \geq 1$. 
Proof. It directly follows from (5.10). □

Proof. Let \((X, t, x) \in \Phi^{-1}(C^ss)\). We claim that when \(\deg X/(m+1) < 8/7\), then every irreducible component of the cycle \([X]\) has multiplicity one; \(X_{\text{red}}\) is a nodal curve, and \(X\) differs from \(X_{\text{red}}\) by embedded points. First, if \(x \in X\) has multiplicity at least 3, then we choose coordinates so that \(x = [1, 0, \cdots, 0]\) and \(\lambda = \text{diag}(t, 1, \cdots, 1)\); in this case the weight

\[
\omega_{\lambda}(\lambda) = \omega(\lambda) + \mu_{\lambda}(\lambda) = \frac{2 \deg X \sum_{i=0}^{m} e(3) - m \cdot \sum_{x=x_0} a_j}{m+1}
\]

\[\leq \frac{2 \deg X \cdot \sum_{i=0}^{m} \rho_i - e(3) - 16 \sum_{i=0}^{m} \rho_i - 3 < 0.\]

Next if \(x \in X\) is a non-ordinary double point then by choosing the coordinates in the proof of [16, Proposition 3.1] and \(\lambda = \text{diag}(t, t^2, t, 1, \cdots, 1)\) accordingly, we obtain

\[
\omega_{\lambda}(\lambda) = \omega(\lambda) + \mu_{\lambda}(\lambda) = \frac{2 \deg X \sum_{i=0}^{m} e(3) - (4m-3) \cdot \sum_{x=x_0} a_j}{m+1}
\]

\[\leq 2 \deg X \cdot \sum_{i=0}^{m} \rho_i - e(3) < 16 \sum_{i=0}^{m} \rho_i - 16 = 0.\]

Both cases contradict our assumption that \((X, t, x) \in \Phi^{-1}(C^ss)\). This proves the claim.

We now show that \(X = X_{\text{red}} \subset \mathbb{P}^m\) and is embedded by a complete nonspecial linear system. Let \(X_{\alpha} \subset X_{\text{red}}\) be an irreducible component, and write \(X_{\text{red}} = X_{\alpha} \cup X_{\alpha}^0\). For \(W_{X_{\alpha}} \subset W\) the linear subspace spanned by \(X_{\alpha}\), we choose a basis \(\{s_i\}\) so that \(W_{X_{\alpha}} = \{s_{m_{\alpha}+1} = \cdots = s_m = 0\}\), and define a 1-PS \(\lambda\) by the rule

\[
\lambda = \text{diag}(t, 1, \cdots, 1, \cdots, 1).
\]

Since Chow \((X, t, x) \in C^ss\), by the same calculation as we did in the proof of Theorem 1.5 (cf. (5.36)) we obtain

\[
0 \leq \frac{\omega_{\lambda}(\lambda)}{2} = \frac{\omega(\lambda) + \mu_{\lambda}(\lambda)}{2}
\]

\[\leq \frac{m_{\alpha} + 1}{m+1} \deg X \left(\deg Y + \frac{\ell_{\alpha}}{2}\right) \cdot \frac{m_{\alpha} + 1}{m+1} \frac{a_X}{2} - \sum_{x_{\alpha} \in \mathbb{P}^m W_{X_{\alpha}}} \frac{a_j}{2}
\]

\[= (m_{\alpha} + 1) \left(\frac{\deg X + \frac{\ell_{\alpha}}{2}}{m+1} - \frac{\deg X + \frac{\ell_{\alpha}}{2} + \frac{a_X}{2}}{m_{\alpha} + 1}\right).
\]

Now we choose \(M_0 \geq 8(g-1) + \frac{a_X}{2}\), and assume \(d = \deg X > M_0\); then

\[
\frac{\deg X_{\alpha} + \frac{\ell_{\alpha}}{2} + \frac{a_X}{2}}{m_{\alpha} + 1} \leq \frac{\deg X + \frac{a_X}{2}}{m+1} \leq \frac{8}{7}.\tag{6.4}
\]

We claim that \(h^1(\mathcal{O}_{X_{\alpha}}(1)) = 0\). Suppose not, then by Saint-Donat’s extension of Clifford’s theorem [9, Lemma 9.1] we have \(h^0(\mathcal{O}_{X_{\alpha}}(1)) \leq \frac{1}{2} \deg X_{\alpha} + 1\), which combined with (6.4) implies

\[
\deg X_{\alpha} \leq \frac{8}{7} h^0(\mathcal{O}_{X_{\alpha}}(1)) \leq \frac{8}{14} \deg X_{\alpha} + \frac{8}{7}.
\]

Note that this is possible only if \(\deg X_{\alpha} \leq 2\), and then \(X_{\alpha} \cong \mathbb{P}^1\) and \(h^1(X_{\alpha}, \mathcal{O}_{X_{\alpha}}(1)) = 0\), a contradiction. This proves the claim.
We next claim that \( h^1(\mathcal{O}_{X_\alpha}(1)\langle-L_\alpha\rangle) = 0 \), where \( L_\alpha = X_\alpha \cap X_\alpha^\circ \) (cf. (3.9)). Indeed, by (6.4) and using \( h^1(\mathcal{O}_{X_\alpha}(1)) = 0 \) just proved, we deduce

\[
\deg X_\alpha + \frac{a X_\alpha}{2} + \frac{\ell X_\alpha}{2} \leq \frac{8}{7}(\deg X_\alpha + 1 - g(X_\alpha)).
\]

Hence

\[
\deg(\mathcal{O}_{X_\alpha}(1)\langle-L_\alpha\rangle) = \deg X_\alpha - \ell X_\alpha \geq 8(g(X_\alpha) - 1) + \frac{5}{2}\ell X_\alpha + \frac{7}{2}a X_\alpha
\]

which is greater than \( 2g(X_\alpha) - 2 \) unless either when \( g(X_\alpha) = 1, \ell X_\alpha = 0 \) and \( a X_\alpha/2 = 0 \), or when \( g(X_\alpha) = 0, \ell X_\alpha = 1 \) or \( 2 \). The first case cannot happen, since \( L_\alpha \neq \emptyset \) by our assumption on the connectedness of \( X \); in the second case, we have \( \mathcal{O}_{X_\alpha}(1)(-L_\alpha) = \mathcal{O}_X(e) \), with \( e \geq -1 \), thus for both cases we will have \( h^1(\mathcal{O}_{X_\alpha}(1)\langle-L_\alpha\rangle) = 0 \). This settles the claim.

Finally, we show that \( h^1(X, \mathcal{O}_X(1)) = 0 \) and \( X = X_{\text{red}} \). First, \( h^1(\mathcal{O}_{X_\alpha}(1)\langle-L_\alpha\rangle) = 0 \) for all \( X_\alpha \) and that \( X_{\text{red}} \) is a nodal curve implies that \( h^1(\mathcal{O}_{X_{\text{red}}}(1)) = 0 \). Since \( X \) differs from \( X_{\text{red}} \) by embedded points, we have \( h^1(\mathcal{O}_X(1)) = 0 \). This proves that \((X, \mathcal{O}_X(1))\) is non-special.

It remains to show that \( X = X_{\text{red}} \). As \((X, \iota, x) \in \mathcal{H}, \) by the vanishing proved, we have \( m + 1 = h^0(X, \mathcal{O}_X(1)) \). Suppose \( X_{\text{red}} \neq X \), then \( X_{\text{red}} \) lies in a hyperplane, say \( \{ s_m = 0 \} \) for a basis \( \{ s_i \} \). Let \( \lambda = \text{diag}[t, \ldots, t, 1] \), then the \( \lambda \)-weight for Chow \((X, \iota, x)\) (by letting \( Y = X_{\text{red}} \) in (5.36)) is

\[
\omega(\lambda) + \mu_\lambda(\lambda) = m\left(\frac{\deg X + a X}{m + 1} - \frac{\deg X + a X}{m}\right) < 0,
\]

contradicting the fact that \((X, \iota, x) \in \Phi^{-1}(C^{ss})\). So \( X = X_{\text{red}} \) is a nodal curve. This implies that \( X \subset PW \) is non-degenerate and is embedded by a complete linear system.

Our next step is to show that \((X, \iota^*\mathcal{O}_{P^m}(1), x, a)\) is a weighted pointed nodal curve. For this, we need to verify that the weighted points are away from the nodes of \( X \), and the total weight at any point is no more than one. Let \( p \in X \) be any point. We choose a 1-PS \( \lambda \) as in Example 2.5; the associated \( \lambda \)-weight for Chow \((X, \iota, x)\) is

\[
\omega(\lambda) + \mu_\lambda(\lambda) = \frac{2\deg X}{m + 1} - \epsilon_p + \frac{1}{m + 1}a X - \sum_{x_j = p} a_j = 2 - \epsilon_p + \frac{2a_{a,g}}{m + 1} - \sum_{x_j = p} a_j,
\]

where \( \epsilon_p = 2 \) if \( p \) is a node and \( 1 \) otherwise. Since Chow \((X, \iota, x)\) is semistable, we must have \( 0 \leq \omega(\lambda) + \mu_\lambda(\lambda) \). Now we choose \( M \) so that \( M \geq g + 2a_{a,g}/\min\{a_i\} \), then \( 0 \leq \omega(\lambda) + \mu_\lambda(\lambda) \) implies that the weighted points must be away from the nodes, and the total weight of marked points at \( p \) does not exceed one.

In the end, Theorem 1.5 implies that such \((X, \iota^*\mathcal{O}_{P^m}(1), x, a)\) is slope semistable. This proves that for the choice \( M_0(g, n, a) := \max\{g + 2a_{a,g}/\min\{a_i\}, 8(g - 1) + n/2\} \), the lemma holds.

We define

\[
\mathcal{H}^{ss} = \Phi^{-1}(C^{ss}) \subset \mathcal{H}.
\]

**Corollary 6.4.** For \( d \geq M \) specified in Proposition 6.2, the restriction

\[
\Phi^{ss} := \Phi|_{\mathcal{H}^{ss}} : \mathcal{H}^{ss} \to C^{ss}
\]

is injective and hence an isomorphism.
Proof. We only need to prove that $\Phi^{ss}$ is injective. Suppose not, say there are $(X, \iota, x) \neq (X', \iota', x') \in \mathcal{H}^{ss}$ such that $\Phi(X, \iota, x) = \Phi(X', \iota', x') \in \mathcal{C}^{ss}$; then by Lemma 6.2, both $X$ and $X'$ are nodal subcurves of $\mathbb{P}^m$. Since $\Phi(X, \iota, x) = \Phi(X', \iota', x') \in \mathcal{C}^{ss}$, the cycles $[X] = [X']$ and $x = x' \subset \mathbb{P}^m$; since both $X$ and $X'$ are nodal, we must have $X = X'$; thus $(X, \iota, x) = (X', \iota', x')$, a contradiction. This proves that $\Phi^{ss}$ is injective. Finally, since $\mathcal{C}^{ss}$ is normal, we conclude that $\Phi^{ss}$ is an isomorphism by Zariski’s main theorem. □

To construct the moduli of weighted pointed curves, taking the $k$ specified before (6.1), we form

$$\mathcal{K} = \{ (X, \iota, x) \in \mathcal{H} \mid X \text{ smooth weighted pointed curves, } \iota^* \mathcal{O}_{\mathbb{P}^m}(1) \cong \omega_X(a \cdot x)^{\otimes k} \}. $$

It is locally closed, and is a smooth subscheme of $\mathcal{H}$. (Note that $\mathcal{H}$ is smooth near $\mathcal{K}$.) Since $X$ in $(X, \iota, x) \in \mathcal{K}$ are smooth, applying Theorem 1.5, we conclude that $\Phi(\mathcal{K}) \subset \mathcal{C}^{ss}$, thus $\mathcal{K} \subset \mathcal{H}^{ss}$. Let $\overline{\mathcal{K}} \subset \mathcal{H}^{ss}$ be the closure of $\mathcal{K}$ in $\mathcal{H}^{ss}$. Because $\Phi^{ss}$ is finite, and $\mathcal{C}$ is projective, the GIT quotients $\mathcal{H}^{ss}/G \to \mathcal{C}^{ss}/G$ exist and the arrow is finite [6, Lemma 4.6], thus $\mathcal{H}^{ss}/G$ is projective. Because $\overline{\mathcal{K}}$ is closed in $\mathcal{H}^{ss}$, the GIT quotient

$$(6.5) \quad q : \overline{\mathcal{K}} \longrightarrow \overline{\mathcal{K}}/G$$

exists and is projective.

There is a natural transformation from the category of flat families of pointed curves in $\overline{\mathcal{K}}$ to the category of stable genus $g$, $a$-weighted pointed nodal curves. For any $(X, \iota, x) \in \overline{\mathcal{K}}$, since the associated weighted pointed nodal curve $(X, x, a)$ is semistable, we can form a new weighted pointed curve by contracting all of its exceptional components (cf. Definition 6.1). We denote the resulting curve by

$$(6.6) \quad (X^{st}, x^{st}, a),$$

and call it the stabilzation of $(X, x, a)$. Since $(X, \iota, x) \in \mathcal{H}^{ss}$ and the marked points never lie on the contracted components, by Lemma 5.2 the stabilization produces a weighted pointed stable curve of the same genus. Furthermore, the stabilization applies to families of quasistable weighted pointed curves. The mentioned transformation is by applying this contraction to the restriction to $\overline{\mathcal{K}}$ of the universal family of $\mathcal{H}$, resulting in a family of weighted pointed stable curves on $\overline{\mathcal{K}}$.

Let $\overline{\mathcal{M}}_{g,a}$ be the coarse moduli space of stable genus $g$, $a$-weighted nodal curves constructed by Hassett [10]. This transformation induces a morphism

$$(6.7) \quad \overline{\nu} : \overline{\mathcal{K}} \longrightarrow \overline{\mathcal{M}}_{g,a}. $$

As this morphism is $G$-equivariant with $G$ acting trivially on $\overline{\mathcal{M}}_{g,a}$, it descends to a morphism

$$(6.8) \quad \psi : \overline{\mathcal{K}}/G \longrightarrow \overline{\mathcal{M}}_{g,a}. $$

**Theorem 6.5.** The morphism $\psi$ is an isomorphism.

It is worth mentioning that the two coarse moduli schemes $\overline{\mathcal{K}}/G$ and $\overline{\mathcal{M}}_{g,a}$ parameterize different moduli objects. For $g$, $a$ and sufficiently divisible $k$, we define $\mathcal{P} \subset \mathcal{H}$ via

$$\mathcal{P} = \{ (X, \iota, x) \in \mathcal{H} \mid (X, x, a) \text{ weighted pointed stable curves, } \omega_X(a \cdot x)^{\otimes k} \cong \iota^* \mathcal{O}_{\mathbb{P}^m}(1) \}. $$

A direct check shows that $\overline{\mathcal{P}}$ with reduced structure is a smooth, locally closed, and $G$-invariant subscheme of $\mathcal{H}$. We let $\mathcal{P} \subset \overline{\mathcal{P}}$ be the open subset of $(X, \iota, x)$ such that $X$ are smooth. By definition, $\mathcal{P} = \mathcal{K}$. However, the following example shows that $\overline{\mathcal{P}} \not\subset \mathcal{H}^{ss}$. The theorem states that this change of moduli objects does not alter the resulting coarse moduli schemes.
We next let the same technique to prove the injectivity, we outline its proof, following [3].

\[ (6.9) \]

The Proposition was essentially proved by Caporaso in [3]. Since we need to use the\[ 6.2. \textbf{Surjectivity.} \text{ Let } (X, x, a) \text{ be a weighted pointed stable curve. We endow it with the polarization } \mathcal{O}_X(1) = \omega_X(a, x)^{\otimes k} \text{ together with the embedding } \iota : X \to \mathfrak{PH}^0(\mathcal{O}_X(1))^\vee. \text{ When } X \text{ is smooth, } (X, \iota, x, a) \text{ lies in } K; \text{ when } X \text{ is singular, this may not necessarily hold. Our solution is to replace } \omega_X(a, x)^{\otimes k} \text{ by its twist, to be defined momentarily.} \text{ Given } (X, x), \text{ we choose a smoothing } \pi : \mathcal{X} \to T \text{ over a pointed curve } 0 \in T \text{ such that } \mathcal{X} \text{ is smooth and } X_0 = \mathcal{X} \times_T 0 \cong X. \text{ By an étale base change of } T, \text{ we can extend the } n\text{-marked points of } X \text{ to sections } \iota_1 : T \to \mathcal{X} \text{ so that, denoting } x = (x_1, \ldots, x_n), (\mathcal{X}, \iota, a) \text{ form a flat family of weighted pointed stable curves. Let } X_1, \ldots, X_r \text{ be the irreducible components of } X. \text{ The following Proposition gives the surjectivity of } \psi. \]

\[ (6.7) \text{ Given } g, n \text{ and } a \in \mathbb{Q}_+^n \text{ satisfying } \chi_{n,g} > 0, \text{ there is a constant } K = K(g, n, a) \text{ so that for a weighted pointed stable curve } (X, x, a) \text{ and sufficiently divisible } k \geq K, \text{ and for } (\mathcal{X}, \iota, a) \text{ the } T\text{-family constructed, there are integers } \{b_\alpha\}_{\alpha=1}^r \text{ independent of } k \text{ so that after letting } \]

\[ (6.9) \]

\[ (X, \mathcal{O}_X(1), s, a) \text{ is a family of slope semistable weighted pointed nodal curves.} \]

The Proposition was essentially proved by Caporaso in [3]. Since we need to use the same technique to prove the injectivity, we outline its proof, following [3].

For any line bundle \( L \) on \( X \), we denote \( \delta_\alpha(L) = \deg L|_{X_\alpha} \), and define the numerical class of \( L \) to be \( \delta(L) := (\delta_1(L), \ldots, \delta_r(L)) \in \mathbb{Z}^{gr} \).

\[ (6.10) \]

We next let \( \ell_{\alpha, \beta} = \ell_{\alpha, \beta}(X) = |X_\alpha \cap X_\beta| \text{ if } \alpha \neq \beta, \text{ and } \ell_{\alpha, \alpha} = \ell_{\alpha, \alpha}(X) = -|X_\alpha \cap X_\alpha| \).

We denote \( \tilde{\ell}_\alpha = \tilde{\ell}_\alpha(X) = (\ell_{\alpha, 1}(X), \ell_{\alpha, 2}(X), \ldots, \ell_{\alpha, r}(X)) \). Letting \( \mathbb{Z}^{gr} = \{ \tilde{v} \in \mathbb{Z}^{gr} \mid \sum_{i=1}^r v_i = 0 \} \),
then every $\ell^r_\alpha \in \mathbb{Z}^{qr}_0$. We define $\Gamma_X \subset \mathbb{Z}^{qr}$ to be the subgroup generated by $\ell_1, \ldots, \ell^r_r$.

**Remark 6.8.** Let $\mathcal{L} = \omega_X(a \cdot x)^{\otimes k}$. Since $X$ is smooth, for the invertible sheaf $\mathcal{O}_X(1)$ defined in (6.9) depending on the integers $b_1, \ldots, b_r$, we have

$$\delta(\mathcal{O}_X(1)|_X) = \delta(\mathcal{L}) + \sum_{\alpha=1}^r b_\alpha \ell^r_\alpha.$$ 

This says that any two choices of $\mathcal{O}_X(1)$ restricted to the central fiber have equivalent numerical classes modulo $\Gamma_X$.

We introduce one more notation. For any vector $v = (v_1, \ldots, v_r) \in \mathbb{Z}^{qr}$ and any subcurve $Y \subset X$, mimicking the notion of degree, we define

$$\text{deg}_Y v = \sum_{X \subset Y} v_\alpha.$$ 

Let $(X, x, a)$ be a weighted pointed nodal curve, and let $d$ be a positive integer. For any subcurve $Y \subset X$, we introduce the $d$-extremes of $Y$ to be

$$(6.11) \quad M^d_Y := \frac{\text{deg}_Y \omega_X(a \cdot x)}{\text{deg} \omega_X(a \cdot x)} \left( d + \frac{a_X}{2} \right) - \frac{a_Y}{2} \pm \frac{\ell_Y}{2}.$$ 

Then Proposition 5.4 can be reformulated as follows

**Proposition 6.9.** Given $g, n$ and $a \in \mathcal{Q}^n_a$ satisfying $\chi_{a, g} > 0$, let $M_3 = M_3(g, n, a)$ be the constant defined in Corollary 5.3. Then for any polarized weighted pointed nodal curve $(X, \mathcal{L}, x, a)$ of $\text{deg} \mathcal{L} = d \geq M_3$ is slope semistable if and only if

$$(6.12) \quad \text{deg}_Y \mathcal{L} \in [M^d_Y^+, M^d_Y^-] \quad \text{for any subcurve } Y \subset X.$$ 

**Proof.** This follows from Proposition 5.4 and the fact that (6.12) is equivalent to (1.6). $\Box$

Let $\mathbb{Z}^{qr}_{\geq 0}$ be those $v = (v_i) \in \mathbb{Z}^{qr}$ so that $v_i \geq 0$. We define

$$\mathfrak{B}^d_{X, x, a} = \{ v \in \mathbb{Z}^{qr}_{\geq 0} \mid \text{deg}_X v = d, \; \text{v satisfies (6.12) with } \text{deg}_Y \mathcal{L} \text{ replaced by } \text{deg}_Y v \}.$$ 

Then we have the following

**Proposition 6.10.** Let $(X, x, a)$ be a weighted pointed quasistable (cf. Definition 6.1) curve, then for any $v \in \mathbb{Z}^{qr}$ we have

$$(\bar{v} + \Gamma_X) \cap \mathfrak{B}^d_{X,x,a} \neq \emptyset.$$ 

**Proof.** Since the proof is completely parallel to [3, Proposition 4.1] we will omit it. $\Box$

**Proof of Proposition 6.7.** By applying Proposition 6.10 to $\bar{v} = \delta(\omega_X(a \cdot x)^{\otimes k})$, one easily see that there are $\{ b_\alpha \}$'s independent of $k$ such that for the $\mathcal{O}_X(1)$ given in (6.9) and $\mathcal{L} = \mathcal{O}_X(1)|_X$, $\delta(\mathcal{L}) \in \mathbb{Z}^{qr}_0$ and satisfies (6.12). Since $X$ has smooth fibers other than the central fiber, we only need to show that the central fiber $(X, \mathcal{L}, x, a)$ is slope semistable. To achieve that, one notice that $\delta(\mathcal{L})$ satisfies (6.12) already, by Proposition 6.9 to show $(X, \mathcal{L}, x, a)$ is a slope semistable polarized weighted pointed nodal curve all we need to show is that $\mathcal{L}$ is ample and $\text{deg}_X \mathcal{L} = k \chi_{a,g} \geq M_3$. First, by our assumption $(X, x, a)$ is a weighted pointed stable curve (cf. Definition 6.1), hence $\omega_X(a \cdot x)$ is ample. It follows from the proof of Lemma 5.2 that $\text{deg}_Y \mathcal{L} \geq C \text{deg} X > 0$ for any $Y \subset X$ as long as $\text{deg} X > M_3 \geq M_2$, in particular, $\mathcal{L}$ is very ample by Corollary 5.3. Now we define $K(g, n, a) := M_3(g, n, a)/\chi_{a,g}$, then in case $k > K = K(g, n, a)$, by Proposition 6.9, $(X, \mathcal{L}, x, a)$ is a polarized slope semistable weighted pointed curve with $\mathcal{L}$ being very ample. $\Box$
6.3. Injectivity. We use the separatedness of $\overline{\mathcal{K}}/G$ to prove that $\psi$ in (6.8) is injective.

Definition 6.11. For $(\bar{X}, \bar{x})$ and $(X, x)$ two pointed nodal curves, we say the former is a blow-up of the latter if there is a morphism $\pi : \bar{X} \to X$ that is derived by contracting some exceptional components of $(X, \bar{x})$.

Since the restriction of $\psi$ to $\mathcal{K}/G$ is an isomorphism and $\mathcal{K}/G$ is irreducible, $\psi$ is a birational morphism. By the deformation theory of pointed nodal curves, we see that $\mathcal{M}_{g,a}$ has only finite quotient singularities, thus is normal. By Zariski’s Main theorem and the properness of $\overline{\mathcal{K}}/G$, the injectivity of $\psi$ follows from

Lemma 6.12. $\psi^{-1}(\psi(\xi))$ is zero dimensional for each $\xi \in \overline{\mathcal{K}}/G$.

Proof. Let $\xi \in \overline{\mathcal{K}}/G \setminus (\mathcal{K}/G)$, and let $\psi(\xi) = (X, x, a) \in \mathcal{M}_{g,a}$ be the associated weighted pointed stable curve. We denote the set $\Theta_\xi = \psi^{-1}(\psi(\xi)) \subset \mathcal{K}$, where $\psi : \mathcal{K} \to \overline{\mathcal{K}}/G$ is the projection.

For any $\eta = (X, \iota, \bar{x}) \in \Theta_\xi \subset \mathcal{K}$, there is a smooth affine curve $\phi : 0 \to \bar{T}$ in $\mathcal{K}$ so that the pull back of the universal family of $\mathcal{K}$, say $\pi : (\mathcal{X}, L, s) \to \bar{T}$, contains $((X, \iota, \bar{x}), \bar{x})$ as its central fiber; $\phi(T \setminus \{0\}) \subset \mathcal{K}$, and the total space $\mathcal{X}$ is smooth.

By Proposition 6.2, the central fiber $(\bar{X}, \bar{x}, a)$ is weighted pointed quasi-stable (cf. Definition 6.1) and is a blow-up of $(X, x, a)$. Since $\mathcal{X}$ is smooth, there are integers $\{b_\alpha\}$ indexed by the irreducible components of $\bar{X}$, such that if we view $\bar{X}$ as divisor in $\mathcal{X}$, then

$$
\iota^* \mathcal{O}_\mathcal{X}(1) = \omega_{\mathcal{X}/\mathcal{T}}(a \cdot x)^{\otimes k} \Sigma b_\alpha \bar{X}_\alpha.
$$

Since the collection of blow-ups of $X$ coupled with integers $\{b_\alpha\}_{\alpha=1}$ is a discrete set, the choices of $(\mathcal{X}, L, \bar{x})$ are discrete. Thus $\{(\bar{X}, \bar{x}, a) \mid (X, x, a) \in \Theta_\xi\}$ is discrete. Finally, any two $(\bar{X}, \bar{x}, a)$ with isomorphic $(\mathcal{X}, \iota, \bar{x})$ lie in the same $G$-orbit. Thus $\Theta_\xi$ consists of a discrete collection of $G$-orbits. Hence $\psi^{-1}(\psi(\xi))$ is discrete. \qed

6.4. The coarse moduli space. We prove that $\overline{\mathcal{K}}/G$ is a coarse moduli space of weighted pointed stable curves, thus proving that $\psi$ is an isomorphism.

Proposition 6.13. Let $T$ be any scheme and $((\mathcal{X}, \mathfrak{r}, a))$ be a $T$-family of weighted pointed stable curves. Then there is a unique morphism $f : T \to \overline{\mathcal{K}}/G$, canonical under base changes, such that for any closed point $c \in T$, the image $\psi(f(c)) \in \mathcal{M}_{g,a}$ is the closed point associated to the weighted pointed stable curve $(\mathcal{X}, \mathfrak{r}, a)|_c$.

We define a subscheme $\mathcal{P} \subset \mathcal{H}$:

$$
\mathcal{P} = \{(X, \iota, x) \in \mathcal{H} \mid (X, x, a) \text{ weighted pointed stable curves, } \omega_X(a \cdot x)^{\otimes k} \cong \iota^* \mathcal{O}_\mathcal{X}(1)\}
$$

A direct check shows that $\mathcal{P}$ is a smooth, locally closed, and $G$-invariant subscheme of $\mathcal{H}$. We let $\mathcal{P} \subset \mathcal{P}$ be the open subset of $(X, \iota, x)$ such that $X$ are smooth. By definition, we have $\mathcal{P} = \mathcal{K}$.

Lemma 6.14. The composition $F : \mathcal{P} \to \mathcal{K} \to \overline{\mathcal{K}}/G$ extends to a unique morphism $\overline{F} : \overline{\mathcal{P}} \to \overline{\mathcal{K}}/G$.

Proof. Applying deformation theory of nodal curves, we know that $\mathcal{P}$ is dense in $\overline{\mathcal{P}}$. Let $\Gamma \subset \mathcal{P} \times \overline{\mathcal{K}}/G$ be the graph of the morphism $\overline{F}$ stated in the Lemma; we let

$$
\Gamma \subset \overline{\mathcal{P}} \times \overline{\mathcal{K}}/G
$$

be the closure of $\Gamma$. Let $p : \Gamma \to \overline{\mathcal{P}}$ be the projection. We claim that $p$ is bijective. Indeed, given $\xi = (X, \iota, x) \in \overline{\mathcal{P}}$, we let $(X, \mathfrak{O}_X(1), \mathfrak{r})$ be the family given by Proposition 6.7, which
shows that $\xi \in p(\Gamma)$. This proves that $p$ is surjective. On the other hand, repeating the proof of Lemma 6.12, we see that $p$ is one-to-one. This proves that $p$ is bijective.

Next, we claim that $p$ is an isomorphism. Since $\overline{\mathcal{P}}$ is smooth, $\mathcal{P} \subset \overline{\mathcal{P}}$ is dense, and $\Gamma$ is isomorphic to $\mathcal{P}$, we conclude that $\Gamma$ is reduced. Then since $p : \Gamma \to \overline{\mathcal{P}}$ is birational, a homeomorphism and $\overline{\mathcal{P}}$ is smooth, $p$ must be étale. Thus $p$ is an isomorphism. Finally, by composing the isomorphism $p^{-1}$ with the projection to the second factor of $\overline{\mathcal{P}} \to \overline{\mathcal{C}}//G$, we obtain the desired extension $\overline{F}$ of $F$.

Proof of Proposition 6.13. We cover $T$ by a collection of affine open $\{T_a\}_{a \in A}$. Let $\pi_a : X_a \to T_a$ with sections $r_{a,i} : T_a \to X_a$ be the restriction of $r_i$ to $T_a$ of the family on $T$. By fixing a trivialization $(\pi_a)_* \omega_{X_a/T_a}(a \cdot r_a)^{\otimes k} \cong \Omega_{T_a}^{(m+1)}$, we obtain morphisms $f_a : T_a \to \overline{\mathcal{P}}$. Composed with the morphism $\overline{F}$ constructed in the previous Lemma, we obtain $\overline{F} \circ f_a : T_a \to \overline{\mathcal{C}}//G$.

Since the choice of the trivializations does not alter the morphism $\overline{F} \circ f_a$, this collection $\{\overline{F} \circ f_a\}_{a \in A}$ patches to a morphism $T \to \overline{\mathcal{C}}//G$. This proves the first part of Proposition 6.13.

Finally, that $\psi(f(e))$ is the point associated to the weighted pointed curve $(X', r, a)|_e$ follows from the construction.

Proof of Theorem 6.5. It follows from Proposition 6.7, 6.13, and Lemma 6.12.

We remark that for convenience, in the proof given above we use the existence of the coarse moduli space $\overline{M}_{g,n}$ constructed by Hassett. A modification of the argument should give an independent GIT construction of it.

For completeness, we give a complete description of polystable points in $C^{ss}$, generalizing the case $x = \emptyset$ proved in [3]. Let the exceptional set $E(X) \subset X$ be the union of exceptional components of $(X, \mathcal{O}_X(1), x, a)$.

Definition 6.15 ([3] when $x = \emptyset$). We call $(X, \mathcal{O}_X(1), x, a)$ extremal if each proper subcurve $Y \subset X$ satisfying $\delta_Y(\mathcal{O}_X(1)) = M_Y^{SS}$ (cf. (6.12)) has $L_Y = Y \cap Y^C \subset E(X)$.

Recall that Chow $(X, x) \in C^{ss}$ is polystable if the $G$-orbit $G \cdot \text{Chow}(X, x)$ is closed in $C^{ss}$. Here is an equivalent characterization of polystable points.

Lemma 6.16. Let $G$ be a reductive group and $(Z, \mathcal{O}_Z(1))$ be a $G$-polarized projective scheme. Then a semistable point $z \in Z^{ss}$ is polystable if and only if for any $1$-PS $\lambda$ either the $\lambda$-weight of $z$ is $> 0$ or $\lim_{t \to 0} \lambda(t) \cdot z \in G \cdot z$.

Proof. Without loss of generality, we may assume that $Z = \mathbb{P} W$ for a $k$-vector space $W$. Let $\lambda : \mathbb{G}_m \to G$ be any $1$-PS, with $W = \bigoplus_{i \in Z} W_i$ its weight decomposition such that $\lambda$ acts on $W_i$ by multiplying by $t^i$. Let $z \in \mathbb{P} W$, and let $0 \neq \hat{z} \in W$ be a lift of $z$, with associated decomposition $\hat{z} = \oplus_i \hat{z}_i$, $\hat{z}_i \in W_i^\vee$. Then the $\lambda$-weight of $\hat{z}$ is $\omega_\lambda(\hat{z}) = \max(-i | \hat{z}_i \neq 0)$.

Suppose $z$ is polystable, then $G \cdot \hat{z}$ is closed in $W^\vee$ (cf. [8]). Suppose $\omega_\lambda(\hat{z}) = 0$, we have $0 \neq \hat{z}_0 = \lim_{t \to 0} \lambda(t) \cdot \hat{z} \in G \cdot \hat{z} = G \cdot \hat{z}$. Thus $z_0 \in G \cdot z$, and this verifies one direction of the Lemma.

Conversely, suppose $z$ is semistable but not polystable, then there are a $1$-PS $\lambda$ and $g \in G$ such that $\lim_{t \to 0} \lambda(t) \cdot (g \cdot z) = z_0$, $z_0$ is polystable and $z_0 \notin G \cdot z$. In particular, since $\lambda^g = (\lambda \cdot g)^{-1} \cdot \lambda$ fixes $z$, the $\lambda^g$-weight of $z$ is $0$ while $\lim_{t \to 0} \lambda^g(t) \cdot z \notin G \cdot z$. This proves the Lemma.
We characterize curves having positive dimensional stabilizers.

**Lemma 6.17.** Given \( g, n \) and \( a \in \mathbb{Q}_+^n \) so that \( \chi_{g,a} > 0 \), let \( (X, \mathcal{O}_X(1), x, a) \) be a genus \( g \) semistable polarized weighted pointed nodal curve such that \( (X, x) \subset \mathbb{P}W \) is invariant under a 1-PS \( \lambda \) and \( \deg X \geq M \), the constant given in Theorem 1.5. Suppose under its diagonalizing basis, \( \lambda \) has \( k \) weights. Then there are \( k \) mutually disjoint subcurves \( Y_1, \ldots, Y_k \subset X \) such that

1. the complement \( \bigcup_{i=1}^k Y_i^c \) is a union of exceptional components of \( X \), and
2. each \( Y_i \) has \( \delta_Y(\mathcal{O}_X(1)) = M^{d-} \), where \( d = \deg X \).

**Proof.** Let \( Y \subset X \) be the union of irreducible components of \( X \) that are fixed by \( \lambda \); we let \( E = Y^c \). As \( E \subset \mathbb{P}W \) is \( \lambda \)-invariant but not fixed, it is a union of \( \mathbb{P}^1 \)s. Since \( (X, \mathcal{O}_X(1), x, a) \) is semistable, by Theorem 1.5, Lemma 5.2 and the assumption \( \deg X \geq M \), \( E \) is a union of exceptional components.

By the assumption that \( \lambda \) has \( k \) weights, we have the weight decomposition \( W = \bigoplus_{i=1}^k W_i \). We let \( Y_i = Y \cap \mathbb{P}W_i \). Because \( Y \) is fixed by \( \lambda \), we have \( Y = \bigcup_{i=1}^k Y_i \) is a disjoint union. Since \( X = Y \cup E \) is connected, and since \( E \) is a union of exceptional components, the first part of the Lemma follows.

For the second part, for each \( Y_i \) let \( \lambda_i \) be the 1-PS that acts on \( W_i \) via multiplying \( t \) and fixes \( \bigcup_{j \neq i} Y_j \). Under such \( \lambda_i \), \( (X, x) \subset \mathbb{P}W \) is invariant, thus by semistability the \( \lambda_i \)-weight of \( (X, \mathcal{O}_X(1), x, a) \) is 0, which is equivalent to \( \delta_{Y_i}(\mathcal{O}_X(1)) = M^{d-} \). This proves the Lemma.

**Proposition 6.18.** Given \( g, n \) and \( a \in \mathbb{Q}_+^n \) so that \( \chi_{g,a} > 0 \), then a stable polarized weighted pointed nodal curve \( (X, \mathcal{O}_X(1), x, a) \) of \( d \geq M \) (the constant given in Theorem 1.5) is polystable if and only if it is extremal.

**Proof.** Suppose Chow \( (X, x) \) is polystable and \( Y \subset X \) such that \( \delta_Y(\mathcal{O}_X(1)) = M^{d-} \). We pick a decomposition \( W_0 \oplus W_1 = W = H^0(X, \mathcal{O}_X(1)) \) so that \( Y = X \cap \mathbb{P}W_0 \), which is possible for \( d \geq M \). We pick a 1-PS \( \lambda \) so that it acts on \( W_0 \) (resp. \( W_1 \)) via multiplication by \( t \) (reps. by 1). Since \( \delta_Y(\mathcal{O}_X(1)) = M^{d-} \), the \( \lambda \)-weight of Chow \( (X, x) \) is 0. By Lemma 6.16, \( (X', x') = \lim_{t \to 0} \lambda(t) \cdot (X, x) \) lies in the \( G \)-orbit of \( (X, x) \). Further, since \( \lambda \) leaves \( (X', x') \) invariant and Chow \( (X', x') \) is semistable, by Lemma 6.17 we have \( L_Y \subset E(X') \), which is equivalent to \( L_Y \subset E(X) \). This proves the sufficient part of the Proposition.

Conversely, suppose \( \xi \) is semistable but not polystable. Then there is a 1-PS \( \lambda \) so that \( \lim_{t \to 0} \lambda(t) \cdot \xi = \xi' \) is polystable. Let \( X' \to \mathcal{X} \) be the total space of this family of curves. Since \( \mathcal{X} \times_{\mathcal{X}} \mathcal{X} = \mathcal{X} \) is a constant family, the special fiber will be a “blow-up” of the general fibers. Because \( \xi' = (X', \mathcal{O}_X'(1), x', a) \) is polystable and is invariant under \( \lambda \), we have the decomposition \( X' = \bigcup_{i=1}^k Y_i' \cup \bigcup_{j \neq j'} E_{ij} \), where \( E_{ij} \) is the union of exceptional components in \( (\bigcup Y_i')^c \) that intersects with both \( Y_i' \) and \( Y_j' \), given by Lemma 6.17. Since \( X' \) is a “blow-down” of \( X' \), and \( \lambda \) does not fixes \( (X, x) \subset \mathbb{P}W \), the blow-down map \( X' \to X \) must contract at least one exceptional component, say, in \( E_{ij} \). Suppose \( j < j' \). We let \( Y \subset X \) be the image of \( \bigcup_{i=1}^k Y_i' \) under \( X' \to X \). Then it is direct to check that \( Y \subset X, \delta_Y(\mathcal{O}_X(1)) = M^{d-} \), and \( L_Y \not\subset E(X) \). This proves the Proposition.

7. **K-stability of nodal curve**

In this section, we apply Theorem 1.5 to study the \( K \)-stability of polarized nodal curves.

**Theorem 7.1.** For a polarized connected nodal curve \( (X, \mathcal{O}_X(1)) \) the following statements are equivalent:

- **Statement 1:** \( (X, \mathcal{O}_X(1)) \) is polystable.
- **Statement 2:** \( (X, \mathcal{O}_X(1)) \) is semistable.
- **Statement 3:** \( (X, \mathcal{O}_X(1)) \) is K-polystable.
- **Statement 4:** \( (X, \mathcal{O}_X(1)) \) is K-semistable.
(1) \((X, \mathcal{O}_X(1))\) is \(K\)-stable;
(2) \((X, \mathcal{O}_X(1))\) is \(K\)-semistable;
(3) \(\mathcal{O}_X(1)\) is numerically proportional to \(\omega_X\).

One direction of the theorem is proved by Odaka who in [19] proved that a nodal curve \(X\) polarized by \(\omega_X^{\otimes k}\) is \(K\)-stable for a \(k \in \mathbb{N}\). He used birational geometry and a weight formula proved by himself and by the second named author independently [28]. He also informed us that he can generalize his method to prove the stated theorem.

7.1. \(K\)-stability of curves. We recall the notion of \(K\)-stability of polarized curves. (See [22, Sect. 3] and [26] for general case.)

Definition 7.2. A test configuration for a polarized curve \((X, \mathcal{O}_X(1))\) consists of a \(\mathbb{G}_m\)-equivariant flat projective morphism \(\pi : \mathcal{X} \to \mathbb{A}^1\), and a \(\mathbb{G}_m\)-linearized \(\pi\)-relative very ample line bundle \(\mathcal{L}\), where \(\mathbb{G}_m\) acts on \(\mathbb{A}^1\) via multiplication, such that for any \(t \neq 0 \in \mathbb{A}^1\), \((\mathcal{X}, \mathcal{L}) \times_{\mathbb{A}^1} \{t\} \cong (X, \mathcal{O}_X(1))\).

For a closed subset \(\Sigma \subset X_0\), we call it a trivial configuration away from \(\Sigma\) if there is a closed subset \(\Sigma_0 \subset X\) such that there is a \(\mathbb{G}_m\)-equivariant isomorphism \(\mathcal{X} - \Sigma \cong X \times \mathbb{A}^1 - \Sigma_0 \times \{0\}\), that the line bundle \(\mathcal{L}|_{\mathcal{X} - \Sigma}\) is the pullback of a line bundle on \(X\), and \(\mathbb{G}_m\)-action on \(X \times \mathbb{A}^1\) is the product action that acts trivially on \(X\). When \(\Sigma \subset \mathcal{X}\) has codimension at least 2, we say \((\mathcal{X}, \mathcal{L})\) is trivial in codimension 2.

Given a test configuration \((\mathcal{X}, \mathcal{L})\) for a polarized curve \((X, \mathcal{O}_X(1))\) as above, we let \(w(l)\) be the weight of the induced \(\mathbb{G}_m\)-action on \(\wedge^{1\text{top}} \pi_* \mathcal{L}^{\otimes l}\). By Riemann-Roch, \(w(l) = a_2 l^2 + a_1 l + a_0\) is quadratic in \(l\) (for \(l \gg 1\)). We expand the following quotient in \(l^{-1}\):

\[
\frac{w(l)}{l \cdot \chi(\mathcal{O}_X(l))} = e_0 + e_{-1} l^{-1} + \ldots
\]

Using \(\chi(\mathcal{O}_X(l)) = b_1 l + b_0\), the Donaldson-Futaki invariant of the test configuration \((\mathcal{X}, \mathcal{L})\) of \((X, \mathcal{O}_X(1))\) is defined to be

\[
\text{DF}(\mathcal{X}, \mathcal{L}) = e_{-1} = -\frac{a_2 b_0 - a_1 b_1}{b_1^2}.
\]

Remark 7.3. Let \((\mathcal{X}, \mathcal{L})\) be a test configuration for \((X, \mathcal{O}_X(1))\), then the \(\mathbb{G}_m\)-linearization of \(\mathcal{L}\) induces a \(\mathbb{G}_m\)-linearization of \(\mathcal{L}^{\otimes l}\), which makes \((\mathcal{X}, \mathcal{L}^{\otimes l})\) a test configuration for \((X, \mathcal{O}_X(l))\), with \(\text{DF}(\mathcal{X}, \mathcal{L}^{\otimes l}) = \text{DF}(\mathcal{X}, \mathcal{L})\).

Definition 7.4. A polarized nodal curve \((X, \mathcal{O}_X(1))\) is \(K\)-stable (resp. \(K\)-semistable) if \(\text{DF}(\mathcal{X}, \mathcal{L}) < 0\) (resp. \(\leq 0\)) for any test configuration \((\mathcal{X}, \mathcal{L})\) of \((X, \mathcal{O}_X(1))\) that is non-trivial in codimension 2.

7.2. Proof of the Main Result. For \((X, \mathcal{O}_X(1))\) and integer \(k\), we let \(W^\vee_{(k)} = H^0(\mathcal{O}_X(k))\) with \(X \subset \mathbb{P}W_{(k)}\) the tautological embedding. Then given any 1-PS subgroup \(\lambda\) of \(\text{Aut} \mathbb{P}W_{(k)}\), the \(\mathbb{G}_m\)-orbit of \(X\) in \(\mathbb{P}W_{(k)} \times \mathbb{A}^1\) via the diagonal action produces a test configuration of \((X, \mathcal{O}_X(k))\); we denote such test configuration by \((\mathcal{X}_{\lambda}, \mathcal{L}_{\lambda})\). Conversely, any test configuration of \((X, \mathcal{O}_X(k))\) can be constructed from a 1-PS of \(\text{Aut} \mathbb{P}W_{(k)}\) (cf. [22, Prop. 3.7]). Thus to prove the \(K\)-stability of \((X, \mathcal{O}_X(1))\), it suffices to show that when \((\mathcal{X}_{\lambda}, \mathcal{L}_{\lambda})\) is non-trivial in codimension 2, the Donaldson-Futaki invariant \(\text{DF}(\mathcal{X}_{\lambda}, \mathcal{L}_{\lambda}) < 0\) for sufficiently large \(k\) and all 1-PS \(\lambda\) of \(\text{Aut} \mathbb{P}W_{(k)}\).

In the following, for notational simplicity, we replace \((X, \mathcal{O}_X(1))\) by \((X, \mathcal{O}_X(k))\), and say that \(\mathcal{O}_X(1)\) is sufficiently ample instead of saying \(k\) being sufficiently large. This way, we only need to study test configuration \((\mathcal{X}_{\lambda}, \mathcal{L}_{\lambda})\) for any 1-PS \(\lambda\) of \(\text{Aut} \mathbb{P}W\), assuming
Lemma 7.6. \(\mathcal{O}_X(1)\) is sufficiently ample. To proceed, we first relate \(\text{DF}(X, \mathcal{O}_X(1))\) to the Chow weights of \((X, \mathcal{O}_X(1))\). We pick a \(\lambda\)-diagonalizing basis \(s = \{s_0, \ldots, s_m\}\) of \(W^\vee\) and represent \(\lambda\) as a 1-PS of \(GL(W^\vee)\) of the form

\[
\lambda(t) := \text{diag}[\rho_0^t, \ldots, \rho_m^t], \quad \rho_0 \geq \rho_1 \geq \cdots \geq \rho_m = 0, \quad \rho_i \in \mathbb{Z}.
\]

By Remark 7.3, replacing \(\mathcal{L}\) by \(\mathcal{L} \otimes \mathcal{O}_X(1)\), the test configuration \((X, \mathcal{L})\) introduces a test configuration \((X, \mathcal{L} \otimes \mathcal{O}_X(1))\), which by [22, Prop. 3.7] is induced by a 1-PS \(\lambda_t\) of \( Aut \mathbb{P}W_{(l)} \). We now construct explicitly such \(\lambda_t\). Since \(\mathcal{O}_X(1)\) is a sufficiently high multiple of an ample line bundle, the tautological

\[
\phi_t : S^l W^\vee \longrightarrow W^\vee_{(l)} = H^0(\mathcal{O}_X(l))
\]

is surjective. We fix our convention. For \(I = (i_0, \ldots, i_m)\), we denote 
\(s^I = s_0^{i_0} \cdots s_m^{i_m}\), which has weight \(\rho(I) = \sum \rho_j \cdot i_j\) under the induced \(\lambda\) action on \(S^l W^\vee\).

We let \(\mathfrak{S}_l\) be the set of monomials in \(S^l W^\vee\). We order \(\mathfrak{S}_l\) as follows: We define \(s^I > s^{I'}\) when either \(\rho(I) < \rho(I')\), or \(\rho(I) = \rho(I')\) and there is a \(0 \leq j_0 \leq m\) such that \(i_j = i'_j\) for all \(j > j_0\) and \(i_{j_0} > i'_{j_0}\). Thus, the set \(\{t^m s_i\}_{i \in \mathfrak{S}_l}\) is ordered increasingly as \(t^m s_0, \ldots, t^m s_m\).

We pick a basis of \(W^\vee_{(l)}\), which will be a diagonalizing basis for \(\lambda_t\). Let \(m_l+1 = \text{dim} W^\vee_{(l)}\) and set \(s_{l,m_l} = s^l\), with weight \(\varrho_{l,m_l} = l \cdot \rho_m\). Suppose for an integer \(0 \leq k < m_l\), we have picked \(s_{l,k+1}, \ldots, s_{l,m_l}\) and their weights \(\varrho_{l,j}\); let \(T_{l,k+1}\) be the linear span of \(\{s_{l,k+1}, \ldots, s_{l,m_l}\}\) and let \(s^{l,k}\) be the largest element in

\[
\{ s^I \in \mathfrak{S}_l | \varrho_I(s^I) \not\in \varrho_I(T_{l,k+1}) \}.
\]

We set \(s_l = \phi_t(s^{l,k})\) and define \(\varrho_{l,k} = \rho(I_k)\) to be the weight of \(s_{l,k}\). Then \(s_{l,0}, \ldots, s_{l,m_l}\) form a basis of \(W^\vee_{(l)}\). We let \(\lambda_t\) be the 1-PS of \( Aut \mathbb{P}W_{(l)} \) with diagonalizing basis \(\{s_{l,0}, \ldots, s_{l,m_l}\}\) and weights

\[
\lambda_t(s_{l,k}) = t^{\varrho_{l,k}} s_{l,k}.
\]

Note that for \(l = 1\), \(s_{1,k} = t^k s_k\) and \(\lambda_1 = \lambda\).

**Lemma 7.5.** Let \((X, \mathcal{L}_X)\) be the test configuration of \(\lambda_t\). Then we have isomorphism of test configurations \((X, \mathcal{L}_X) \cong (X, \mathcal{L}_X^\otimes t)\).

**Proof.** We let \(R_{l,k}\) be the \(k[t]\)-submodule of \(H^0(\mathcal{O}_X(kl)) \otimes k[t]\) generated by monomials of degree \(k\) in elements in \(\{t^{\varrho_{l,k}} s_i\}\), and let \(R_l = \oplus_{k \geq 0} R_{l,k}\), where \(R_{l,0} = k[t]\). Clearly, \(R_l\) is a graded \(k[t]\)-algebra, and is generated by \(R_{l,1}\). Following [16, Page 28], we have

\[
X_{\lambda} = \text{Proj}_k R_l \subset \mathbb{P}W^\vee_{(l)} \times k^1.
\]

For \(l = 1\), we have \(X_{\lambda} = \text{Proj}_k R_1 \subset \mathbb{P}W \times k^1\).

We claim that for any \(k \geq 1\), \(R_{l,k} = R_{l,1,k} \subset H^0(\mathcal{O}(kl)) \otimes k[t]\). Indeed, by definition, we have \(R_{l,1,k} = R_{l,1,t}\). Since \(R_{l,k}\) is generated by \(R_{l,1}\) and \(R_{l,1}\) is generated by \(R_{l,1,t}\), we conclude that \(R_{l,k} = R_{l,1,k}\) as \(k[t]\)-submodules of \(H^0(\mathcal{O}(kl)) \otimes k[t]\). Consequently, they induce a homomorphism of graded \(k[t]\)-algebra \(R_1 \to R_{l,1}\), which induces a \(\mathfrak{S}_m\)-equivariant isomorphism \((X, \mathcal{L}_X^\otimes t) \cong (X, \mathcal{L}_X, \lambda_{l,t})\). This proves the Lemma.

**Lemma 7.6.** Let the notation be as before. Then

\[
\lim_{l \to \infty} l^{-1} \cdot \omega(\lambda_l) = -b_1^{-1} \cdot \text{DF}(X, \mathcal{L}_X) < \infty.
\]
Proof. It follows from Lemma 7.5 that \( \omega(\lambda_l) \) is the Chow weight for the test configuration \((X_\lambda, \mathcal{L}_\lambda^\mathfrak{a})\). By [22, Theorem 3.9], we know that this Chow weight \( \omega(\lambda_l) \) is a linear function of the form \(-b_1^{-1} \text{DF}(X_\lambda, \mathcal{L}_\lambda) \cdot l + \text{constant} \). Dividing by \( l \) and taking limit, we complete the proof of the Lemma.

Thus to prove \( \text{DF}(X_\lambda, \mathcal{L}_\lambda) < 0 \), it suffices to show that

\[
\lim_{l \to \infty} l^{-1} \cdot \omega(\lambda_l) > 0.
\]

Proof of Theorem 7.1. We will prove the Theorem in the following order:

\[ (3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3). \]

Since the middle arrow trivially follows from Definition 7.4, we only need to establish the 1st and 3rd arrows.

Suppose \( X \) is a stable (nodal) curve and \( \mathcal{O}_X(1) \) is numerically proportional to \( \omega_X \), then \((X, \mathcal{O}_X(1))\) is slope stable. We will show in this case that for any 1-PS \( \lambda \subset SL(W) \) we have \( \text{DF}(X_\lambda, \mathcal{L}_\lambda) < 0 \) unless \((X_\lambda, \mathcal{L}_\lambda)\) is trivial in codimension 2.

We divide the study into two cases. The first case is when \( e(\mathcal{O}(\lambda)) = 0 \). In this case, we claim that there is a \( 0 < i_0 < m \) such that \( q_{i_0} = 0 \) and \( \bigcap_{k > i_0} \{ s_k = 0 \} = \emptyset \). Indeed, let \( i_0 \) be the smallest index such that \( \rho_{i_0} = 0 \). Suppose \( q \in \bigcap_{k > i_0} \{ s_k = 0 \} \neq \emptyset \), then we have \( \rho_{\alpha(q)} > 0 \) and \( \Delta_q \neq 0 \), hence \( e(\mathcal{O}(\lambda)) = 0 \). By Corollary 2.8 and Lemma 2.4 we obtain \( e(\mathcal{O}(\lambda)) > 0 \), a contradiction. Therefore we have \( \bigcap_{k > i_0} \{ s_k = 0 \} = \emptyset \), and then \( i_0 < m \). This proves the claim. To continue, we quote a result of Stoppa ([25, page 1405-1406], [26]) that in this case either the test configuration \((X_\lambda, \mathcal{L}_\lambda)\) of \( \lambda \) is trivial in codimension two or \( \text{DF}(X_\lambda, \mathcal{L}_\lambda) < 0 \). This proves the theorem in this case.

The other case is when \( e(\mathcal{O}(\lambda)) > 0 \). We let \( \lambda'_l \) be the staircase constructed from \( \lambda_l \) using Proposition 3.5, and with weights \( \hat{g}_{l,i} = g_{l,i} \). We let \( \hat{g}_{l,i} \) be the shifted weights according to the rule (5.14) applied to \( \lambda'_l \); namely, \( \hat{g}_{l,i} = \min_i \{ g_{l,i} - \hat{g}_{l,i}(\lambda'_l) \mid i \in I(\lambda'_l) \} \). Since \((X, \mathcal{O}_X(1))\) is slope stable, applying Theorem 5.10 and Proposition 3.5, we can find an \( \epsilon > 0 \) so that for \( l \) sufficiently large,

\[
(7.6) \quad l^{-1} \cdot \omega(\lambda_l) \geq l^{-1} \cdot \omega(\lambda'_l) \geq \frac{1}{\deg X + l^{-1}(1-g_X)} \cdot \frac{\epsilon}{t^2} \cdot \sum_{i=0}^{m_l} \hat{g}_{l,i}.
\]

We state a sublemma which we will prove shortly.

Sublemma 7.7. Suppose \( e(\mathcal{O}(\lambda)) > 0 \). Then \( \lim_{l \to \infty} \frac{1}{t^2} \cdot \sum_{i=0}^{m_l} \hat{g}_{l,i} > 0 \).

Applying Sublemma 7.7, we obtain \( l^{-1} \cdot \omega(\lambda_l) > 0 \), which by Lemma 7.6 is equivalent to \( \text{DF}(X_\lambda, \mathcal{L}_\lambda) < 0 \). Since \( \lambda \) is arbitrary, we conclude that \((X, \mathcal{O}_X(1))\) is K-stable. This proves one direction of the theorem.

Conversely, suppose \((X, \mathcal{O}_X(1))\) is K-semistable. Since \((X, \mathcal{O}_X(1))\) being K-semistable is equivalent to \((X, \mathcal{O}_X(k))\) being K-stable for all large \( k \) (cf. Remark 7.3), without loss of generality we assume that \( \mathcal{O}_X(1) \) is sufficiently ample. In particular, this implies that \( X \subset \mathbb{P}^W \) contains no line. We claim that \((X, \mathcal{O}_X(1))\) satisfies (1.6) with \( a = 0 \). Suppose not, then there is a subcurve \( Y \subset X \) destabilizing \((X, \mathcal{O}_X(1))\), that is,

\[
(7.7) \quad \frac{\deg Y \cdot \omega_X}{\deg \omega_X} \cdot \deg \mathcal{O}_X(1) - \deg \mathcal{O}_Y(1) - \frac{\ell_Y}{2} > 0.
\]

Let \( W_Y = H^0(\mathcal{O}_Y(1))^{\vee} \subset W = H^0(\mathcal{O}_X(1))^{\vee} \), which is the linear subspace spanned by \( Y \); let \( m_0 + 1 = \dim H^0(\mathcal{O}_Y(1)) \). We choose a two-weight 1-PS \( \lambda \) as in the proof of Theorem 1.5 (at the end of Section 5) so that \( \lambda \) acts with weight 1 on \( W_Y \subset W \) and acts with weight
0 on a complement \( W_Y^1 \) of \( W_Y \subset W \). Let \((X_\lambda, L_\lambda)\) be the test configuration associated to \( \lambda \). We now evaluate

\[
\frac{\text{DF}(X_\lambda, L_\lambda)}{b_1} = \lim_{l \to \infty} -\frac{\omega(\lambda)}{l} = -\lim_{l \to \infty} \frac{1}{l} \left( \frac{2l \deg X \sum_{i=0}^{m_l} g_{l,i} - e(\beta(\lambda))}{l \deg X + 1 - g} \right).
\]

To evaluate this term, we identify the central fiber \((X_\lambda)_0\) of \( X_\lambda \). As \( O_X(1) \) is sufficiently ample, \( X \subset \mathbb{P} W \) contains no line. Further, as \( \lambda \) is a two-weight 1-PS, and the weight one eigenspace is \( W_Y \), we see that \((X_\lambda)_0 = Y \cup E \cup Y' \), derived by inserting \( \ell_Y \)-lines (whose union is \( E \)) into \( X \) at the nodes \( Y \cap Y' \), and \( Y' \subset \mathbb{P} W_Y^1 \) is isomorphic to \( Y^0 \subset X \), because \( O_X(1) \) is sufficiently ample. Consequently,

\[
H^0(L_\lambda^\otimes l|(X_\lambda)_0) = H^0(O_Y(1)) \oplus H^0(O_E(l)(-(Y \cup Y') \cap E)) \oplus H^0(O_Y(l)),
\]

and elements in \( H^0(O_X(l)|Y) \) (resp. \( H^0(O_E(l)(-(Y \cup Y') \cap E)) \); resp. \( H^0(O_Y(l)) \)) have weights \( l \) (resp. \( l-1, \ldots, 0 \); resp. \( 0 \)). Therefore

\[
\sum g_{l,i} = h^0(O_Y(l)) \cdot l + \ell_Y \cdot \frac{ll(l-1)}{2} = (\deg O_Y(1) + \ell_Y) \cdot l^2 + (1 - g(Y) - \ell_Y/2) \cdot l.
\]

By Definition 2.4 and Lemma 7.5, \( \sum_{i=0}^{m_l} g_{l,i} = e(\beta(\lambda)) \cdot \ell_Y/2 + O(l) \) is the weight of the \( \mathbb{G}_m \)-action on \( \lambda^{\text{top}}(H^0(L_\lambda^\otimes l)/tH^0(L_\lambda^\otimes l)) \). Thus

\[
e(\beta(\lambda)) = \ell_Y e(\beta(\lambda)) = 2\ell_Y^2 (\deg Y + \ell_Y/2).
\]

Combining and simplifying by using \( \deg \omega_X = g - 1 \), etc., we obtain

\[
\frac{\text{DF}(X_\lambda, L_\lambda)}{b_1} = -\lim_{l \to \infty} \frac{1}{l} \left( \frac{2l \deg X \sum_{i=0}^{m_l} g_{l,i} - e(\beta(\lambda))}{l \deg X + 1 - g} \right) = -\lim_{l \to \infty} \frac{1}{l} \left( \frac{2l^2(g - 1) \deg Y + \ell_Y/2 - l^2 \deg X \omega_X}{l \deg X + 1 - g} \right) = \frac{g - 1}{\deg X} \left( \frac{\deg Y \omega_X}{\deg X} - \deg X - \deg Y - \ell_Y/2 \right).
\]

Since \( Y \subset X \) is destabilizing, by (7.7) we have \( \text{DF}(X_\lambda, L_\lambda) > 0 \), contradicting \((X, O_X(1))\) being \( K \)-semistable. This proves that \((X, O_X(1))\) satisfies (1.6) with \( a = 0 \). In particular, we obtain that \((X, O_X(l))\) satisfies (1.6) for all large \( l \) since \((X, O_X(l))\) is also \( K \)-semistable for any \( l > 0 \). This forces \( O_X(1) \) to be numerically proportional to \( \omega_X \). This proves the other direction of the theorem. \( \square \)

It remains to prove Sublemma 7.7. We introduce a few more notations. Following the discussion in Section 2, we define (for \( q \in X \) and \( \tilde{s}_{l,k} \) the lift of \( s_{l,k} \) to the normalization \( \tilde{X} \) of \( X \))

\[
h_\alpha(\lambda_l) = \min\{i \mid \tilde{s}_{l,i+1}|_{\tilde{X}_\alpha} = 0\}, \quad h(\lambda_l, q) = \max\{i \mid q(\tilde{s}_{l,i}, q) \neq \infty\},
\]

and

\[
\hat{\Lambda}_\alpha(\lambda_l) = \{ p \in \tilde{X}_\alpha \mid \tilde{s}_{l,h_\alpha(\lambda_l)}(p) = 0 \}, \quad \hat{\Lambda}(\lambda_l) = \bigcup_{\alpha} \hat{\Lambda}_\alpha(\lambda_l).
\]

(Here \( r \) is the number of irreducible components of \( X \).) We claim that

\[
\hat{\Lambda}(\lambda_l) = \hat{\Lambda}(\lambda) \quad \text{and} \quad e(\beta(\lambda_l)) = \text{n.l.c.} \chi(O_{X_{\lambda_l}}(k)/3(\lambda)^k).
\]

Indeed, by our choice of the basis \( \{s_{l,0}, \cdots, s_{l,m_l}\} \), we have \( \tilde{s}_{l,h_\alpha(\lambda_l)} = \tilde{s}_{h_\alpha}^l \) and \( g_{l,h_\alpha(\lambda_l)} = l \rho_{h_\alpha} \) for all \( X_\alpha \subset X \), from which we deduce

\[
\hat{\Lambda}_\alpha(\lambda_l) = (\tilde{s}_{l,h_\alpha(\lambda_l)})^{-1}(0) = (\tilde{s}_{h_\alpha}^l)^{-1}(0) = \hat{\Lambda}_\alpha(\lambda) \subset \tilde{X}_\alpha
\]
and hence $\tilde{\lambda}(\lambda_i) = \tilde{\lambda}(\lambda)$ by Definition 3.1. Also by the construction of $\lambda_i$, we have the middle identity

$$\text{(7.12)} \quad (t^{\Theta_{\mathfrak{p}_0}s_{l,0}, \ldots, t^{\Theta_{\mathfrak{p}_m}s_{l,m}}}) = \mathcal{I}(\lambda_i) = \mathcal{I}(\lambda)' = (t^{\Theta_{\mathfrak{p}_0}s_0, \ldots, t^{\Theta_{\mathfrak{p}_m}s_m}) \subset \mathcal{O}_{X \times \mathcal{H}}(l),$$

where the first and the third are by the definition. This implies the second part of (7.11), hence our claim. Furthermore, (7.12) together with (2.11), (2.12) and Lemma 2.6, actually imply $\Delta_q(\lambda_i) = l \cdot \Delta_q(\lambda)$ for each $q \in \tilde{\lambda}(\lambda)$.

With those in hand, we conclude that, for $\lambda_j$, the staircase 1-PS obtained from $\lambda_i$ by applying Proposition 3.5, we have (1) $\tilde{\lambda}(\lambda_i) = \tilde{\lambda}(\lambda_j)$, and for each $q \in \tilde{\lambda}(\lambda_i)$, $w(\tilde{\mathcal{I}}(\lambda_i), q) = w(\tilde{\mathcal{I}}(\lambda_j), q)$; and (2) for each $q \in \tilde{\lambda}(\lambda_i)$, $\Delta_q(\lambda_i) = l \cdot \Delta_q(\lambda) \subset \Delta_q(\lambda_j)$.

**Proof of Sublemma 7.7.** We first prove when $\rho_{h_\alpha} = 0$ for all irreducible components $X_\alpha$, then the sublemma holds. Indeed, applying [16, Prop. 2.11], we have

$$\sum_{i=0}^{m_i} \varrho_{l,i} = e(\mathcal{I}(\lambda)) \cdot \frac{l^2}{2} + a_1 \cdot l + a_2, \quad a_i \text{ depending only on } \lambda.$$

Since all $\rho_{h_\alpha} = 0$, we have $\varrho_{l,i} = \hat{\varrho}_{l,i}$. Therefore,

$$\liminf_{l \to \infty} \frac{1}{l^2} \sum_{i=0}^{m_i} \hat{\varrho}_{l,i} = \liminf_{l \to \infty} \frac{1}{l^2} \sum_{i=0}^{m_i} \varrho_{l,i} = \frac{e(\mathcal{I}(\lambda))}{2} > 0.$$

We now prove the general case. We claim that there is an irreducible component $X_\beta$ and a $q \in \bar{X}_\beta$ so that

$$\text{(7.14)} \quad |\Delta_q(\lambda)| - \rho_{h_\alpha}(\lambda) \cdot w(\tilde{\mathcal{I}}(\lambda), q) > 0.$$

Suppose not. Since the $\geq$ for (7.14) always holds, we will have $\varrho_{l,i} = \varrho_{h_\alpha}$ for every $i \in \mathcal{I}_\alpha$. Because of the prior discussion, we must have an $X_\alpha$ such that $\varrho_{h_\alpha} > 0$. Since $X$ is connected, we can find a pair $X_\alpha \neq X_\beta$ so that $X_\alpha \cap X_\beta \neq \emptyset$ and $\rho_{h_\alpha}(\lambda) > \rho_{h_\beta}(\lambda) = 0$.

Let $\pi : \bar{X} \to X$ be the projection and let $q \in \bar{X}_\beta \cap \pi^{-1}(X_\alpha \cap X_\beta)$. We claim that the pair $(\beta, q)$ satisfies the inequality (7.14). Since $\pi(q) \in X_\alpha$, we have $\tilde{s}_j(q) = 0$ for all $j > h_\alpha(\lambda)$, hence $i_0(q) \leq h_\alpha(\lambda)$. Since $\rho_{h_\alpha}(\lambda) > \rho_{h_\beta}(\lambda) = 0$, we have $\rho_{i_0}(q) \geq \rho_{h_\alpha}(\lambda) > 0$, thus $|\Delta_q(\lambda)| > 0$, contradicting the assumption that (7.14) never holds and $\rho_{h_\beta}(\lambda) = 0$.

So a pair $q \in \bar{X}_\beta$ satisfying (7.14) exists.

Let $(\beta, q)$ be such a pair. We will show that

$$\sum_i \hat{\varrho}_{l,i} \geq \frac{1}{2} \left( |\Delta_q(\lambda_i)| - \left. \varrho_{l,h_\beta}(\lambda_i) \cdot w(\tilde{\mathcal{I}}(\lambda_i), q) - \rho_{l} \cdot \lambda \right) \right)$$

and

$$\liminf_{l \to \infty} \frac{1}{l^2} \left( |\Delta_q(\lambda_i)| - \left. \varrho_{l,h_\beta}(\lambda_i) \cdot w(\tilde{\mathcal{I}}(\lambda_i), q) \right) \geq |\Delta_q(\lambda)| - \left. \varrho_{l,h_\beta}(\lambda) \cdot w(\tilde{\mathcal{I}}(\lambda), q) \right)$$

The sublemma follows after these two inequalities are established.

We prove (7.15). Following the notation introduced in Section 4, we have $\sum_{i=0}^{m_i} \hat{\varrho}_{l,i} \geq \sum_{i \in \mathcal{I}_\beta(\lambda_i)} \varrho_{l,i} \geq \sum_{i \in \mathcal{I}_\beta(\lambda_i)} \hat{\varrho}_{l,i}$, where $\mathcal{I}_\beta(\lambda_i)$ is the set of indices for $\bar{X}_\beta$, and $\mathcal{I}_\beta(\lambda_i)$ is the set of primary indices for $q \in \bar{X}_\beta$, both with respect to the staircase $\lambda_i$.

By Proposition 3.9 and 3.11, we know that for $i_0(q) \neq i \in \mathcal{I}_\beta(\lambda_i)$, we have $\hat{\varrho}_{l,i} = \varrho_{l,i} - \varrho_{l,h_\beta}(\lambda_i)$ (Note that it is possible that $\hat{\varrho}_{l,i}(q) = \varrho_{l,i}(q) - \varrho_{l,h_\beta}(\lambda_i) < \varrho_{l,i}(q) - \varrho_{l,h_\alpha}(\lambda_i)$ for some $\alpha' \neq \beta$ (cf. (5.14)) and Figure 6.)
By the proof of Lemma 4.2, we have

\[
\sum_{i \in \mathbb{I}^\text{pr}(\lambda'_i) \setminus \{i_0(q)\}} \hat{\theta}_{i,i} \geq |\Delta^\text{pr}_q(\lambda'_i) \cap ([1, w^\text{pr}(q, \lambda'_i)] \times \mathbb{R})| - \theta_{l,h_\beta}(\lambda'_i) \cdot (w^\text{pr}(q, \lambda'_i) - 1).
\]

Following (4.9), we continue to denote \( j_q(\lambda'_i) = \max\{i \in \mathbb{I}^\text{pr}_q(\lambda'_i)\} \) and \( w^\text{pr}(q, \lambda'_i) = w(\tilde{E}_{j_q(\lambda'_i) + 1}(\lambda'_i), q) \). By the boundedness result from Corollary 3.12, for sufficiently large \( l \), since the number of secondary indices is bounded by a uniform constant depend only on \( g \) and \( n \) (cf. Definition 3.10 and Corollary 5.3), the effects to the shape of \( \Delta_q(\lambda'_i) \) from the secondary indices \( \mathbb{I}_q(\lambda'_i) \setminus \mathbb{I}^\text{pr}_q(\lambda'_i) \) are marginal, thus for large \( l \) we have

\[
|\Delta^\text{pr}_q(\lambda'_i) \cap ([1, w^\text{pr}(q, \lambda'_i)] \times \mathbb{R})| - \theta_{l,h_\beta}(\lambda'_i) \cdot (w^\text{pr}(q, \lambda'_i) - 1) \\
\geq \frac{1}{2} \left( |\Delta_q(\lambda'_i) \cap ([1, w(\tilde{\lambda}(\lambda'_i), q)] \times \mathbb{R})| - \theta_{l,h_\beta}(\lambda'_i) \cdot (w(\tilde{\lambda}(\lambda'_i), q) - 1) \right) \\
\geq \frac{1}{2} \left( |\Delta_q(\lambda'_i)| - \theta_{l,h_\beta}(\lambda'_i) \cdot w(\tilde{\lambda}(\lambda'_i), q) - \vartheta_{l,i_0(q)} \right).
\]

On the other hand, by our construction \( \vartheta_{l,i_0(q)} \leq \vartheta_{l,0} \leq \rho_0 \cdot l \). Combined, and adding \( \vartheta_{l,i_0(p)} > 0 \), we obtain

\[
\sum_{i \in \mathbb{I}^\text{pr}_q(\lambda'_i)} \hat{\theta}_{i,i} \geq \frac{1}{2} \left( |\Delta_q(\lambda'_i)| - \theta_{l,h_\beta}(\lambda'_i) \cdot w(\tilde{\lambda}(\lambda'_i), q) - \rho_0 \cdot l \right).
\]

This proves (7.15).

Before we move to (7.16), we claim that

\[
A_\beta := \limsup_{l \to \infty} \frac{\theta_{l,h_\beta}(\lambda'_i) - \theta_{l,h_\beta}(\lambda_i)}{l} = 0.
\]

Suppose not, say \( A_\beta > 0 \) (it is non-negative, by our construction of staircase in Proposition 3.5), then for \( l \) large, \( \theta_{l,h_\beta}(\lambda'_i) - \theta_{l,h_\beta}(\lambda_i) \geq \frac{1}{2} \cdot l \cdot A_\beta \).
By examining the geometry of $\Delta_q(\lambda_l) \subset \Delta_q(\lambda'_l)$ (cf. Figure 7), we obtain

$$|\Delta_q(\lambda'_l)| - |\Delta_q(\lambda_l)| \geq |T| = \frac{1}{2} \cdot \frac{\left( \varrho_{l,h_\beta}(\lambda'_l) - \varrho_{l,h_\beta}(\lambda_l) \right)^2 \cdot l \cdot w(\tilde{3}(\lambda), q)}{\varrho_{l,i_0(q)} - \varrho_{l,h_\beta}(\lambda_l)} \geq \frac{1}{2} \cdot \frac{A_\beta \cdot l^2 \cdot l \cdot w(\tilde{3}(\lambda), q)}{l \cdot (\rho_{i_0(q)} - \rho_{h_\beta}(\lambda_l))} := C \cdot A_\beta^2 \cdot l^2 > 0,$$

where we have used $l \cdot (\rho_{i_0(q)} - \rho_{h_\beta}(\lambda_l)) = \varrho_{l,i_0(q)} - \varrho_{l,h_\beta}(\lambda_l)$ because of Lemma 7.11. Therefore,

$$(7.18) \quad l^{-1} \cdot \omega(\lambda_l) = l^{-1} \cdot \omega(\lambda'_l) + l^{-1} \cdot (e(\tilde{3}(\lambda'_l) - e(\tilde{3}(\lambda_l))) \geq l^{-1} \cdot (e(\tilde{3}(\lambda'_l) - e(\tilde{3}(\lambda_l)))$$

where we have used Theorem 1.5 to deduce $\omega(\lambda'_l) \geq 0$.

By Corollary 2.8 and our construction of staircase using Proposition 3.5, we deduce

$$l^{-1} \cdot (e(\tilde{3}(\lambda'_l) - e(\tilde{3}(\lambda_l))) \geq l^{-1} \cdot (|\Delta_q(\lambda'_l)| - |\Delta_q(\lambda_l)|) \geq C \cdot A_\beta^2 \cdot l.$$

This is impossible since Lemma 7.6 implies that the left-hand-side of (7.18) remains bounded for large $l$. This proves $A_\beta = 0$.

We now prove (7.16). Because $A_\beta = 0$, $|\Delta_q(\lambda'_l)| \geq |\Delta_q(\lambda_l)|$, and by (2) after (7.12), we obtain

$$|\Delta_q(\lambda'_l)| = \varrho_{l,i_0(q)} - \varrho_{l,h_\beta}(\lambda'_l) \cdot w(\tilde{3}(\lambda'_l), q)$$

$$= |\Delta_q(\lambda'_l)| + \varrho_{l,h_\beta}(\lambda_l) \cdot w(\tilde{3}(\lambda_l), q) + \varrho_{l,h_\beta}(\lambda_l) \cdot w(\tilde{3}(\lambda_l), q) - \varrho_{l,h_\beta}(\lambda'_l) \cdot w(\tilde{3}(\lambda'_l), q)$$

$$\geq |\Delta_q(\lambda_l)| - \varrho_{l,h_\beta}(\lambda_l) \cdot w(\tilde{3}(\lambda_l), q) + (\varrho_{l,h_\beta}(\lambda_l) - \varrho_{l,h_\beta}(\lambda'_l)) \cdot w(\tilde{3}(\lambda'_l), q)$$

$$= l^2 \cdot \left( |\Delta_q(\lambda_l)| - \varrho_{l,h_\beta}(\lambda_l) \cdot w(\tilde{3}(\lambda_l), q) \right) + \frac{\varrho_{l,h_\beta}(\lambda_l) - \varrho_{l,h_\beta}(\lambda'_l)}{l} \cdot w(\tilde{3}(\lambda), q).$$

Taking lim inf as $l \to \infty$, and using $A_\beta = 0$, we obtain (7.16).

Finally, by (7.13), and that $0 \leq \varrho_{l,i} \leq \varrho_{l,i}$, we conclude that the lim inf in the statement of the lemma is finite; thus the lim inf is finite and positive by (7.14). This proves the lemma.
Remark 7.8. It follows from the proof of Lemma 7.7 that \( \omega(\lambda_l) \geq c \cdot \rho_0 \cdot l \) for \( l \) large (cf. [25]). This can be viewed as a version of uniform Chow stability, an advantage of the GIT approach compared with that of [18].

Remark 7.9. Theorem 7.1 implies that the Deligne-Mumford compactification \( \overline{M}_g \) (for \( g \geq 2 \)) is a \( K \)-stable compactification the moduli of smooth curves. As \( K \)-stability is an analytic version of GIT stability via a CM-line bundle \( \Lambda^{g-1} \otimes \delta^{-1} \) on the moduli of curves defined by Paul and Tian [23], it is interesting to see this generalized to moduli of high dimensional polarized varieties. For recent progress, see Odaka [20].

Remark 7.10. Theorem 7.1 can be easily generalized to the weighted pointed stable curve, that is, although a weighted pointed stable curve in general is not asymptotic Chow stable with respect to the polarization \( \omega_X(a \cdot x)^{\otimes k} \), it is log \( K \)-stable (cf. [21] for the definition). In another word, the asymptotic of Chow instability behaves in a controlled manner.

References


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