

Lecture 5: Abelian Subgroups

26. LAST TIME

- Last time we proved:

Main Proposition. Suppose H is a generic limit group. Then, there is a factor set \mathcal{S} with the following property:

- for all $h \in \text{Hom}(H, \mathbb{F})$, there is $a \in \text{Mod}(H)$, such that $h \circ a$ factors through \mathcal{S} .

27. PROOF OF THEOREM MRD

- For fg H , $\text{rank}(H)$ is the minimal number of generators.
- H is *simpler* than H' if there is an epimorphism $H' \twoheadrightarrow H$ and either
 1. $\text{rank}(H) < \text{rank}(H')$; or
 2. $\text{rank}(H) = \text{rank}(H')$ and $\text{Hom}(H, \mathbb{F}) \subsetneq \text{Hom}(H', \mathbb{F})$.
- We have seen (Section 4 of Lecture 1) that every sequence $\{H_i\}$ with H_{i+1} simpler than H_i is finite.

- For fg H , $RF(H)$ is the universal residually free quotient of H , i.e. $RF(H)$ is the quotient of H by the normal subgroup of elements killed by every element of $Hom(H, \mathbb{F})$.
- $Hom(H, \mathbb{F}) = Hom(RF(H), \mathbb{F})$
- If H' is a proper quotient of $RF(H)$, then $Hom(H', \mathbb{F}) \subsetneq Hom(H, \mathbb{F})$.

Proof of Theorem MRD.

- Assume that H is fg and not free.
- We may assume Theorem MRD holds for non-free groups simpler than H .
- Replacing H with $RF(H)$, we may assume H is residually free.
- We have seen the MRD Theorem holds for surface and abelian groups.
- If $H = U * V$ with U non-free and freely indecomposable and V non-trivial, then U is simpler than H . A factor set for H can be constructed from one for U .

- If H is not a limit group, then by definition there is a finite subset $\{\eta_1, \dots, \eta_n\}$ such that any $h \in \text{Hom}(H, \mathbb{F})$ kills some η_i . So, $\mathcal{S} = \{H \rightarrow H/\langle\langle \eta_i \rangle\rangle \mid 1 \leq i \leq n\}$ works.
- The remaining case is H is a generic limit group. The Main Proposition gives a factor set $\{q : H \rightarrow H_q\}$ for H . Each H_q is simpler than H . By our assumption, MRD Theorem holds for each non-free H_q and this implies the result for H . \square

28. ABELIAN SUBGROUPS OF LIMIT GROUPS ARE FG

• For a limit group H , $a \in \text{Mod}(H)$ is a *generalized Dehn twist* if either:

1. a is a Dehn twist; or

2. $H = A *_C B$ with A abelian, $a|_B = \text{id}_B$, and $a|_A$ is a unimodular automorphism of A .

A generalized Dehn twist as in 2 is called *unimodular*.

• $\text{Mod}(H)$ is generated by generalized Dehn twists.

Lemma. Let M be a non-cyclic maximal abelian subgroup of the limit group H .

1. If $H = A *_C B$ with C abelian, then M is conjugate into A or B .
2. If $H = A *_C$ with C abelian, then either M is conjugate into A or there is a stable letter t such that M is conjugate to $M' = \langle C, t \rangle$ and $H = A *_C M'$.

Proof. Only need fact that maximal abelian subgroups in limit groups are malnormal and standard Bass-Serre theory. (See for example Henry's solution manual.) □

Corollary. If $a \in \text{Mod}(H)$ is a generalized Dehn twist, and if $a|M$ is non-trivial (i.e. not inner), then (up to an inner automorphism) a is unimodular and defined by a GAD of the form $M *_C B$. \square

Lemma (Injects). Let $\{q : H \rightarrow H_q\}$ be a factor set for the limit group H . If M is a subgroup of H such that, for every $h \in \text{Hom}(H, \mathbb{F})$, $h|M$ factors through some $q|M$, then some $q|M$ is injective.

Proof.

- Suppose not and let $1 \neq \mu_q \in \text{Ker}(q|M)$.
- Choose $h \in \text{Hom}(H, \mathbb{F})$ that is injective on $\{\mu_q\}$.
- By hypothesis, $h|M$ factors through some q and so kills μ_q , contradiction. \square

Lemma. Let M be a non-cyclic maximal abelian subgroup of the limit group H . There is an epimorphism $r : H \twoheadrightarrow A$ such that A is free abelian and every $a \in \text{Mod}(H)$ is trivial when restricted to $M \cap \text{Ker}(r)$.

Proof.

- Enough to consider generalized Dehn twists $a \in \text{Mod}(H)$.
- By Lemma, WMA a is unimodular and defined by $M *_C B$.
- Define $r_a : H \twoheadrightarrow M_a := M/\bar{P}(M)$, $r := \prod_a r_a : H \rightarrow \prod_a M_a$, and $A := \text{Im}(r)$. □

Corollary of Theorem MRD. Abelian subgroups of a limit group H are fg.

Proof.

- Let M be a non-cyclic maximal abelian subgroup of H .
- Using $r|M$, $M = A \oplus A'$ with A fg and each $a|A'$ trivial.
- If $\{H \rightarrow H_q\}$ is a factor set as in Theorem MRD, then Lemma (Injects) implies A' injects into some H_q . Apply induction. \square