Lecture 1: Introduction.

• This course is an exposition of some of Zlil's ideas and techniques and is based on the joint paper with Mladen Bestvina:

"Notes on Sela's work: Limit groups and Makanin-Razborov diagrams"

References can be found there.

• There are exercises in the paper and Henry Wilton produced a solution manual:

"Solutions to Bestvina & Feighn's exercises on limit groups"

1. INTRODUCTION

How much group theory can a logician understand if the logician is only allowed to probe groups with statements?

- *Statement in in a group G*:

There exist x, y such that $xy \neq yx$.

- Not a statement in G:

For all x there exists n such that $x^n = 1$.

- Write $G \sim G'$ if G and G' have the same true statements.

- $\mathbb{Z}^m \not\sim \mathbb{Z}^n$ if $m \neq n$

For example, in $\mathbb{Z}$, elements are even or odd, but not so in $\mathbb{Z}^m$, $m > 1$.

- $\exists x \forall y \exists z$ such that $y = z + z$ or $y = z + z + x$

Tarski 1945: $F_m \overset{?}{\sim} F_n$ if $m, n > 1$ where F_m is a free group of rank m.

Sela: Yes.

Kharlampovich-Myasnikov

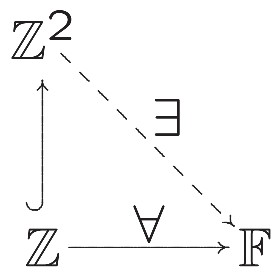
- Surprisingly, Sela's techniques are geometric.

• $\mathbb{F}$ non-abelian free group with basis B

• Statements in $\mathbb{F}$ are interpreted as (generalized) extension problems:

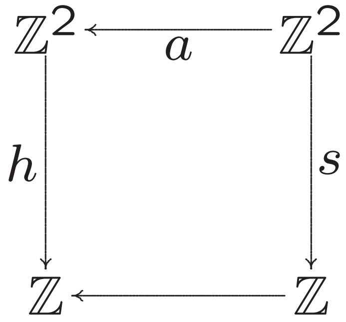
$$\forall x \exists y \text{ such that } xy = yx$$

$$\Updownarrow$$

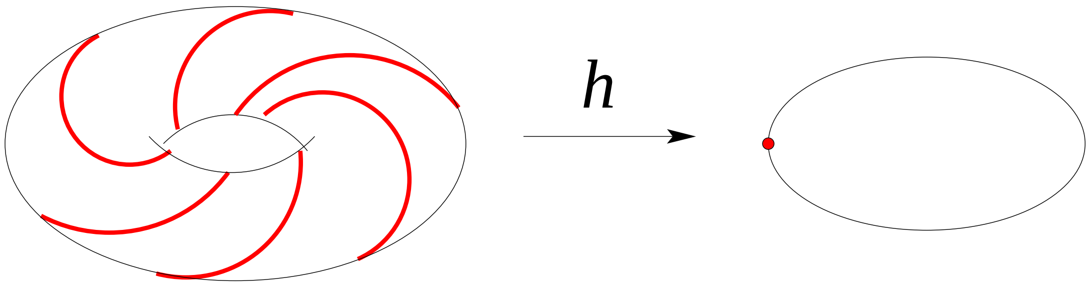


2. $Hom(H, \mathbb{F})$?

Proposition (Makanin-Razborov Diagrams; baby version). Let $s : \mathbb{Z}^2 \to \mathbb{Z}$ be the standard projection. For all $h \in Hom(\mathbb{Z}^2, \mathbb{Z})$ there is $a \in Aut(\mathbb{Z}^2)$ such that $h \circ a$ factors through s.



Proof. Shorten.



Proof.

- Fix basis $S \subset \mathbb{Z}^2$.

- Define $\|h\| = \max_{s \in S} |h(s)|$.

- Unless h kills a basis element, there is $a \in Aut(\mathbb{Z}^2)$ such that $\|h \circ a\| < \|h\|$. $\square$

Theorem (Kharlampovich-Myasnikov, Sela) (MRD-version one).

For finitely generated and non-free H, there is a finite set $\mathcal{S} = \{s : H \twoheadrightarrow H_s\}$ of proper epimorphisms such that:

- for all $h \in Hom(H, \mathbb{F})$, there exists $a \in Aut(H)$ such that $h \circ a$ factors through* $\mathcal{S}$.

$$\begin{array}{ccc} H & \xleftarrow[a]{} & H \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle s} \\ \mathbb{F} & \longleftarrow & H_s \end{array}$$

Proof. Shorten.

*i.e. $h \circ a$ factors through some element of $\mathcal{S}$

Proof.(Details later)

- $S \subset H$ finite generating set.

- $h \in Hom(H, \mathbb{F})$ is *shortest* if $\|h\| \leq \|i_\phi \circ h \circ a\|$, $a \in Aut(H)$, $\phi \in \mathbb{F}$.

- T_h is Cayley tree for $\mathbb{F}$ with H-action given via h.

- $\mathcal{T}' = Closure\{T_h \mid 1 \neq h \text{ is shortest}\}$

- There are no faithful trees in $\mathcal{T}'$ (shorten).

- $\mathcal{T}'$ is compact.

- For $\eta \in H$, $\{T \in \mathcal{T}' \mid \eta \in Ker(T)\}$ is open. $\square$

- Summary:

Logic: sentences in $\mathbb{F}$

$$\Updownarrow$$

Group Theory: extension problems

$$\Updownarrow$$

Geometry: spaces of trees

3. LIMIT GROUPS (Motivation)

• Where do *limit groups* come in? In the baby case, $\mathcal{S} = \{s : \mathbb{Z}^2 \to \mathbb{Z}\}$ and we understand $Hom(\mathbb{Z}, \mathbb{Z})$. But, what about $Hom(H_s, \mathbb{F})$ for general $\mathcal{S} = \{s : H \to H_s\}$?

• **(Induction)** Any sequence

$$H_0 \to H_1 \to \cdots$$

of proper epimorphisms between limit groups is finite. (See Section 5.)

• A finite set $\mathcal{S} = \{s : H \to H_s\}$ of proper epimorphisms is a *factor set for H*.

• For free H:

$$Hom(H, \mathbb{F}) \cong \mathbb{F}^{rank(H)}$$

Theorem (MRD version two). For finitely generated and non-free H, there is a factor set $\mathcal{S} = \{s : H \twoheadrightarrow H_s\}$ such that:

• each H_s is a limit group; and

• for all $h \in Hom(H, \mathbb{F})$, there exists $a \in Aut(H)$ such that $h \circ a$ factors through $\mathcal{S}$.

4. LIMIT GROUPS (Definition and basic properties)

• H fg, a sequence $\{h_i\}$ in $Hom(H, \mathbb{F})$ is *stable* if, for all $\eta \in H$, $\{h_i(\eta)\}$ is eventually always 1 or eventually never 1.

• $\underrightarrow{Ker}\ h_i$ is:

$$\{\eta \in H \mid h_i(\eta) = 1 \text{ for almost all } i\}$$

• Γ is a *limit group* if there is a fg H and a stable sequence $\{h_i\}$ such that:

$$\Gamma \cong H / \underrightarrow{Ker}\ h_i$$

• fg Γ is *residually free* if for every $\gamma \in \Gamma$, there is $g \in Hom(\Gamma, \mathbb{F})$ such that $g(\gamma) \neq 1$. Γ is *ω-residually free* if, for every finite $X \subset \Gamma$, there is $g \in Hom(\Gamma, \mathbb{F})$ such that $g|X$ is injective.

Easy exercises.

- Residually free $\implies$ torsion free.
- Fg free groups and fg free abelian groups are ω-residually free.
- ω-residually free groups are limit groups.
- Fg subgroups of ω-residually free groups are ω-residually free.

Easy exercises (continued).

• A group G is *commutative transitive* if, for all γ_1, γ_2 in G the following property holds:

$$\exists \gamma_3 \neq 1 \text{ such that } [\gamma_1, \gamma_3] = 1 = [\gamma_2, \gamma_3]$$

$$\implies [\gamma_1, \gamma_2] = 1$$

• ω-residually free groups are commutative transitive. (And so, every non-trivial abelian subgroup is contained in a unique maximal abelian subgroup.)

• $H \times \mathbb{Z}$ is not ω-residually free if H is non-abelian.

• If H_1 and H_2 are ω-residually free, then so is $H_1 * H_2$.

• If H is residually free and rank two, then $H \cong F_2$ or $\mathbb{Z}^2$.

Lemma. Let $H_0 \to H_1 \to \cdots$ be a sequence of epimorphisms between fg groups. Then, the sequence

$$Hom(H_0, \mathbb{F}) \supset Hom(H_1, \mathbb{F}) \supset \cdots$$

eventually stabilizes.

Proof.

- Embed $\mathbb{F}$ in $SL_2(\mathbb{R})$.

- The sequence of algebraic varieties

$$Hom(H_0, SL_2(\mathbb{R})) \supset Hom(H_1, SL_2(\mathbb{R})) \supset \cdots$$

stabilizes. $\square$

Corollary. Any sequence of proper epimorphisms between (ω-) residually free groups is finite. $\square$

Lemma. Limit groups are ω-residually free.

Proof.

• Suppose $\Gamma = H/\underrightarrow{Ker}\ h_i$ as in definition. WLOG H is fp.

• Add a relation one-at-a-time to H to obtain:

$$H \to H_1 \to H_2 \to \cdots \to \Gamma$$

• We may assume $Hom(H, \mathbb{F}) = Hom(\Gamma, \mathbb{F})$ and hence that the h_i's are defined on Γ.

• Each non-trivial element of Γ is mapped to 1 by only finitely many h_i. $\square$

5. SUMMARY

Proposition. The following are equivalent:

- H is a limit group.
- H is ω-residually free. $\square$

Proposition (Induction). Any sequence of proper epimorphisms between limit groups is finite. $\square$