

# Lecture 3: Picturing Trees

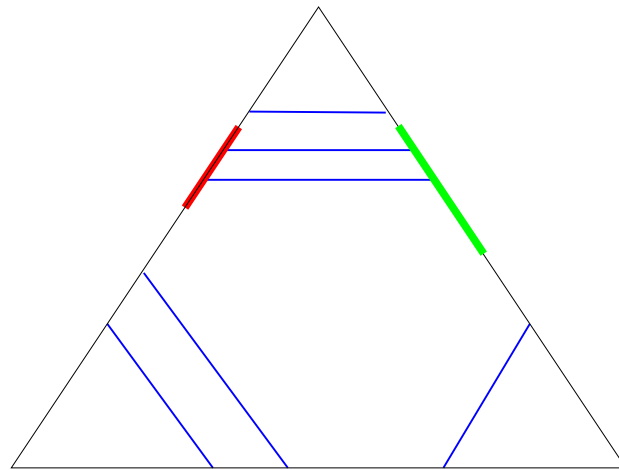
## 14. LAST TIME

- To  $h \in \text{Hom}(H, \mathbb{F})$  we associate  $T_h \in \mathcal{PA}(H)$ .  $X \subset \text{Hom}(H, \mathbb{F})$  consists of those homomorphisms with non-cyclic image. The closure  $\mathcal{T}(H)$  of  $\{T_h \mid h \in X\}$  in  $\mathcal{PA}(H)$  is compact and consists of very small trees. We want to understand faithful trees in  $\mathcal{T}(H)$ .

## 15. MEASURED LAMINATIONS

- A *measured lamination*  $(\Lambda, \mu)$  on a 2-complex  $K$ , a closed subset  $\Lambda$  of  $K \setminus K^{(0)}$ , and a measure  $\mu$  on  $K^{(1)}$  such that:
  1.  $\Lambda$  intersects an edge in a (possibly empty) Cantor set;
  2.  $\Lambda$  intersects each closed 2-simplex  $\Delta$  in straight line segments spanning distinct sides;
  3. (compatibility) if intervals  $I$  and  $J$  in two sides of  $\Delta$  meet the same components of  $\Lambda \cap \Delta$ , then  $\int_I \mu = \int_J \mu$ .

4. (regularity)  $\mu|_{\text{edge}}$  is equivalent under a “Cantor function” to the Lebesgue measure on an interval of  $\mathbb{R}$ .



- We often suppress  $\mu$ .
- A *leaf* is a connected component of  $\Lambda$ .

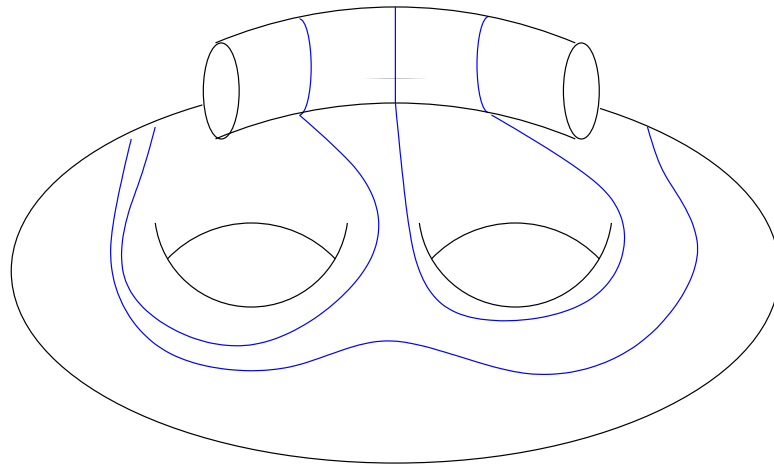
**Proposition (Morgan-Shalen).** Suppose that  $\Lambda$  is a measured lamination on compact  $K$ . Then,

$$\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_n$$

such that, for each  $i$ , either:

1. each leaf of  $\Lambda_i$  is dense in  $\Lambda_i$ ; or
2.  $\Lambda_i$  is simplicial, i.e. each leaf is compact.

**Example (Surface-type).** Filling\* geodesic laminations without isolated leaves on a hyperbolic surface  $K$  with geodesic boundary. More generally, add annuli  $S^1 \times I$  to  $K$  along  $* \times I$  transverse to  $\Lambda$ .

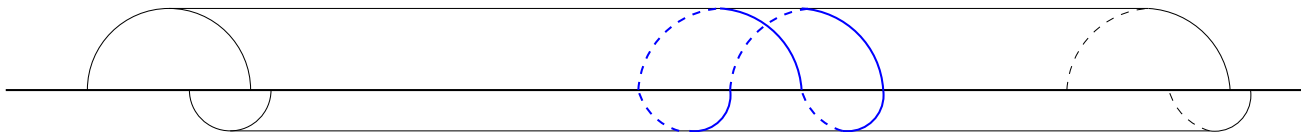


\*complementary components are simply-connected or retract to a boundary curve

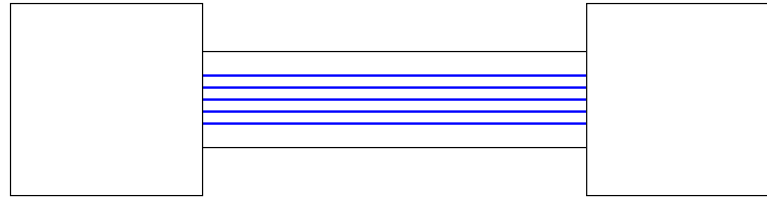
**Example (Toral-type).** Fix an interval  $\mathbb{I} \subset \mathbb{R}$  and isometries  $\alpha_i : J_i \rightarrow J'_i$  between subintervals. Suppose:

1. if  $\alpha_i$  is orientation-reversing, then  $J_i = J'_i$ ;
2.  $|J := \cap_i (J_i \cap J'_i)| > 2 \max_i \{\ell(\alpha_i)\}$ ; and
3. fixed points of orientation-reversing  $\alpha_i$  are in the middle third of  $J$ .

Let  $B_i = J_i \times I$  be a laminated band with leaves  $* \times I$ . Form  $K$  by adding the  $B_i$ 's to  $\mathbb{I}$  by gluing  $J_i \times 0$  to  $J_i$  and  $J_i \times 1$  to  $J'_i$ .



**Example (Thin-type).** Generic leaves are qi to one-ended trees.  $K$  is obtained by adding annuli to a complex  $\bar{K}$  as in surface-type case. Up to *Rips moves*,  $\bar{K}$  has a *naked* band.



**Theorem (“Rips machine”).** Let  $\Lambda$  be a measured lamination on a compact 2-complex  $K$  and let  $\Lambda_i$  be a non-simplicial component of  $\Lambda$ . Then, there is a neighborhood  $N$  of  $\Lambda_i$ , a compact 2-complex  $N'$  with a measured lamination  $\Lambda'$  as in one of the 3 examples, and a  $\pi_1$ -isomorphism  $f : N \rightarrow N'$  such that  $f^*(\mu') = \mu_i$ .

## 16. PICTURING TREES

- Let  $\hat{\Lambda}$  be a measured lamination on simply-connected  $\hat{K}$ . The pseudometric on  $\hat{K}$ :

$$d(x, x') = \inf \left\{ \int_{\alpha} \mu : \alpha \text{ is a path from } x \text{ to } x' \right\}$$

has associated metric space a real  $\pi_1(K)$ -tree called the *dual tree*  $T(\hat{\Lambda})$ . (Hint: Show  $d$  is 0-hyperbolic.)

- There is a natural map  $\hat{K} \rightarrow T(\hat{\Lambda})$ .

- Two laminations on  $\hat{K}$  are *equivalent* if they assign the same length to each 1-simplex. Equivalent laminations have the same dual trees.
- Let  $\hat{E}$  be the edges of  $\hat{K}$ . A point of  $[0, \infty)^{\hat{E}}$  such that there is a triangle inequality for each 2-simplex determines a measured lamination uniquely up to equivalence.
- A *generalized leaf* is a class of the relation:  $x \sim x'$  if there is a measure 0 path between  $x$  and  $x'$ . The image of a generalized leaf in the dual is a point.

• Let  $T$  be an  $H$ -tree. We say that  $(\hat{K}, \hat{\Lambda})$  is a *model for  $T$*  if  $T = T(\Lambda)$  where:

1.  $\hat{K}$  is a simply-connected 2-dimensional  $H$ -complex;
2.  $\hat{\Lambda}$  is  $H$ -invariant;
3.  $Stab(\hat{\Delta}) = Fix(\hat{\Delta})$  for each simplex  $\hat{\Delta}$ ;
4. for each edge  $\hat{e}$  of  $\hat{K}$ ,  $T(\hat{\Lambda}|\hat{e}) \rightarrow T(\hat{\Lambda})$  is an isometric embedding; and
5.  $K := \hat{K}/H$  is compact.

- $(\hat{K}, \hat{\Lambda})$  is an *exact model* if additionally:

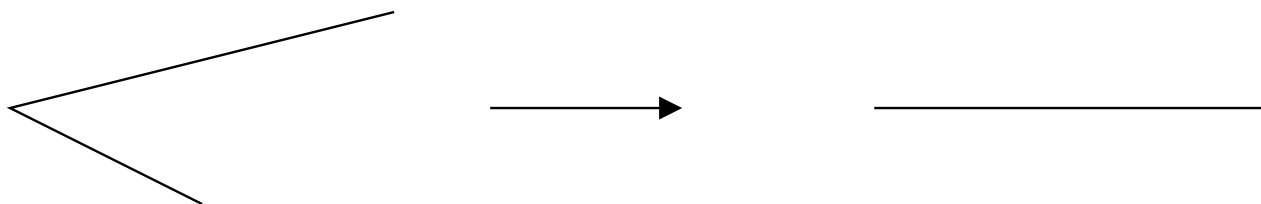
6. The natural map {generalized leaves}  $\rightarrow T(\hat{\Lambda})$  is injective.

- We also call  $K$  with the induced  $\Lambda$  a model.

**Open Problem?** Find a model that is not an exact model.

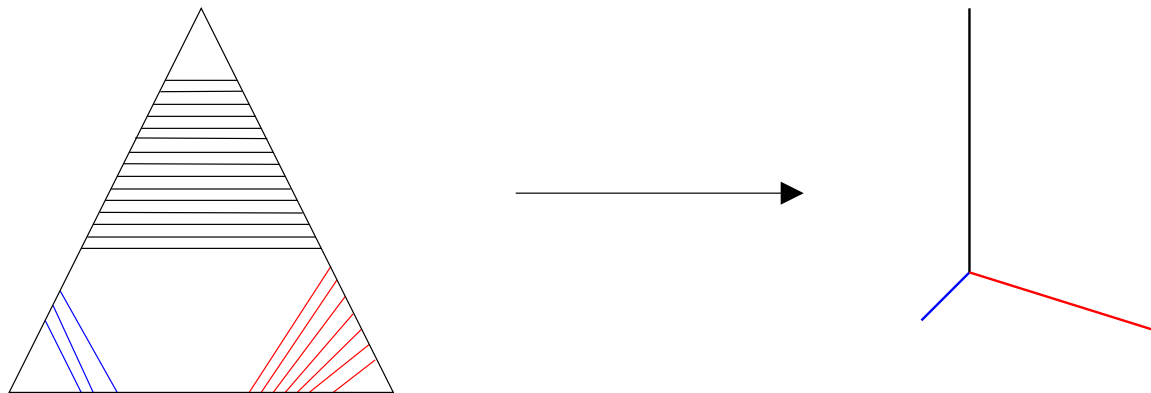
- There are trees that have no model.

- A map  $S \rightarrow T$  of trees is a *morphism* if every compact arc  $\sigma$  in  $S$  is a union of finitely many subarcs  $\sigma_i$  such that the restriction to  $\sigma_i$  is an isometric embedding.



**Proposition.** Suppose  $H$  is fp. Every  $H$ -tree  $S$  admits a resolution  $T \rightarrow S$  where  $T$  has a model.

**Proof.** Let  $\hat{K}$  be the universal cover of a compact 2-complex with  $\pi_1(K) \cong H$ . Choose for each orbit representative  $x \in \hat{K}^{(0)}$  an image  $s \in S$ . Extend to  $\hat{K}^{(0)}$  by  $H$ -equivariance. Extend to  $\hat{K}^{(1)}$  linearly. Extend to  $\hat{K}$  as:



## 17. EXAMPLES

- *Simplicial*:  $T$  is a simplicial real tree and  $\hat{K} = T$ .
- *Surface*:  $T$  is dual to a filling lamination on a compact surface. We may also use a model where boundary components are crushed to points (and are labelled  $\mathbb{Z}$ ).
- *Linear*:  $H$  is abelian and there is a free  $\mathbb{Z}^n$ -action on a line  $T$  such that  $H$  acts via  $H \twoheadrightarrow \mathbb{Z}^n$ . The model is a linear lamination on an  $n$ -torus (restricted to the 2-skeleton).

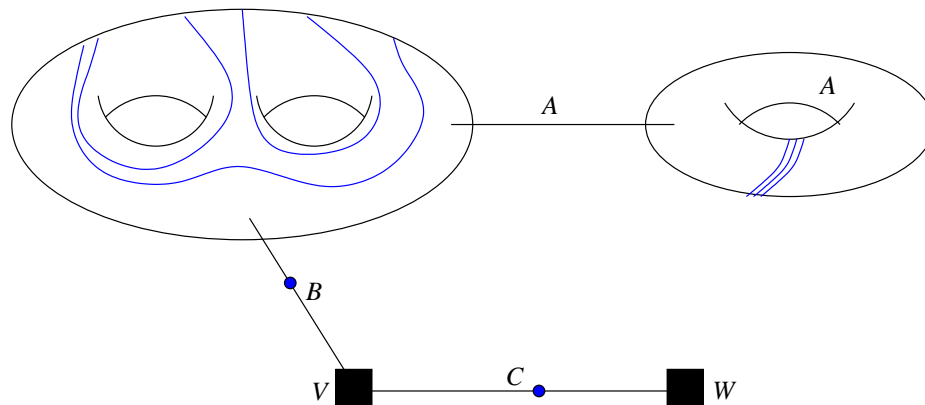
## 18. STRUCTURE THEOREM

- $H$  is *generic* if  $H$  is fg, torsion-free, freely indecomposable, not a closed surface group, and non-abelian.

**Theorem (Structure Theorem) (Sela).** Suppose  $H$  is generic and  $T$  is a very small  $H$ -tree. Then,  $T$  has an exact model  $(K, \Lambda)$ .

Moreover,  $(K, \Lambda)$  can be chosen to be a graph of spaces where each edge space is a point with non-trivial abelian stabilizer and each vertex space with restricted lamination is:

1. a point with non-trivial stabilizer;
2. non-faithful linear; or
3. a surface with non-empty boundary (represented by points).



**Corollary.** If generic  $H$  admits a stable, very small action on a tree, then  $H$  has a  $GAD^*$ , i.e. a non-trivial graph of groups decomposition with abelian edge groups in which some vertices are linear and some are surface. Remaining vertices are *rigid*.  $\square$

**Warning.** Edge and vertex groups of  $GADs$  need not be fg.

- We now have some understanding of the faithful actions in  $\mathcal{T}(H)$ . What good is it?

\*Generalized Abelian Decomposition