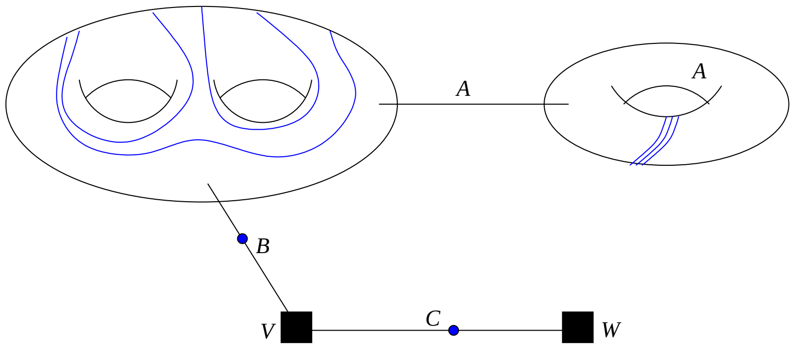
Lecture 4: Shortening

19. LAST TIME



• If generic H admits a very small action on a tree, then H has a *GAD*, i.e. a non-trivial graph of groups decomposition with abelian edge groups in which some vertices are linear and some are surface. Remaining vertices are *rigid*. □

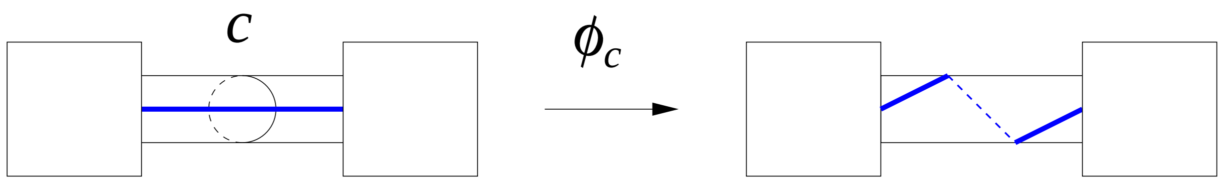
Warning. Edge and vertex groups of GADs need not be fg.

• We now have some understanding of the faithful actions in $\mathcal{T}(H)$. What good is it?

20. MODULAR AUTOMORPHISMS

• If $H = A *_C B$ and z is in the centralizer $Z_A(C)$, then $\phi_z \in Aut(H)$, called the *Dehn twist in z*, is given by:

$$\phi_z(\eta) = \begin{cases} \eta, & \text{if } \eta \in A; \\ z\eta z^{-1}, & \text{if } \eta \in B. \end{cases}$$



• If $C \subset A$, $\psi : C \to A$ is a monomorphism, $H = A*_C = \langle A, t \mid tat^{-1} = \psi(a), a \in A\rangle$, and $z \in Z_A(C)$, then:

$$\phi_z(\eta) = \begin{cases} \eta, & \text{if } \eta \in A; \\ \eta z, & \text{if } \eta = t. \end{cases}$$

• Let A be an abelian vertex group of a GAD. $P(A)$ is the subgroup of A generated by incident edge groups.

• The *peripheral subgroup* $\bar{P}(A)$ *of* A is the subgroup of A that dies under every homomorphism from A to $\mathbb{Z}$ that kills $P(A)$.

Lemma. Any automorphism of A that fixes $P(A)$ also fixes $\bar{P}(A)$. $\square$

• Let Δ be a GAD of fg H. The *modular group* $Mod(\Delta)$ is the subgroup of $Aut(H)$ generated by:

1. inner automorphisms of H;

2. Dehn twists in elements of H that centralize an edge group of Δ;

3. unimodular* automorphisms of an abelian vertex group of Δ that fix incident edge groups and all other vertex groups; and

4. $Mod(\Sigma, \partial\Sigma)$ where Σ corresponds to a surface vertex of Δ.

*induced auto of $A/\bar{P}(A)$ has det=1

• Intuitively, $Mod(\Delta)$ is generated by automorphisms that are supported in and edge or a vertex group.

• If H is generic, then $Mod(H)$ is generated by $Mod(\Delta)$, Δ a GAD of H.

• If $H = \pi_1(\Sigma)$ where Σ is a closed surface, then $Mod(H)$ "is" $Mod(\Sigma)$. If H is fg free abelian, then $Mod(H)$ consists of unimodular autos. If H is freely decomposable, then H is generated by modular automorphisms of its freely indecomposable factors.

Theorem (MRD final version). For finitely generated and non-free H, there is a factor set $\mathcal{S} = \{s : H \twoheadrightarrow H_s\}$ such that:

1. each H_s is a limit group; and

2. for all $h \in Hom(H, \mathbb{F})$, there exists $a \in Mod(H)$ such that $h \circ a$ factors through $\mathcal{S}$.

Moreover, if H is not a limit group, then 2 may be replaced by:

2'. all $h \in Hom(H, \mathbb{F})$ factor through $\mathcal{S}$.

21. SHORTENING

• Let H be a generic group with a fixed finite generating set S. For $h \in Hom(H, \mathbb{F})$, $\|h\|$:= $\max_{s \in S} |h(s)|$.

• $h \sim h'$ if $h' = i_\phi \circ h \circ a$ where $a \in Mod(H)$.

• h is *shortest* if $h = \min_{h' \sim h} \|h'\|$.

Theorem (Shortening) (Sela). If H is generic and $T = \lim T_{h_i}$ in $\mathcal{T}(H)$ where $\{h_i\}$ is a sequence of shortest elements, then T is not faithful.

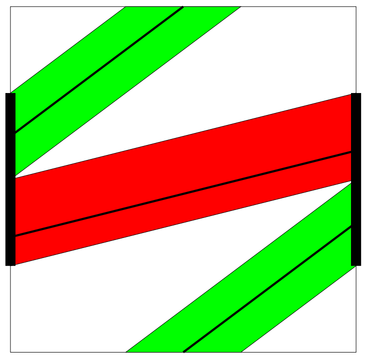
• To prove this, we assume T is faithful and show how to shorten nearby T_h. There are 3 cases to understand. Here are the ideas.

22. T IS DUAL TO A SURFACE LAMINATION

• Λ is a filling geodesic lamination on the compact surface $(\Sigma, *)$ with $\pi_1(\Sigma, *) \cong H$.

• Given an arc in Σ transverse to Λ there is a decomposition of a neighborhood of Λ into a union of maximal foliated rectangles. (Use first return map.)

• Generators in a *standard generating set for H* correspond to the rectangles in such a decomposition.



• There are only finitely many standard generating sets up to the action of $Mod(\Sigma, \partial\Sigma)$. So we may pretend there is only one and that our fixed generating set S is standard.

$$\frac{1}{C} \cdot \|h\|_{S'} - C \leq \|h\|_S \leq C \cdot \|h\|_{S'} + C$$

• S is represented as a set of based geodesics. $\|T\|_S := \max_{s \in S} \int_s \mu$.

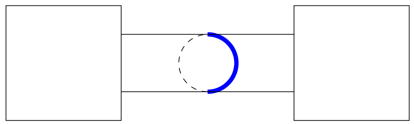
• Given $\epsilon > 0$, there are standard generating sets S' with $\|T\|_{S'} < \epsilon$. (If S' is constructed from the arc σ, then $\|T\|_{S'} \leq 2 \int_\sigma \mu$.)

• So, there is $a \in Mod(F)$ such that $\|T\|_{a(S)} < \epsilon \cdot \|T\|_S$ and for nearby T_h, $\|h \circ a\| \leq 2\epsilon \cdot \|h\|$.

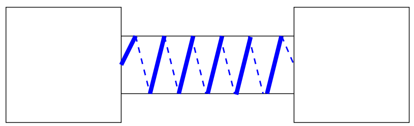
• Some care with basepoints is needed and has been suppressed.

23. T IS SIMPLICIAL

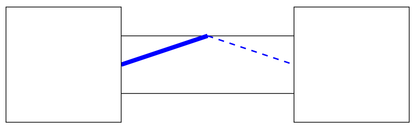
- Assume the Bass-Serre presentation for T is $A *_C B$ with C maximal cyclic in A and B, i.e. T is dual to:



- Nearby T_h have a resolution:

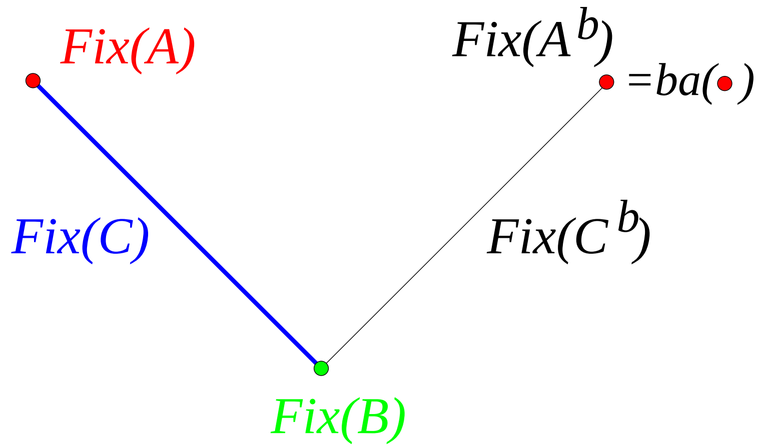


- which after Dehn twisting looks like:

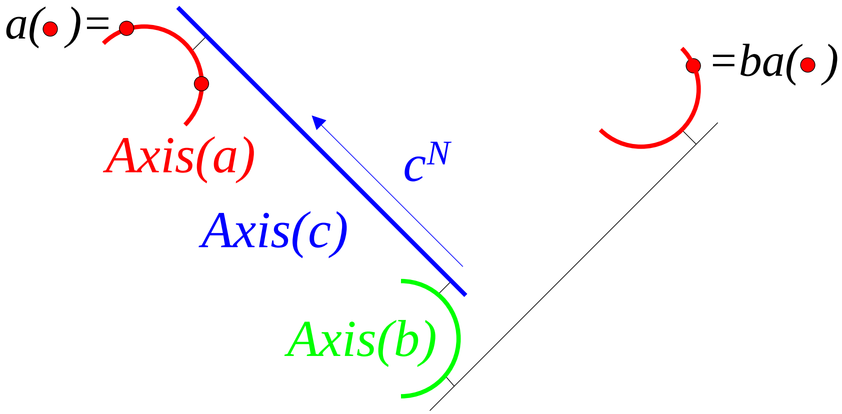


• In case C is not fg, the previous argument is best made in T. The Bass-Serre presentation of T is:

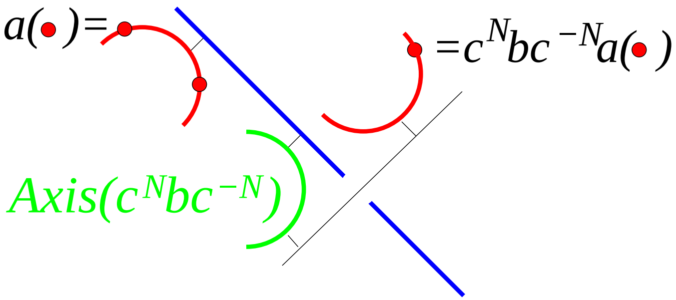
• As an example, we will shorten a fixed element $ba \in H$ where $a \in A$ and $b \in B$. Our basepoint $*$ is $Fix(A) \in T$ and the arc $[*, ba(*)]$ is pictured:



- Pick $c \in C$. In nearby T_h, c has very short translation length:



- which after Dehn twisting by c^N looks like:



24. GENERAL CASE

- In the general simplicial case, T has more than one edge.

- The linear case is very similar to the surface case.

- In general, a model for T has simplicial edges as well as both surface and linear vertices. Since each of our automorphisms is supported on an edge, on a surface vertex, or on a linear vertex, we may compose and shorten in this case as well.

- This ends the discussion of the shortening theorem.

25. THE MAIN PROPOSITION

• We are now ready to prove the final version of the MRD-theorem. It will follow from:

Main Proposition. Suppose H is a generic limit group. Then, there is a factor set $\mathcal{S}$ with the following property:

• for all $h \in Hom(H, \mathbb{F})$, there is $a \in Mod(H)$, such that $h \circ a$ factors through $\mathcal{S}$.

Proof. Let $Y := \{h \in X \mid h \text{ is shortest}\}$ and let $\mathcal{T}'(H)$ be the closure in $\mathcal{T}(H)$ of $\{T_h \mid h \in Y\}$. By the shortening theorem, there are no faithful trees in $T'(H)$. By Section 13 of Lecture 2, Y has a factor set. □

• Cornelius Reinfeldt and Richard Weidmann point out that a factor set for Y can be shown to exist as follows. Let $\{\eta_1, \ldots\}$ enumerate the non-trivial elements of H. If

$$\mathcal{Q}_i := \{H/\langle\langle\eta_1\rangle\rangle, \ldots, H/\langle\langle\eta_i\rangle\rangle\}$$

then there is $h_i \in Y$ that is injective on $\{\eta_1, \ldots, \eta_i\}$. A subsequence of $\{h_i\}$ converges to a faithful tree contradicting the fact that the h_i's are shortest.