Lecture 2: Real Trees

6. LAST TIME

• H fg, a sequence $\{h_i\}$ in $Hom(H, \mathbb{F})$ is *stable* if, for all $\eta \in H$, $\{h_i(\eta)\}$ is eventually always 1 or eventually never 1.

• $\underrightarrow{Ker}\ h_i$ is:

$$\{\eta \in H \mid h_i(\eta) = 1 \text{ for almost all } i\}$$

• Γ is a *limit group* if there is a fg H and a stable sequence $\{h_i\}$ such that:

$$\Gamma \cong H/\underrightarrow{Ker}\ h_i$$

• fg Γ is *ω-residually free* if, for every finite $X \subset \Gamma$, there is $g \in Hom(\Gamma, \mathbb{F})$ such that $g|X$ is injective.

Proposition. The following are equivalent:

- H is a limit group.
- H is ω-residually free.

Proposition (Induction). Any sequence of proper epimorphisms between limit groups is finite.

7. EXAMPLES OF LIMIT GROUPS

- fg free groups: $F_n \hookrightarrow \mathbb{F}$

- fg free abelian groups are $\mathbb{Z}$-ω-residually free: A generic projection $\mathbb{Z}^n \to \mathbb{Z}$ is injective on a pre-specified finite set.

- free products of limit groups

- fg subgroups of limit groups

• doubles: If F is free and $\gamma \in F$ has no proper roots, then $H = F *_{\langle\gamma\rangle} F$ is a limit group.

Hint: Use h's of the form $id_F * id_F : H \to F$ pre-composed with Dehn twists in γ and the fact that if, in the Cayley tree for $\mathbb{F}$,

$$|A_\phi \cap A_{\phi'}| > \ell(\phi) + \ell(\phi')$$

then $[\phi, \phi'] = 1$ and $A_\phi = A_{\phi'}$.

• closed surface groups (except $n\mathbb{P}$, $n = 1, 2, 3$): See next section.

8. EXAMPLES OF FACTOR SETS

• There aren't many explicit examples of factors sets satisfying the conclusion of Theorem MRD v2.

• If $H = H_1 * H_2$, then $\mathcal{S}(H) = \mathcal{S}(H_1) * \mathcal{S}(H_2) := \{s_1 * s_2 \mid s_i \in \mathcal{S}(H_i)\}$.

• $\mathcal{S}(\mathbb{Z}^n) = \{s : \mathbb{Z}^n \to \mathbb{Z}\}$

• If H is a closed ,orientable, genus g surface group and if $s : H \to F_g$ represents the standard retraction of the surface onto a rank g graph, then $\mathcal{S}(H) = \{s\}$. (Zieschang, Stallings)

• The non-orientable version is due to Grigorchuk and Kurchanov and one map does not suffice. Note: for $n = 1$, 2, or 3, $n\mathbb{P}$ is not a limit group.

9. REAL TREES

• A real tree (T, d_T) is a metric space such that between any two points $t, t' \in T$, there is a unique arc* from t to t' and this arc is the image of an isometric embedding of an interval.

Example. A *finite simplicial real tree* is a finite tree with each edge identified with an interval.

Example. A countable increasing union of finite simplicial real trees

Example. 0-hyperbolic spaces embed into real trees.

*the image of an embedding $\sigma : [x, x'] \to T$ with $\sigma(x) = t$ and $\sigma(x') = t'$

10. ISOMETRIES OF REAL TREES

• An isometry η of a real tree T either *elliptic* or *hyperbolic*.

• Elliptic η fixes a point of T. The *axis of* η is $Fix(\eta)$.

• Hyperbolic η leaves invariant an isometrically embedded $\mathbb{R}$ (its *axis* A_η). Points on A_η are translated by

$$\ell_T(\eta) := \min\{d_T(t, \eta(t)) \mid t \in T\}.$$

- We will be interested in actions of a fg group H on T.

- The H-tree T is *minimal* if T contains no proper invariant H-subtrees.

Lemma. *If H is fg and T is a minimal H-tree, then T is either a point or the union of the axes of the hyperbolic elements of H.* $\square$

11. SPACES OF REAL TREES

• *$\mathcal{A}(H)$* is the set of isometry classes of non-trivial minimal H-trees.

• *Gromov topology*: $\lim\{(T_n, d_n)\} = (T, d)$ iff for all finite $K \subset T$, $\epsilon > 0$, and finite $P \subset H$ there are, for all large n, subsets K_n of T_n and bijections $f_n : K_n \to K$ such that:

$$|d(\eta f_n(s_n), f_n(t_n)) - d_n(\eta s_n, t_n)| < \epsilon$$

for all $s_n, t_n \in K_n$ and $\eta \in P$.

• Intuitively, larger and larger pieces of the limit tree with their restricted actions (approximately) appear in nearby trees.

- *Length topology*: $\mathcal{A}(H) \to [0,\infty)^H$, $(T,d) \mapsto (\ell_T(\eta))_{\eta \in H}$.

- A neighborhood basis for T consists of sets of the form

$$\{T' : |\ell_{T'}(\eta) - \ell_T(\eta)| < \epsilon, \eta \in P\}$$

where finite $P \subset H$ and $\epsilon > 0$.

- (Serre) $\vec{0}$ is not in the image.

- (Culler-Morgan) This map is injective on *semi-simple actions*, i.e. those that are linear† or don't fix an end.

†isometric to $\mathbb{R}$

• We are interested in projectivized spaces of non-trivial H-trees, i.e. $(T,d) \sim (T, \lambda d)$, $\lambda > 0$.

$$\mathcal{PA}(H) := \mathcal{A}(H)/(0,\infty)$$

• (Culler-Morgan) $\mathcal{PA}(H)$ is compact wrt the length topology.

• $T \in \mathcal{A}(H)$ is *irreducible* if it is not linear and doesn't have a fixed end.

• (Paulin) The Gromov and length topologies agree on the set of irreducible trees.

12. $\mathcal{T}(H)$

• Non-trivial $h \in Hom(H, \mathbb{F})$ gives an action of H on the Cayley tree for $\mathbb{F}$ and so determines $T_h \in \mathcal{PA}(H)$.

• Since the Cayley tree is a free $\mathbb{F}$-tree, if $\eta \in H$ fixes a point in T_h, then $h(\eta) = 1$.

• In particular, $Ker(T_h) = Ker(h)$.

• T_h and $T_{i_\phi \circ h}$ are isometric.

• $X := \{h \in Hom(H, \mathbb{F}) \mid Image(h) \text{ is not cyclic}\}$

• $\mathcal{T}(H)$ is the closure in $\mathcal{PA}(H)$ of $\{T_h \mid h \in X\}$

- Trees in $\mathcal{T}(H)$ have nice properties.

- An H-tree T is *super stable* if the following property holds: if $J \subset I$ are non-degenerate arcs with $Fix_T(I)$ non-trivial, then $Fix_T(J) = Fix_T(I)$.

- An H-tree T is *very small* if it is non-trivial (i.e. not a point), minimal, super-stable, non-degenerate arc stabilizers are primitive abelian, and non-degenerate tripod stabilizers are trivial.

Proposition. Suppose $T \in \mathcal{T}(H)$. Set $\bar{H} = H/Ker(T)$ and let $\bar{T}$ denote T with the action of $\bar{H}$. Then:

1. T is irreducible.

2. $Ker(T) = \varinjlim Ker\ h_i$.

3. $\bar{T}$ is very small.

4. For $\eta \in H$,

$$U(\eta) := \{T \in \mathcal{T}(H) \mid \eta \in Ker(T)\}$$

   is clopen (i.e. open and closed).

Proof. We will show tripod stabilizers are trivial. The rest is similar.

- Assume η stabilizes the endpoints of a tripod.

- Nearby T_{h_i} have tripods with endpoints nearly stabilized by η.

- So, η fixes the cone point and $h_i(\eta) = 1$. $\square$

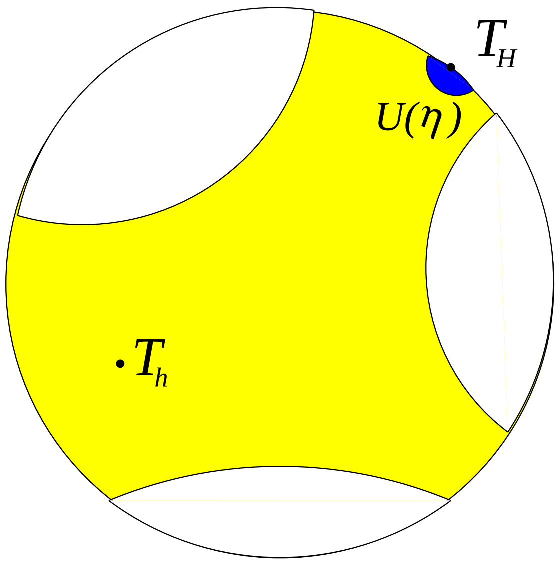
We will need the following corollary of Paulin's Convergence Theorem.

Proposition. Suppose $\{h_i\}$ is a sequence in X. Then, $\{T_{h_i}\}$ has a subsequence converging to a tree in $\mathcal{PA}(H)$ with the Gromov topology. $\square$

Corollary. $\mathcal{T}(H)$ is compact and metrizable.

Proof. $\mathcal{T}(H)$ consists of irreducible trees and is homeomorphic to its image in $\Big([0,\infty)^H \setminus \{0\}\Big)/(0,\infty)$. In particular, $\mathcal{T}(H)$ is metrizable. In light of this, the preceding proposition implies that $\mathcal{T}(H)$ is compact.

13. PAUSE



$\mathcal{T}_Y \subset \mathcal{T}(H)$ is yellow

- Goal: Given $Y \subset Hom(H, \mathbb{F})$, we want a strategy to show Y *has a factor set* $\mathcal{S}$, i.e. a set through which every $h \in Y$ factors.

- WMA no $h \in Y$ has abelian image.

- Strategy: Show there are no faithful trees in the closure $\bar{\mathcal{T}}_Y$ in $\mathcal{T}(H)$ of $\mathcal{T}_Y = \{T_h \mid h \in Y\}$. Since:

- No faithful trees in $\bar{\mathcal{T}}_Y$

- $\iff$ for every $T \in \bar{\mathcal{T}}_Y$ there is $1 \neq \eta_T \in Ker(T)$

- $\iff$ for every $T \in \bar{\mathcal{T}}_Y$ there is $1 \neq \eta_T$ such that $T \in U(\eta_T)$

- $\implies$ there are $\eta_1, \ldots, \eta_n$ such that $\{U(\eta_1), \ldots, U(\eta_n)\}$ covers $\bar{\mathcal{T}}_Y$

- $\implies$ $\mathcal{S} = \{H/Ab(H), H/\langle\langle\eta_1\rangle\rangle, \ldots, H/\langle\langle\eta_n\rangle\rangle\}$ is a factor set for Y.

- So, next we have to understand faithful trees in $\mathcal{T}(H)$.