

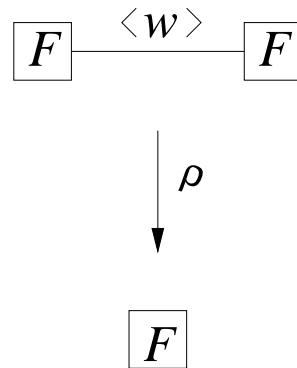
Lecture 6: Constructible Limit Groups

29. MOTIVATION

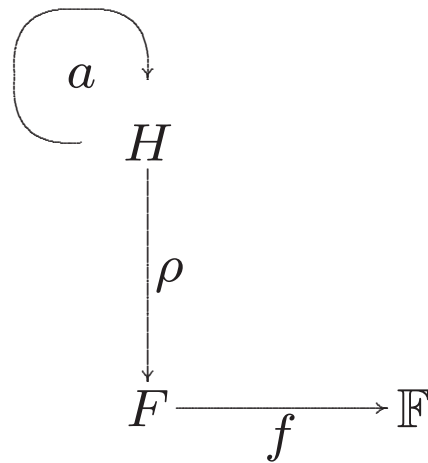
• How are all limit groups are constructed? The motivation is the case of a double. Let $H = F *_{\langle w \rangle} F$ be a double where F is free. We have:

1. the retraction $\rho := id_F * id_F : H \rightarrow F$; and

2. the *GAD* $\Delta := F *_{\langle w \rangle} F$ for H with two rigid vertices.



- Is $\mathcal{F} := \{f \circ \rho \circ a \mid f \in \text{Hom}(F, \mathbb{F}), a \in \text{Mod}(\Delta)\}$ big enough to show that H is a limit group?



- There is a necessary condition: w must have no proper roots in F . Indeed, if say $v^2 = w$, then $x = v * v^{-1} \neq 1$ in H . But

$$f \circ \rho \circ a(v * v^{-1}) = f \circ \rho(v * v^{-1}) = f(v \cdot v^{-1}) = 1.$$

- We saw in Lecture 1 that this condition is sufficient.

30. CONSTRUCTIBLE LIMIT GROUPS

- We describe how to build the groups in the class of *CLGs*. This class will turn out to coincide with the class of limit groups. Among the consequences, limit groups are fp.
- The *envelope* \hat{B} of a rigid vertex group B of a *GAD* Δ of H is the subgroup of H defined as follows:
 1. form a graph of groups $\hat{\Delta}$ for a subgroup of H by replacing each abelian vertex group of Δ with its peripheral subgroup; then
 2. \hat{B} is generated by B and centralizers of edges in $\hat{\Delta}$ that are incident to B .

- $\text{fg } H$ is a *CLG of level 0* if it is free.
- $\text{fg } H$ is a *CLG of level $\leq n + 1$* if either $H = H_1 * H_2$ where H_i are each level $\leq n$ or there is $\rho : H \rightarrow H'$ where H' is level $\leq n$ and a *GAD* for H such that:

1. ρ is injective on the peripheral subgroup of each abelian vertex group;
2. at least one of the inclusions of each edge group E in a vertex group of the one-edge splitting of H induced by E is maximal abelian;
3. ρ is injective on each edge group;
4. the ρ -image of each surface vertex group is non-abelian; and
5. ρ is injective on the envelope of each rigid vertex group.

- Conditions 1–5 are the generalizations of our condition of Section 29.

Example. \mathbb{Z}^n has level 1. ($\rho : \mathbb{Z}^n \rightarrow \langle 0 \rangle$)

Example. A double $H := F *_{\langle w \rangle} F$ where F is free and w has no proper roots in F . (See Section 29.)

Example. Many surface groups are doubles.

Example. A subgroup of a level n group has level $\leq n + 1$.

- Some *CLG* facts that can be proved by induction on levels:

1. *CLGs* are *coherent*, i.e. fg subgroups are fp.

2. *CLGs* H are *CSA*, i.e. maximal abelian M subgroups are *malnormal* ($\eta \in H, M \cap M^\eta \neq 1 \implies \eta \in M$).
3. Abelian subgroups of *CLG*'s are free abelian. Their rank is uniformly bounded.
4. *CLGs* have finite Eilenberg-MacLane spaces.
5. Non-abelian freely indecomposable *CLGs* H have *principal splittings over \mathbb{Z}* , i.e. $H = A *_{\mathbb{Z}} B$ or $H = A *_{\mathbb{Z}}$ and in the *HNN* case \mathbb{Z} is maximal in H .
6. *CLGs* are limit groups.

Sample proof. *CLGs* are coherent.

- A group is *slender* if all subgroups are fg. (Example: \mathbb{Z}^n).
- A finite graph of groups with fp vertex groups and fg edge groups is fp.
- A finite graph of groups with coherent vertex groups and slender edge groups is coherent. (Consider the induced graph of groups for the subgroup.)
- Given $\rho : H \rightarrow H'$ and Δ as in *CLG* definition, Δ is a graph of groups presentation for H with slender edge groups.
- Vertex groups are surface or abelian (therefore coherent) or rigid. Rigid vertex groups inject via ρ into H' , so are coherent by induction. \square

31. LIMIT GROUPS ARE *CLGs*

- A limit group H has a *GAD* Δ_{JSJ} such that $Mod(\Delta_{JSJ}) = Mod(H)$ (more in Section 33).

Proposition. Limit groups H are *CLGs*.

Proof.

- WMA H is generic.
- Let $\{h_i\}$ be a sequence in $Hom(H, \mathbb{F})$ such that h_i is injective on elements of length at most i .
- Let h'_i be shortest in the equivalence class of h_i .

• By Theorem (Shortening), $\rho : H \twoheadrightarrow H' := H / \underline{\text{Ker}} h'_i$ is proper. By induction, H' is a *CLG* of some level n .

• Claim: ρ and $\Delta := \Delta_{JSJ}$ satisfy the conditions in the definition of *CLGs*:

1. Generators of $\text{Mod}(\Delta)$ are trivial on peripheral subgroups. So, ρ is injective on these.
2. Limit groups are commutative transitive. So, edge groups are maximal abelian in one of the incident vertex groups.
3. Generators of $\text{Mod}(\Delta)$ are trivial on edge groups. So, ρ is injective on these.

4. The generators of $Mod(\Delta)$ leave surface vertex groups F invariant (up to conjugacy). So,

$$\rho(F) \text{ abelian} \implies \text{eventually } h'_i(F) \text{ abelian}$$

$$\implies \text{eventually } h_i(F) \text{ abelian,}$$

contradicting $\underline{Ker} h_i = 1$.

5. Similarly, generators of $Mod(\Delta)$ are trivial on envelopes. \square

32. THE GRUSHKO DECOMPOSITION

- As a warm-up to *JSJ*, let's discuss the Grushko decomposition of $\text{fg } H$.
- Suppose Δ and Δ' are graph of groups decompositions of H with trivial edge groups.
- A vertex group V of Δ inherits a graph of groups decomposition from its action on $T_{\Delta'}$.
- If V is not elliptic in $T_{\Delta'}$, i.e. doesn't fix a point, then we may refine Δ by replacing the vertex labeled V by its new graph of groups.

- This process stops since:

Theorem (Grushko). If $H = A * B$, then $\text{rank}(H) = \text{rank}(A) + \text{rank}(B)$.

Corollary (Grushko Decomposition). If H is fg, then $H = H_1 * \cdots * H_n * F$ where each H_i is freely indecomposable and F is free.

- More generally, if Δ and Δ' are arbitrary and if the edge groups of Δ act elliptically on $T_{\Delta'}$, then we may still refine Δ .

33. THE JSJ DECOMPOSITION

- The JSJ -decomposition of a fg group is due to Rips-Sela. We follow Fujiwara-Papasoglou. Another good reference is Dunwoody-Sageev.
- Throughout H is a freely indecomposable limit group. All $GADs$ for H that we consider are assumed to satisfy (*):
 1. edge groups are cyclic; and
 2. maximal non-cyclic abelian subgroups are elliptic.

Lemma. If Δ and Δ' are two one-edge *GADs* for H , then they are elliptic-elliptic or hyperbolic-hyperbolic.

Proof.

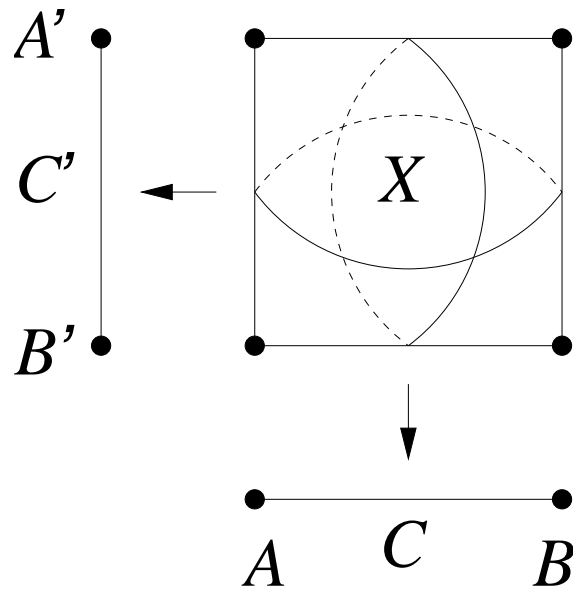
- WMA $\Delta = A *_C B$ and $\Delta' = A' *_C B'$.
- Suppose C is hyperbolic (i.e. not elliptic) in $T_{\Delta'}$ and C' is elliptic in T_{Δ} .
- Consider the action of A' on T_{Δ} . The new edge groups are conjugate to $C \cap C'$ which is trivial.
- Refining Δ' by replacing the vertex A' with its new graph of groups decomposition gives a free decomposition of H , contradiction. \square

Lemma. Suppose that Δ and Δ' are one-edge hyperbolic-hyperbolic GAD s for H . Then, there is a GAD for H with a surface vertex.

Proof idea.

- WMA $\Delta = A *_C B$ and $\Delta' = A' *_C B'$.
- Let T'_A be the minimal A -tree induced by the action of A on $T_{\Delta'}$. Similarly define T'_B and T'_C .
- T'_C/C is a circle.
- Assemble an (orbi)-2-complex X for H by gluing $T'_C/C \times I$ to $(T'_A/A) \sqcup (T'_B/B)$. This complex is a subcomplex of $(T_{\Delta} \times T_{\Delta'})/H$.

- $X \setminus X^{(0)}$ is a surface. □



- Given a finite collection of splittings of H we can refine as in Grushko. Elliptic-elliptic splittings enlarge the graph. Hyperbolic-hyperbolic splittings create surface vertices.

- We want to produce a maximal GAD from the family of one-edge splittings satisfying (*). Why does the refining process stop in this case?
- A simplicial H -tree is *k -acylindrical* if, for $\eta \neq 1$,

$$Diam(Fix(\eta)) \leq k$$

Theorem (Acylindrical Accessibility) (Sela) (Weidmann).

Let H be non-cyclic, fg, and freely indecomposable. Let T be a minimal and k -acylindrical H -tree. Then, T/H has at most $1 + 2k(rank(H) - 1)$ vertices.

- Sela showed that the combinatorics of T/H are bounded. Weidmann provided the explicit bound.

- Our one-edge splittings satisfying (*) are 2-acylindrical and the refinements can be rearranged to be 2-acylindrical.
- Any maximal *GAD* Δ_{JSJ} obtained from one-edge splittings satisfying (*) is a *JSJ GAD*.

Corollary. *JSJ GADs* exist.

Lemma. $Mod(\Delta_{JSJ}) = Mod(H)$

Proof. Generators for $Mod(H)$ are defined in terms of one-edge splittings. □