Spaces of Kleinian Groups
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Boundaries for two-parabolic Schottky groups

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1. Introduction

In this paper we survey various results about the Schottky parameter space for a two parabolic generator group and the smooth boundary for the classical Schottky parameter space lying inside what is known as the Riley slice (of Schottky space).

The problem can be formulated in a number of equivalent different settings: in terms of the topology and the geometry of hyperbolic three-manifolds, in purely algebraic terms, or in a combination of these. Since each of these use terminology that has whose exact meaning has evolved over time, we survey some of the basic terminology for Schottky groups, non-separating disjoint circle groups, noded surfaces and their various representation spaces. This is done in sections 2, 5, 3, and 4 respectively. In the introduction (sections 1.1, 1.2, and 1.3) we state the main theorems taking the liberty of using some terms whose precise definitions are deferred to the later sections. We begin with the algebraic formulation which may be the quickest way to approach the problem.

1.1. Two Parabolics after Lyndon–Ulman

In the Lyndon–Ulman formulation, we consider two by two matrices, S and T where

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix}.$$

We write $T = T_\lambda$ to emphasize that T depends upon the complex number λ. We let G_λ be the group generated by S and T_λ so that $G_\lambda = \langle S, T_\lambda \rangle$.

In 1969, Lyndon and Ulman asked the question, "for what values of λ is G_λ a free group?". They found certain regions in the complex λ-plane that assured that G_λ would be free for λ in one of these regions. The regions are symmetric about the real and imaginary axes. The portions in the first quadrant of three of the five regions they found, F_1, F_2, F_3, are shown in the diagram of figure 1.

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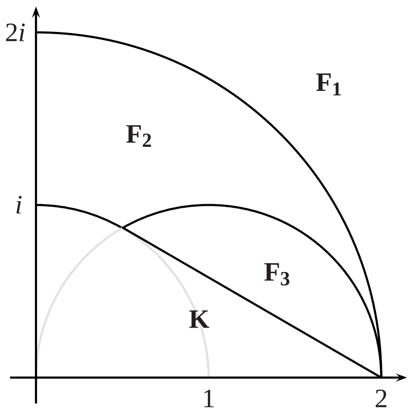

Figure 1: Some of the Lyndon-Ullman Free Region(s) in $\mathbb{C}$.

Around 1980 David Wright wrote a computer program to plot all the points λ for which G_λ is free and discrete. His picture (shown as figure 2) has come to be known as the *Riley Slice of Schottky Space* although under the original definitions of Schottky space it is not technically a slice of Schottky space or even a subset of Schottky space, but rather the quotient of a slice of Schottky space, which we will term *Schottky Parameter Space.*

In Wright's plot the boundary appears to be fractal in nature, but this is not known for a fact [MSW02]. Keen and Series [KS94] have studied moving towards the boundary along the lines that are roughly orthogonal to the boundary and are faintly visible in figure 2. These are called *pleating rays* and geometrically these are points where the pleating locus of the convex hull in $\mathbb{H}^3$ has a particular form (see [KS94]). The geometric condition implies the algebraic condition that the trace of certain words in the generators of G_λ take on real values, but this algebraic condition alone is not sufficient for a point to be on a pleating ray.

1.2. Classical and Non-classical groups

In [GW02] Gilman and Waterman described the parameter space of classical Schottky groups affording two parabolic generators within the larger parameter space of all Schottky groups with two parabolic generators. Two parabolic Schottky groups belong to the boundary of purely loxodromic Schottky space of rank two. The Gilman-Waterman result uses the most general definition of Schottky group is used (see section 2). This gave a smooth boundary (except for two points) whose boundary equations are given by portions of two intersecting parabolas. These are depicted superimposed upon the Wright plot in figure 3 and also in figure 4.

The existence of non-classical Schottky groups was proved by A. Marden in his 1974 paper [Mar74]. However, his proof was not constructive. In 1975 Zarrow ([Zar75]) claimed to give an example of a non-classical Schottky group, but in 1988 Sato showed that Zarrow's example was erroneous. Zarrow's construction only gave a group that was not marked classical (on the given set of generators) but might be classical on a different set of generators [Sat88]. This showed that the verification that an example of a non-classical Schottky group was what it claimed to be would require two steps: show (1) that it was non-classical on the given set of generators and (2) that it remained non-classical under any change of generators.

In 1990 Yamamoto gave an example of a non-classical Schottky group on two loxodromic generators [Yam91]. Unfortunately this has never been well enough understood to lead to any further examples. In [GW02] an explicit construction is given for a one complex parameter family of non-classical two parabolic generated Schottky groups. The construction is not related to Yamamoto's in any obvious manner. Recently Hidalgo and Maskit have given a theoretical construction of other non-classical Schottky groups [HM04].

1.3. Group Theoretic statement

The easiest way to state the main result of [GW02] is group theoretically:

Theorem 1.1. *Let λ be a complex number with $\lambda = |\lambda|e^{i\theta}$ where $0 < \theta < \pi$. The group G_λ is a classical Schottky group*

$$\Leftrightarrow |\lambda|(1+\sin\theta) \geq 2.$$

Here $|\ |$ denotes absolute value.

A two-by-two matrix with complex entries acts as a fractional linear transformation on $\hat{\mathbb{C}}$ and if it has determinant one, we let tr denote its trace. While there are many variations on trace inequalities that imply discreteness, necessary and sufficient trace inequalities for discreteness are not common. Equivalent to Theorem 1.1 is

Theorem 1.2. *Let A and B be two-by-two complex matrices each with determinant one with $A \neq \mathrm{I}$ and $B \neq \mathrm{I}$. The group $G = \langle A, B \rangle$ generated by A and B with $tr(A) = tr(B) = 2$ is a classical Schottky group*

$$\Leftrightarrow |tr(AB) - 2| + |Im[tr(AB)]| \geq 4.$$

Here Im denotes the imaginary part of a complex number and I *denotes the identity matrix.*

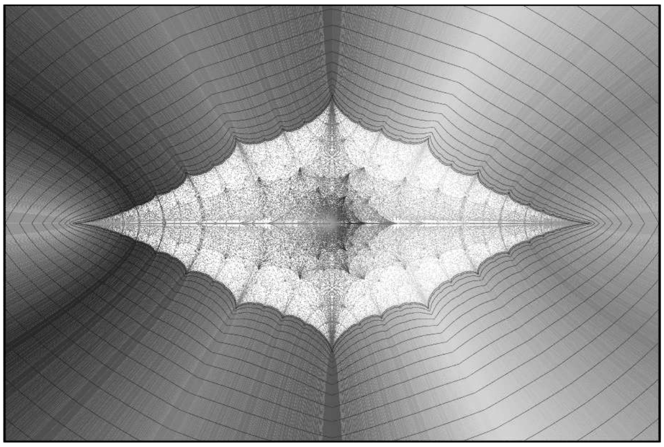

Figure 2: The Riley Slice

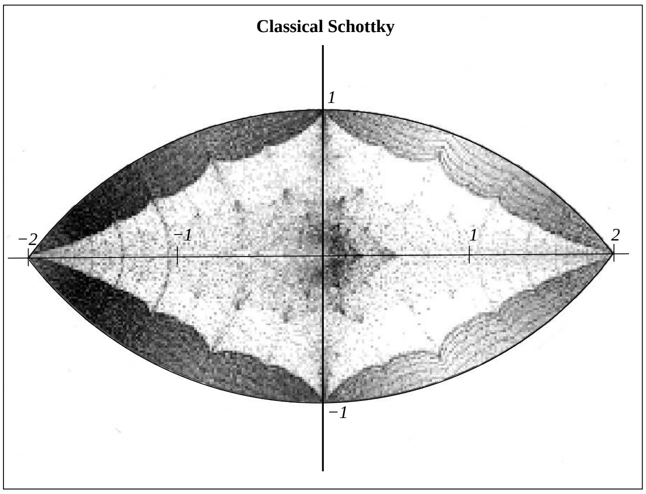

Figure 3: Classical Schottky boundary superimposed on the Riley Slice

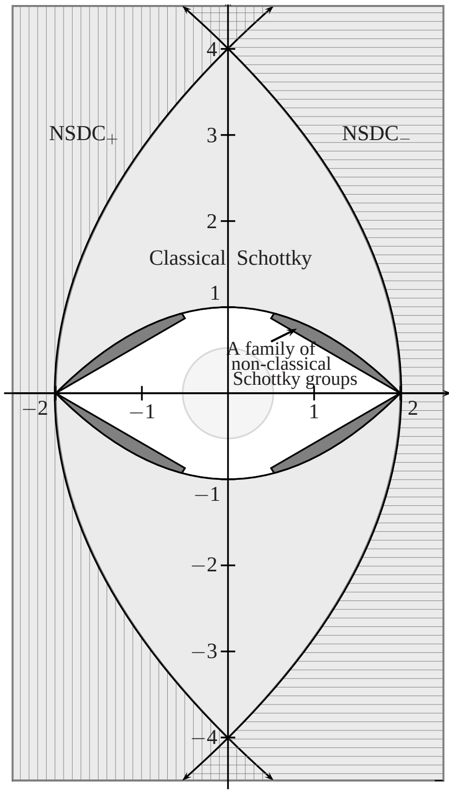

Figure 4: Superimposed Boundary Parabolas. Each point $\lambda \in \mathbb{C}$ corresponds to a two-generator group. The darkest region shows a one parameter family of non-classical Schottky groups. The line-shaded subset of the classical Schottky groups comprises the non-separating disjoint circle groups (NSDC groups). The unshaded region consists of additional non-classical Schottky groups together with degenerate groups, isolated discrete groups and non-discrete groups. Points inside the Shimizu-Leutchbecher-Jørgensen circle ($|\lambda| < \frac{1}{2}$) are non-discrete groups.

Theorem 1.3. *Let $\lambda = x + iy$. The group G_λ is a classical Schottky group and lies on the boundary of classical Schottky parameter space*

$$\Leftrightarrow |y| = 1 - \frac{x^2}{4}.$$

In 1996 Hidalgo [Hid96] proved that for arbitrary genus, the space of Schottky groups with no allowed tangencies between Schottky circles is dense in the space of noded Schottky groups, that is, where tangencies at parabolic fixed points are allowed. Related to the density of Schottky groups with no allowed tangencies between Schottky circles is the work of B. Maskit [Mas81] where it is shown that geometrically finite

Kleinian groups which are free groups belong to the closure of Schottky space.

The proof of Theorem 1.1 in [GW02] shows that using the most general definition of Schottky group (section 2) does not change the interiors or the boundaries of the classical Schottky space for groups generated by a pair of parabolics.

Theorem 1.4. *All variations on the definition of Schottky group yield spaces with the same interior for a group generated by two parabolics. Further, the closures are the same so that the boundaries are the same. For a group generated by two parabolics, the classical Schottky parameter space is closed when the most general definition of Schottky group is used.*

Following the notation of Lyndon-Ullman [LU69], we let K be the convex hull of the set in the λ-plane consisting of the circle $|z| = 1$ and the points $z = \pm 2$. It is shown in [GW02] that if λ is not in the interior of K, then G_λ is Schottky. More specifically, it is proved that

Theorem 1.5. The Non-classical Schottky family. *If λ lies in the upper-half plane below the Schottky parabola $y = 1 - \frac{x^2}{4}$ but exterior to K, then G_λ is a non-classical Schottky group.*

2. Definitions for Schottky groups

A Schottky group is defined as a group where the generators have a certain geometric action on the complex sphere, $\hat{\mathbb{C}}$, that is easily stated. However, the definition of Schottky group has changed over time. Initially a Schottky group of rank n, $n > 0$ and $n \in \mathbb{Z}$, was defined to be a group generated by elements $g_1, \ldots., g_n$ for which there where $2n$ disjoint oriented circles $C_1, C_1', \ldots., C_n, C_n'$ such that $g_i(C_i) = C_i'$ with g_i mapping the exterior of C_i to the interior of C_i', where the common exteriors of the circles bounded a connected region in $\hat{\mathbb{C}}$.

For purely loxodromic groups this definition is a natural extension of the idea of isometric circle. However, for a parabolic transformation, its isometric circle and that of its inverse will be tangent at the fixed point of the transformation. Thus the first generalizations of Schottky group allowed tangencies of paired circles at parabolic fixed points, sometimes called *groups of Schottky type.* Later *marked noded Schottky groups* were studied by Hidalgo. Most recently Hidalgo and Maskit(see [Mas88, Hid96]) have obtained results about neo-classical Schottky groups, which allowed tangencies of Schottky circles, paired or not, as long as the points of tangency are parabolic fixed points. Groups where tangencies of paired Schottky circles at non-parabolic fixed points are allowed were sometimes dubbed [MSW02] *kissing Schottky groups.* Eventually [Rat94] topologists dropped even this restriction and we now allow any tangencies between any Schottky circles, paired or not, at parabolic fixed points or not. It is this definition that allows one to more easily identify the boundary of Schottky space or Schottky parameter space for groups generated by two parabolics. The interiors of these spaces and their boundaries turn out to be the same once the restriction of two parabolic generators is made. Further, some of the technical problems about fundamental domains that motivated the original more restrictive definition can be overcome in this case ([GW02]).

Definition 2.1. A *marked classical Schottky group* is a group G together with a set of n generators $g_1,....,g_n$ for which there are $2n$ circles $C_1,C_1',....,C_n,C_n'$ in $\hat{\mathbb{C}}$ whose interiors are all pairwise disjoint and such that $g_i(C_i)=C_i'$ with g_i mapping the exterior of C_i to the interior of C_i'. A finitely generated group of Möbius transformations G is a *classical Schottky group* if it is a marked classical Schottky group on *some* set of generators.

If the requirement that the C_i,C_i' be circles is dropped and the C_i are only required to be Jordan curves, with non-empty common exterior, then G is called a *marked Schottky group* and a *Schottky group* respectively. A *marked non-classical Schottky group* is a marked Schottky group that is not marked classical Schottky with the given marking. A Schottky group G is a *non-classical Schottky group* if it is not a marked classical Schottky on any set of generators.

Further, the region exterior to the $2n$ circles is called a *(classical) Schottky domain* or *a (classical) Schottky configuration* and the transformations g_i are called the *Schottky pairings.*

Remark 2.2. We observe that classical Schottky domains are not necessarily fundamental domains, but we can push out at tangencies that are not at parabolic fixed points to get a non-classical domain that is a fundamental domain. In [GW02] it is shown that the so called *extreme* domains for two parabolic generator classical Schottky groups are fundamental domains.

Remark 2.3. All Schottky groups are geometrically finite, free, discrete groups by the Klein-Maskit combination theorems or the Poincaré Polyhedron theorem [Mas88].

3. The Geometry and Topology: pinching and nodes

We remind the reader that the hyperbolic three-space is given by,

$$\mathbb{H}^3 = \{\ (x,y,t) \mid x,y,t \in \mathbb{R} \text{ with } t > 0\}$$

together with the metric

$$ds = \frac{\sqrt{dx^2+dy^2+dt^2}}{t}.$$

The boundary of $\mathbb{H}^3$, $\hat{\mathbb{C}} = \{(x,y,0) \in \mathbb{R}^3\} \cup \{\infty\}$ is the complex sphere and is also called the *sphere at* ∞.

We recall that a discrete group of Möbius transformations G acts on $\hat{\mathbb{C}}$ and divides the complex plane into two disjoint regions, the region consisting of points where G acts discontinuously, called *the region of discontinuity of* G and denoted $\Omega(G)$ and its complement $\Lambda(G)$, *the limit set of* G. A group G acts *discontinuously* at a point $z \in \hat{\mathbb{C}}$ if z has a neighborhood U such that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G$ and *freely discontinuously* at the point if $g(U) \cap U \neq \emptyset \Rightarrow g =$ the identity. The group G is *Kleinian* if it acts freely discontinuously at some point z. If the group is Kleinian and the stabilizer of a point $z \in \Omega(G)$ is finite, then U can be replaced by a *precisely invariant* neighborhood of z (i.e. a neighborhood U where $g(U) \cap U \neq \emptyset \Rightarrow g(U) = U$).

Since the discrete group of Möbius transformations G acts discontinously on $\mathbb{H}^3$, one can form the quotient, $\overline{M(G)} = (\mathbb{H}^3 \cup \Omega(G))/G$. The groups we consider are finitely generated so that by the Ahlfors finiteness theorem [Ahl64] $S = \Omega(G)/G$, called the ideal boundary of $M(G)$, is a Riemann surface of finite type or a finite union of such surfaces where every boundary component has a neighborhood conformally equivalent to a punctured disc. It may be that $\Omega(G)$ is empty. If G is torsion free, $M(G) = \mathbb{H}^3/G$ is a hyperbolic three manifold. If G has finite torsion, $M(G)$ is a hyperbolic three-orbifold.

In this paper the groups we consider will be finitely generated and either torsion free or have torsion elements of order two. In the later case appropriate modifications of the following can be made as needed.

If G has a parabolic element with fixed point $p \in \hat{\mathbb{C}}$ and G is torsion free Kleinian, then G_p, the stabilizer of p, is an abelian group of rank 1 or 2.

Let $\pi : \Omega(G) \to S$ be the projection. If p_0 is a puncture on the quotient, it has a neighborhood that lifts to a closed circular disc D_p containing p, a parabolic fixed point. D_p is fixed by every element of G_p, moved to a disjoint disc by every element of G not in G_p, and $D_p \cap \Lambda(G) = \{p\}$. The point p belongs to the boundary of D_p. Such a point p is said to *support a horocycle* and D_p is called a horocyclic neighborhood of p. For every puncture on the quotient there is a lift of the point that supports a horocycle. If p supports two horocyclic neighborhoods that are disjoint except for p, then p is said to support a *double horocycle* or be a *double cusp* and the quotient manifold is said to be doubly cusped at p_0.

A Kleinian group is termed *geometrically finite* if there is a finite sided fundamental polyhedron for its action on hyperbolic three space, $\mathbb{H}^3$. By the Poincaré polyhedron theorem, a geometrically finite group is finitely generated.

In terms of the limit set, G is geometrically finite if and only if every point in the limit set is of one of three types: (i) a rank two parabolic fixed point, (ii) a doubly

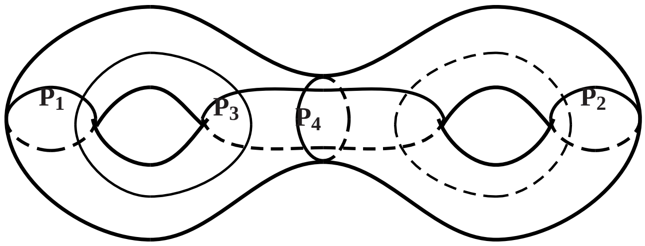

Figure 5: The surface is the quotient of a rank two Schottky group. A noded surface is obtained if the generators are parabolic. In that case the curves P_1 and P_2 are pinched to form a noded surface with two double cusps.

cusped parabolic fixed points or (iii) a point of approximation. It is shown in [Mas88, p. 123] that a parabolic fixed point is not a point of approximation and it is proved in [JMM79] that a finitely generated free Kleinian group is geometrically finite if and only if each of its parabolic fixed points supports a double horocycle. Thus when we consider free, geometrically finite Kleinian groups, we have that all parabolic fixed points are rank one, doubly cusped.

A *noded surface* is one where every point has a neighborhood which is either isomorphic to a disc in $\mathbb{C}$ or to the set $|z| < 1, |w| < 1, zw = 0$ where (z, w) are coordinates in $\mathbb{C}^2$ [Ber74]. The later points are called *nodes.* Geometrically, a parabolic fixed point corresponds to a node if the parabolic is not *accidental* [Mar77].

If G is a rank two Schottky group with no parabolics, topologically $\overline{M(G)}$ is a solid handlebody of genus two and S is a genus two Riemann surface (see figure 5). If G is *generated* by two parabolics, it will have at least two double cusps and topologically (if it has only those two cusps) S will be a sphere with four punctures. The four punctures are identified in pairs. The two double nodes are obtained from the surface in the figure by *pinching* the curves P_1 and P_2 to points. This is the case we study.

For any given group the surface may or may not be further pinched along some curve(s), but not all curves are *pinchable* (see [HM04]). While every noded surface can be obtained by taking algebraic limits of groups corresponding topologically to pinching [Yam79], the results of [GW02] show that

Theorem 3.1. *The boundary of classical two-parabolic Schottky (parameter) space (see section 4) does not generically come from additional pinching. In particular, there are only four points on the boundary, the points $\lambda = \pm i$ and $\lambda = \pm 2$ where the group G_λ corresponds to additional pinching to a parabolic.*

Related to this theorem is the observation of [HM04] that there are only finitely many different topological types of neo-classical Schottky groups of a fixed genus.

4. Schottky Space and the Schottky Parameter Space

We recall the definition of Schottky space from [Mar74, Mar77] and [JMM79] and refer the reader to Marden's papers for an excellent fuller background. We alert the reader to the fact that the use of the term *Schottky space* has also varied over time. Original papers use the term Schottky space for the representation space, which is how we define it below. Some, but not all, recent papers use the term *Schottky space* to refer to the quotient of the space under the conjugation action of $PSL(2,\mathbb{C})$. We term the latter the call the Schottky *parameter* space to avoid confusion. We define these space with some care.

4.1. The Representation Variety

Following Marden [Mar77] and the Jørgensen-Marden-Maskit paper [JMM79] we discuss the representation space of a finitely generated group and complex projective coordinates for these spaces. To shorten the exposition, we assume the group in question is of rank two, but the statements hold for any group (see [GW02]).

• *The space V of representations for a two generator group.* Let G be the group generated by A_1, A_2 a pair of 2×2 of non-singular matrices. If A_i is the matrix $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ then it determines a point in complex projective space of dimension 3 under the map $A_i \mapsto [(a_i, b_i, c_i, d_i)]$ where $[\,]$ denotes the projective equivalence class of the four-tuple. If $\mathbb{CP}_3$ denotes complex projective space of dimension 3, the ordered pair (A_1, A_2) determines a point in ${\mathbb{CP}_3}^2$ as does any ordered pair of matrices in $PSL(2,\mathbb{C})$. The image of $(PSL(2,\mathbb{C}))^2$ is the open set $V = {\mathbb{CP}_3}^2 - \mathbf{P}$ where $\mathbf{P}$ is the subvariety given by $\Pi_{i=1}^2\,(a_i d_i - b_i c_i) = 0$.

• *The space $V(G)$ for a finitely presented group, G.* More generally, let G be a finitely presented group $G = \langle A_1, A_2 | R_m(A_1, A_2) = 1, \mathrm{tr}^2(\mathrm{P_j}) = 4\rangle$ where $P_j = P_j(A_1, A_2)$, $j = 1, \ldots, r$ is a complete set of unique representatives for the conjugacy classes of maximal parabolic elements and $R_m, m = 1, \ldots, k$, is a complete set of relations for the group. Let $V(G)$ be the set of images of A_1, A_2 in ${\mathbb{CP}_3}^2$ as above and let $V(G)^*$ be the set of points in ${\mathbb{CP}_3}^2$ satisfying the polynomial equations induced by the relations R_i and P_j. Then $V(G)^*$ is an algebraic sub-variety, and $V(G)$ is Zariski open with $V(G) = V(G)^* - \mathbf{P}$. $V(G)$ can be represented as an affine algebraic variety, in fact a domain in complex number space of appropriate dimensions.

Each point of $V(G)$ corresponds to a pair of Möbius transformations, (B_1, B_2) satisfying the relations and thus is the image of a homomorphism of G onto the group H generated by the B_i. A homomorphism $\Theta : A_i \mapsto B_i$ that sends each parabolic element either to a parabolic or the identity is called an *allowable* homomorphism. From now on we assume all homomorphisms Θ are allowable and note that $V(G)$ can be written as $V(G) = \{(H,\Theta)|\ \Theta$ is a homomorphism of G onto $H\}$.

The natural topology of $V(G)$ is the topology of point-wise convergence, also called the topology of algebraic convergence. $V(G)$ with this topology is called *the representation space of G*.

• *The space $V_0(G)$ of $PSL(2,\mathbb{C})$ conjugacy classes of representations.* An element $h \in PSL(2,\mathbb{C})$ induces an action on $V(G)$ given by conjugation, (that is, $(H,\Theta) \mapsto (hHh^{-1}, h\circ\Theta\circ h^{-1})$). We let $V_0(G)$ be the quotient of $V(G)$ under this action and call it the *representation parameter space.*

4.2. Spaces of two-generator groups

We can define additional spaces related to $V(G)$ by putting conditions on the group G or the group H or the map Θ and we use the notation $V(G)_X$ to denote that H satisfies condition X. We want to attach names to the spaces defined using this notation. (See section 5 for the definition of nsdc group.)

Consider a fixed group $\widehat{G} = \langle A_1, A_2\rangle$ so that $\widehat{G}$ has no R_i or P_j relations, that is, a free group of rank two. We define

NSDC-space: $V(\widehat{G})_{nsdc} = \{(H,\Theta) \in V(\widehat{G})|\ H$ is of nsdc-type$\}$

and its quotient NSDC-parameter space: $V_0(\widehat{G})_{nsdc} = V(\widehat{G})_{nsdc}/PSL(2,\mathbb{C})$.

Schottky space: $V(\widehat{G})_{Schottky} = \{(H,\Theta) \in V(\widehat{G}) \mid H$ is a Schottky Group$\}$

and its quotient *Schottky parameter space*: $V_0(\widehat{G})_{Schottky} = V(\widehat{G})_{Schottky}/PSL(2,\mathbb{C})$

Classical Schottky Space: $V(\widehat{G})_{classical} = \{(H,\Theta) \in V(\widehat{G}) \mid H$ is a Classical Schottky Group.$\}$

and its quotient *Classical Schottky parameter space*, $V_0(\widehat{G})_{classical} = V(\widehat{G})_{classical}/PSL(2,\mathbb{C})$.

Next, consider a non-elementary free Kleinian group $G^{\mathcal{P}}$ generated by two parabolic transformations A_1 and A_2 without any additional parabolic transformations (i.e. any parabolic transformation is conjugate to a power of A_1 or A_2). $G^{\mathcal{P}} = \langle A_1, A_2 \mid \mathrm{tr}^2\, \mathrm{A}_1 = \mathrm{tr}^2\, \mathrm{A}_2 = 4\rangle$. Such a group is often called a *Riley group* [MSW02]. We call $V(G^{\mathcal{P}})$ the *two-parabolic representation space.* It is a slice of the representation space. We have

Two-parabolic Schottky space:

$V(G^{\mathcal{P}})_{Schottky} = \{(H,\Theta) \in V(G^{\mathcal{P}})\ |H$ is a Schottky group$\}$

Two-parabolic classical Schottky space:

$$V(G^{\mathcal{P}})_{classical} = \{(H,\Theta) \in V(G^{\mathcal{P}}) \, | H \text{ is a classical Schottky group}\}.$$

Marked two parabolic classical Schottky space:

$$V(G^{\mathcal{P}})_{marked\ classical} = \{(H,\Theta) \in V(G^{\mathcal{P}}) \mid H \text{ is marked classical on } (\Theta(A_1),\Theta(A_2))\}$$

Non-classical two-parabolic Schottky space:

$$V(G^{\mathcal{P}})_{non-classical} = \{(H,\Theta) \in V(G^{\mathcal{P}}) \, | H \text{ is a non-classical Schottky group}\}$$

It is clear that the two parabolic representation space, $V(G^{\mathcal{P}})$ is a slice of the rank two representation space $V(\widehat{G})$ and that two-parabolic Schottky space $V(G^{\mathcal{P}})_{Schottky}$ is a slice of Schottky space $V(\widehat{G})_{Schottky}$. Since trace is a conjugacy invariant, we have $V_0(G^{\mathcal{P}})_{Schottky} \subset V_0(\widehat{G})_{Schottky}$. It is easy to see that any non-elementary Möbius group generated by two parabolics is conjugate to a group of the form G_λ for some λ where $\mathrm{tr}[\mathrm{A}_1,\mathrm{A}_2] = 4\lambda^2$. One can take λ to be the complex parameter for the Schottky parameter space $V_0(G^{\mathcal{P}})_{Schottky}$.

5. Non-separating disjoint circle groups

The precise definition of non-separating disjoint circle groups appears below in section 5.2. We end this paper with a summary of some related results for non-separating disjoint circle groups for two reasons: (1) the theory here motivated the theory for two-parabolic Schottky groups by indicating how allowing tangencies between any Schottky circles, at non-parabolic points or at parabolic fixed points, made it possible to identify the boundary of the space and (2) it is easy to outline the method here so as to illustrate when non-parabolic tangencies can be *pulled apart.* Troels Jørgensen pointed out the fact that allowing such arbitrary tangencies would yield the tear drop boundary of figure 7.

5.1. $\mathbb{H}^3$ geometry

A hyperbolic line is determined by two points and has two ends on the sphere at ∞. We follow the notation of [Fen89] and denote a hyperbolic line by $[v,v']$ where v and v' may lie on $\hat{\mathbb{C}}$ or in $\mathbb{H}^3$. An elliptic or loxodromic transformation fixes a unique hyperbolic line, known as its *axis.* A parabolic transformation fixes a unique point point on $\hat{\mathbb{C}}$. We may see that parabolic fixed point as degeneration of the axis of either a loxodromic or elliptic transformation.

Given any hyperbolic line, L, there is a unique hyperbolic isometry that fixes the line and rotates points in $\mathbb{H}^3$ by an angle of π about the line and which also acts on the boundary. We call this transformation the *half-turn* about L and denote it by H_L.

5.2. Definitions and basic facts

Non-separating disjoint circle groups, known as *nsdc groups* for short, were first defined in [Gil97]. Like Schottky groups, nsdc groups are geometrically defined groups. To define them, we begin with the definition of the *ortho-end* of a group.

For a two generator group $G = \langle A, B\rangle$, we define N, *the perpendicular to A and B* to be the hyperbolic line in $\mathbb{H}^3$ that is the common perpendicular to the axis of A and B if A and B are either loxodromic or elliptic. If either A or B or both are parabolic, we define N to be the perpendicular from the parabolic fix point to the other axis or to the other parabolic fixed point. We assume that G is non-elementary so that the axes of A and B always have a common perpendicular.

We associate to (A,B) its *ortho-end*, the six-tuple of complex numbers (a,a',n,n',b,b') where A factors as $H_{[a,a']}H_{[n,n']}$ and B factors as $H_{[n,n']}H_{[b,b']}$ where N has ends n and n', the axis of A has ends a and a' and the axis of B has ends b and b' (see [Fen89] for more details about this notation). Conversely, an ordered six-tuple of numbers $(a,a',n,n',b,b') \in \hat{\mathbb{C}}^6$ determines an ordered pair of matrices (A,B) where $A = H_{[a,a']}H_{[n,n']}$ and $B = H_{[n,n']}H_{[b,b']}$.

Definition 5.1. We say that a point in $\hat{\mathbb{C}}^6$ has *the non-separating disjoint circle property* if there exist circles C_A, C_D and C_B, with disjoint interiors where C_A passes through a and a', C_B passes through b and b' and C_D passes through n and n' and where no one of the three circles separates the other two. We allow the possibility that some of the three circles are tangent to others.

We say that G is a *marked nsdc group* if its ortho-end has the nsdc property and G is an *nsdc group* if some ortho-end has the nsdc property.

For an non-elementary group G the ortho-end of the group can be defined whether or not the group is discrete. It is shown in [Gil97] that an nsdc group is always discrete. Further it is shown that an nsdc group is free and is always a classical Schottky group. It is clear that one can easily pass back and forth between the matrix entries for A and B to the ortho-ends.

Geometrically when G is nsdc, the three generator group $3G = \langle H_{[a,a']}, H_{[n,n']}, H_{[b,b']}\rangle$ has as its quotient an orbifold whose singular set is the image of three hyperbolic lines. There is a natural projection from $\mathbb{H}^3/G \to \mathbb{H}^3/(3G)$ that comes from factoring out by the action of a hyperbolic isometry of order two.

Remark 5.2. If we begin with a group $G = \langle A, B\rangle$ for which the three circles C_A, C_B and C_D are pairwise disjoint and no one separates the other two, then the group $3G$ is a geometrically finite function group uniformizing a sphere with exactly six points of order two. Such a group is called a Whittaker group of genus two. These have been studied by [Kee80] and more recently by [GGD04]. Each Whittaker group of genus two contains a Schottky group of genus two as an index two subgroup. The hyperellptic involution of the uniformized genus two surface is induced by any of the elements of order two in the Whittaker group. Conversely each genus two surface can be obtained in this way. As any simple closed geodesic on a genus two surface is invariant under the hyperelliptic involution, we have this phenomena is still valid after degeneration to the two parabolic Schottky group: the Whittaker groups then degenerate to $3G$ groups with tangencies.

5.3. Methods

The nsdc boundary tear drop (figure 7) is found by analyzing configurations of circles, attaching appropriate parameters to the configurations, and then relating their geometry to the entries in the matrices of the generators for the group. This method is also used to find the boundary for Riley groups that are Schottky groups, but the latter proof involves many more technicalities. The main tools for finding the boundary in the nsdc case are analytic geometry and the inverse function theorem. We give some of the details of this case to illustrate the method and the idea of *pulling tangencies apart.*

An nsdc group generated by two parabolics involves one free parameter $d = x + iy$. The six-tuple of points are $(-2, 0, 2, d, 0, 2)$. The circles C_A and C_B are tangent at the point 0 and, therefore, determine an angle τ. This is the angle that the line connecting their centers makes with the positive x-axis moving in a counter-clockwise direction (see figure 6). Conversely, any angle τ between $\pi/4$ and $-\pi/4$ determines such a pair of tangent circles. Any circle C_D passing through 2 and d will have a center with coordinates (M, N) and radius r. If C_D is tangent to C_A at a point T, then the coordinates of the point (M, N) can be computed as explicit functions of τ as can the radius r. Points on the circle C_D, known as the τ-circle are given by $(x, y) = (M(\tau) + r_\tau \cdot \cos t, N(\tau) + r_\tau \cdot \sin t)$, for some real parameter t, $0 \leq t \leq 2\pi$.

If C_D is not required to be tangent to C_A but only to pass through d and be tangent to C_B so that we have an nsdc triple, then it is clear that for $d' = (x', y')$ in a small circular neighborhood of $d = (x, y)$ there are circles through (x', y') so that $(-2, 0, 2, d', 0, 2)$ are still nsdc. This shows that non-tangent C_D's correspond to interior points of *NSDC*-space.

If C_D is required to be tangent to C_A, we have that for some d' near the point on $d = (x, y)$ on the τ-circle, C_D, this may or may not be the case.

We consider all points on the τ-circle. One has $x = x(\tau, t) = M(\tau) + r_\tau \cdot \cos t$ and $y = y(\tau, t) = N(\tau) + r_\tau \cdot \sin t$.

Thus one has a map from $\mathbb{R}^2$ to itself: $(\tau, t) \mapsto (x, y)$. One can calculate the Ja-

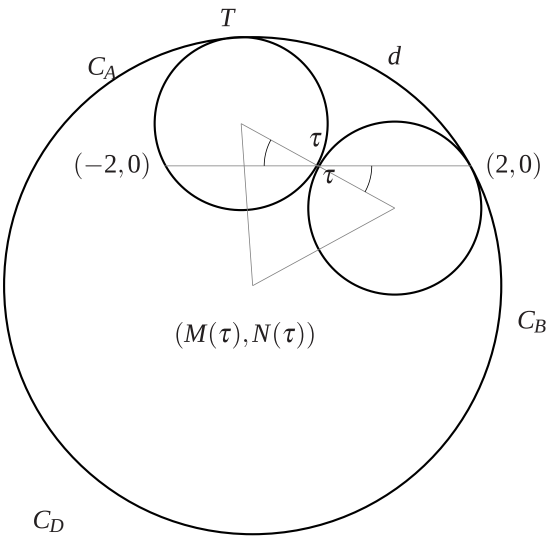

Figure 6: τ - circle configuration with C_D and C_A tangent

cobian of this map to show that (x_0, y_0) is a boundary point of NSDC space precisely when $x_0 = 2 - \frac{\sin^2\tau}{1+\sin^2\tau}$ and $y_0 = 8\frac{\sin^3\tau\cos\tau}{1+\sin^2\tau}$. The plot of this boundary is the tear drop shown in figure 7. Points $d = x + iy$ in the interior of the tear drop corresponds to (marked) groups G_d that are not NSDC and points in the exterior to G_d that are NSDC groups. A change of parameters maps the tear drop into a parabola and replaces the parameter d by $\lambda = \frac{4d}{d-2}$. Taking marked and non-marked nsdc groups into account yields the region bounded by the two parabolas pictured in figure 4 as $\mathcal{NSDC}$-space, that is $\{G_\lambda \mid G_\lambda$ or $G_{-\lambda}$ is an nsdc group$\}$.

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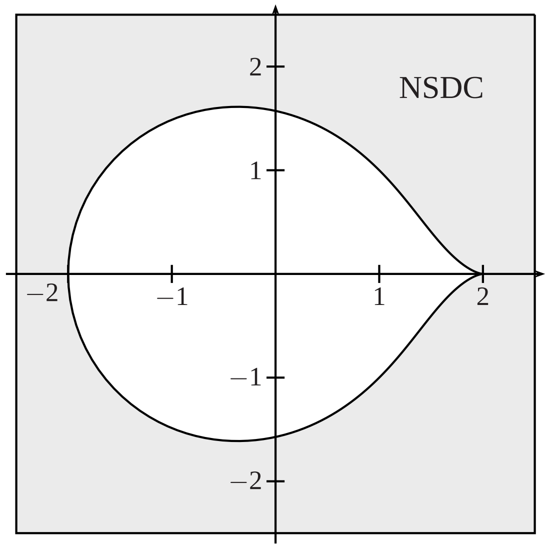

Figure 7: The NSDC tear drop. Points *d* in the complex plane that lie exterior to the tear drop parameterize groups that are discrete and of non-separating disjoint circle type.

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