

## Informative Words and Discreteness

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*Dedicated to Gerhard Rosenberger on his 60<sup>th</sup> birthday.*

ABSTRACT. There are certain families of words and word sequences (words in the generators of a two-generator group) that arise frequently in the Teichmüller theory of hyperbolic three-manifolds and Kleinian and Fuchsian groups and in the discreteness problem for two generator matrix groups. We survey some of the families of such words and sequences: the semigroup of so called *good* words of Gehring-Martin, the so called *killer* words of Gabai-Meyerhoff-NThurston, the Farey words of Keen-Series and Minsky, the discreteness-algorithm Fibonacci sequences of Gilman-Jiang and *parabolic dust* words. We survey connections between the families and establish a new connection between good words and Farey words.

In 1970's and 80's Gerhard Rosenberger and his collaborators developed a theory of discreteness for two generator subgroups of  $PSL(2, \mathbb{R})$ . Essential to the theory was the concept of replacing pairs of successive generators by Nielsen equivalent pairs in a *trace minimizing* manner (in order to reach a pair where a discreteness determination could be made). Although they did not use the word *algorithm*, algorithms and sequences of words were central to their conceptualization of the discreteness problem. In this paper we survey a number of areas where their concepts have had an impact upon the discreteness problem both directly or indirectly. A partial list of the papers they wrote are included in the bibliography.

### 1. Introduction

Classical Riemann surface theory is the basis of the development of a lot of modern mathematics. In the last few decades greater interest has grown in the theory of hyperbolic geometry and three-manifolds. Instead of studying the surfaces or hyperbolic three-manifolds, one can equivalently study the theory of their uniformizing groups, Fuchsian groups and Kleinian groups, respectively. Because questions about these discrete matrix groups are often intractable, families of surfaces or three-manifolds or their groups are studied by studying their Teichmüller

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spaces or their representation spaces or parameter spaces. Recent important results about (appropriately defined) parameter spaces and their boundaries have been obtained by Gabai-Meyerhoff-N.Thurston, Gehring-Martin, Keen-Series-Maskit, and Minsky. (See [7], [8], [9], [20], [25], [26], and the references given there.) During this period, algebraic geometers and researchers in symbolic computation have been working on creating user friendly programs for computing with on algebraic curves. (See the work of Buser and Seppälä [1], [3], [4], and the references given there.) A further thread has been the development of algorithms for determining the discreteness or non-discreteness of two-generator subgroups of real matrix groups. (See the work of Gilman, Gilman-Maskit and Jiang in addition to the papers of Purzitsky and Rosenberger [12], [13], [17], [18], [27], [28], and [30].) Here one of the main results is the existence of algorithms that solve the  $PSL(2, \mathbb{R})$  discreteness problem. The algorithms can be given in several different forms. In some cases they can be implemented on a computer if, for example, the entries in the matrix are assumed to be algebraic numbers lying in a finite extension of the rationals given in terms of their minimal polynomials [12]. It can also be shown that (when appropriately defined) the discreteness problem can not be solved without an algorithm [13].

We view the results of Gerhing-Martin, Gilman-Maskit-Jiang, Keen-Series, Gabai-Meyerhoff-N.Thurston, Minsky, et al as being connected to the results of Buser, Seppälä, et al and their project of computing on algebraic Riemann surfaces (CARS) and algorithm questions through the observation that all are studying families of words in two-generator groups and that these are precisely the families of words that any algorithm in  $PSL(2, \mathbb{C})$  would require.

The families of words studied include *Farey words*, so called *good words*, so called *killer words*, Fibonacci and non-Fibonacci word sequences, the algorithmic words and sequences of pairs primitive associates in free groups. Precise definitions are given in section 3.

A current goal is to put all of the different families of words into a unified algebraic context. That is, is to establish connections between the different families of words and to apply these connections to obtain further results in the theory of discrete groups, especially for  $PSL(2, \mathbb{C})$  and results applicable to computing on three-manifolds and algebraic curves. For example, to address such problems as finding short geodesics on Riemann surfaces or finding a discreteness algorithms in the case of complex matrix entries.

In this paper we survey some of the families of words that are currently in use and known connections among them. We also establish some new connections. The words begin with Rosenberger and the concepts of replacing generators by Nielsen equivalent generators in a *trace minimizing manner*. The replacement in a trace minimizing manner is an algorithm. Both the algorithmic nature of this problem and the trace minimizing are significant.

## 2. Overview

Let  $\mathbb{M}$  denote the group of all Möbius transformations of the extended complex plane  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . We associate with a Möbius transformation

$$f = \frac{az + b}{cz + d} \in \mathbb{M}, \quad ad - bc = 1,$$

one of the two matrices that induce the action of  $f$  on the extended plane

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$$

and set  $\text{tr}(f) = \text{tr}(A)$  where  $\text{tr}(A)$  denotes the trace of  $A$ . For a two generator group, once a matrix corresponding to each generator is chosen, the matrices associated to all other elements of the group are determined and thus their traces are determined. Elements of  $\mathbb{M}$  are classified as loxodromic, elliptic, hyperbolic or parabolic according to the nature of the square of their traces.

A subgroup  $G$  of  $\mathbb{M}$  is discrete if no sequence of distinct elements of  $G$  converges to the identity. A discrete non-elementary subgroup of  $\mathbb{M}$  is a *Kleinian* group. If the matrix entries are real (or if the group is conjugate to such a group), then such a discrete group is called a *Fuchsian* group.

The problem of determining discreteness for even a two-generator real matrix group is non-trivial. It turns out, that if  $G = \langle A, B \rangle$  is generated by  $A$  and  $B$  in  $\mathbb{M}$ , an algorithm is needed to determine discreteness [13]. The discreteness algorithm inputs  $A$  and  $B$  and outputs one of two statements: the group  $G$  generated by the matrices is *discrete* or the group is *not discrete*. The algorithm appears in a number of different forms. A corresponding algorithm for two matrices in  $PSL(2, \mathbb{C})$  is not known to exist.

### 3. Informative Words

We summarize types words that have been used to obtain significant results about Kleinian groups and their representation space(s).

**3.1. Primitive words.** Let  $W = W(A, B)$  denote a word in the generators  $A$  and  $B$  of  $G = \langle A, B \rangle$  so that

$$W(A, B) = A^{u_1} B^{v_1} A^{u_2} \dots B^{v_t} \text{ for some integers } u_1, \dots, u_t; v_1, \dots, v_t. \quad (*)$$

We are primarily interested in *primitive* words, words that can be extended to a minimal set of generators for the group. In the case of primitive words that generate a two-generator free group, we may assume that up to cyclic permutation and interchanging the generators there is a unique shortest word where the  $u_i$  are all equal to each other and are all  $\pm 1$ . That is, we can apply results from [24] to see that if  $G = \langle A, B \rangle$  is a free group on two generators, then a primitive word can be written in a unique canonical form  $A^{-1} B^{v_1} A^{-1} B^{v_2} A \dots$  or the equivalent with  $A$  replacing  $A^{-1}$  and/or  $A$  and  $B$  interchanged. We call the  $v_i$  the *primitive exponents*.

**3.2. Algorithmic words.** In the geometric algorithm [12, 13] if discreteness or non-discreteness cannot be determined directly from the pair of generators, then the pair is replaced by a new pair. At step  $k$ , the generating pair  $A_k, B_k$  is replaced by a new pair  $A_{k+1}, B_{k+1}$ . However, one of the new generators is the same as one of the previous generator. Thus the sequence of the *algorithm words* refers to the sequence of new generators.

The fact that the replacement procedure stops to give an algorithm as opposed to a procedure (something that does not necessarily stop) depends upon the fact that the algorithm is trace minimizing.

There are two types of steps that occur in the algorithm, steps that increase the word length in a linear manner and steps that make the word length increase as

a Fibonacci sequence [13]. They are termed Fibonacci/non-Fibonacci steps. The size of the input to the algorithm is measured by the maximal initial trace. In the analysis of the algorithm, Jiang shows that when the word length increases exponentially in a Fibonacci step, the absolute value of trace of the word ([18]) decreases logarithmically. For non-Fibonacci steps both are linear. This is key to proving that the algorithm has polynomial complexity.

**3.3. Farey Words.** Farey words have been used by Keen and Series [20] to find the boundary of the Riley slice of Schottky space. Farey words also play a role in recent work of Minsky [26].

*Farey words* (apres the mathematician Farey) are related to continued fraction expansions and the tessellation generated by the action of  $PSL(2, \mathbb{Z})$  on the upper half plane [32]. If  $p/q$  is a rational number between 0 and 1 and if  $[a_0, \dots, a_k]$  is the continued fraction expansion so that  $\frac{p}{q} = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}}$ , then there is a word,

denoted by  $W_{p/q}$  which is the *Farey word* corresponding to  $[a_0, \dots, a_k]$ . Further it is shown in [20] that there is another sequence of integers  $[v_1, \dots, v_t]$  obtained from  $[a_0, \dots, a_k]$  so that  $W_{p/q}$  is equal to the word  $W(A, B)$  with primitive exponents  $[v_1, \dots, v_t]$  where  $t = p$ ,  $u_i = -1 \forall i$  and  $\sum v_i = q$ . Farey words that are neighbors, (i.e. that satisfy  $|ps - rq| = 1$ ), can be added with the following Farey addition (when  $\frac{p}{q} < \frac{r}{s}$ ):  $W_{\frac{p}{q}} +_{\mathcal{F}} W_{\frac{r}{s}} = W_{\frac{p+r}{q+s}}$ .

We write  $\frac{p}{q} = [a_0, a_1, \dots, a_k]$  to denote the fact that  $\frac{p}{q}$  has continued fraction expansion  $[a_0, a_1, \dots, a_k]$ .

**3.4. Trace Polynomials.** We recall the notion of a *trace polynomial*. The trace of a word  $W = W(A, B)$  can be described by a polynomial in  $\text{tr}(A)$ ,  $\text{tr}(B)$  and  $\text{tr}([A, B])$  or  $\text{tr}(AB)$  where  $[A, B]$  is the multiplicative commutator of  $A$  and  $B$ . For example, for  $h \in \mathbb{M}$  as above,  $\text{tr}(h^2)$  is given by the polynomial  $p(x) = x^2 - 2$  since  $\text{tr}(h^2) = (\text{tr}h)^2 - 2$ . So families of words give rise to families of so called trace polynomials, polynomials in the three variables:  $\text{tr}A$ ,  $\text{tr}B$ ,  $\text{tr}[A, B]$ . In addition to using different families of words, different families of parameters are also used in different settings. Thus the trace polynomials can also be viewed as polynomials in the variables  $\beta(f) = (\text{tr}^2 f - 4)$  and  $\gamma(f, g) = \text{tr}[f, g] - 2$ .

In the case that  $G$  is discrete and Fuchsian, then the axis of a hyperbolic element of  $G$  projects onto a geodesic on the quotient surface and  $\text{tr}(A)$  gives the length  $T$  of this geodesic (specifically,  $|\text{tr}A| = 2 \cosh(\frac{1}{2}T)$ ). Thus trace polynomials are essentially lengths of geodesics. A similar statement holds for the complex length of a loxodromic element.

**3.5. Good words.** *Good Words* were defined by Gehring and Martin and used to obtain results about the  $\gamma\beta$  parameter space and minimal volume three-manifolds. If  $G = \langle A, B \rangle$  is the free group on  $A$  and  $B$ , then a *good word* is a word in  $A$  and  $B$  that starts and ends with a power of  $A$  or its inverse and the exponents of the  $A$ 's oscillate in sign. Thus using the notation in formula \*,  $u_1 = \pm 1$  and  $u_{i+1} = -u_i$  and  $g_i = v_i$ ,  $v_i \neq 0$  if  $i \neq t$ , for example, a good word can be written as  $A^{-1}B^{g_1}AB^{g_2}A^{-1}B^{g_3} \dots A^{-1}$  [25].

We let  $\mathcal{W}$  denote the family of *good words*. These good words give rise to a family of trace polynomials,  $\{\mathcal{P}_{W \text{ good}}\}$ . Good words can be composed and this gives  $\mathcal{W}$  a semi-group structure. If  $W_1(A, B)$  and  $W_2(A, B)$  are good words, then

$W_1 *_{\mathcal{G}} W_2 = W_1(W_2(A, B), B)$ . That is  $W_2(A, B)$  is substituted into  $W_1(A, B)$  wherever  $A$  occurs in  $W_1$ . To be more precise

**THEOREM 3.1 (Gehring-Martin).** *Given  $W \in \mathcal{W}$ , then  $\exists P_W$ , a polynomial with integer coefficients, such that if  $\rho : F_2 \rightarrow \mathbb{M}$  with  $f = \rho(a), g = \rho(b), h = \rho(w)$  with  $\gamma = \text{tr}[f, g] - 2, \beta = \beta(f)$ , then  $P_W(\gamma, \beta) = \gamma(f, h)$ .*

Thus  $P_{W_1 *_{\mathcal{G}} W_2}(\gamma, \beta) = P_{W_1}(P_{W_2}(\gamma, \beta), \beta)$ . Since for any  $f$  and  $g$  in  $\mathbb{M}$  one can regard the trace parameters, as being  $\beta(f) = (\text{tr}f)^2 - 4$ ,  $\beta(g) = (\text{tr}g)^2 - 4$  and  $\gamma(f, g) = \text{tr}[f, g] - 2$ , This gives a new  $\gamma$  parameter, a new way of moving around parameter space and it differs from the traditional way of producing new  $\gamma$ 's.

**3.6. Killer Words.** *Killer words* arise in [7], historically the first paper to prove a result about three-manifold theory using computer implementations. That work builds upon ideas initially developed by Riley [29]. The technique in [7] is to divide the potential representation space into grids. A *killer word* is a word in the generators whose trace becomes very small on a given grid. Jørgensen's inequality shows that a group is not discrete if the trace(s) of some word(s) are small enough [19]. A procedure is used to find killer words for appropriate grids, thus ruling out points in a grid possessing a killer word as potential parameters for a discrete group. An exact description of the *killer words* is not really known for not all of the killer words used in [7] come from a fixed algorithm. The program produces some by trial and error. It would be nice to bring a theory to the killer words. Gaven Martin and T. Marshall are in the process of finding sets of killer words that are good words.

**3.7. The question.** The main question is

Can one establish connections between all of these disparate families of informative words?

Typical of the type of connection we hope to make is the following. We observe that a good word in  $A$  and  $B$  is a Farey word  $W(C, D)$  in the generators of the group  $G_0$  where  $C = A, D = BAB^{-1}$  and  $G_0 = \langle A, BAB^{-1} \rangle$ . Thus results applied to Farey words in  $G$  and the boundary of its representation space can be applied to the Farey words in  $G_0$  and the boundary of the representation space of  $G_0$  and vice-versa. A related question is,

Is there a corresponding connection between  $+_{\mathcal{F}}$ , addition of Farey neighbors, and  $*_{\mathcal{G}}$  good word multiplication?

## 4. Connections

As noted earlier G. Martin and T. Marshall are working on connections between good words and killer words. Connections between algorithmic words and Farey words and between short geodesics and the complexity of the algorithm were found in [16]. They are summarized below after some additional notation and terminology are introduced. Connections between good words and Farey words are mentioned. More details of the latter will appear in [14].

**4.1. The  $F$ -sequence.** The algorithm begins with a pair of generators  $(A, B)$  (which have been appropriately normalized by interchanging  $A$  and  $B$  and replacing  $A$  and  $B$  by  $A^{\pm 1}$  and  $B^{\pm 1}$  as necessary. If it does not stop and say " $G$  is discrete" or " $G$  is not discrete", it determines *the* next pair of generators, which is either

$((A^{-1}B)^{-1}, B)$  or  $(B^{-1}, A^{-1}B)$ . The former type of replacement is called a linear or non-Fibonacci step and the latter a Fibonacci step (see [13] and [18]). The algorithm dictates which type of step is the next step. Thus one can associate to the algorithm its *F-sequence* the numbers  $[n_1, n_2, \dots, n_k]$  where  $n_i$  denotes the number of consecutive linear steps before the next Fibonacci step (see [16]).

If we have a primitive word in  $A$  and  $B$  given by its F-sequence, we can multiply the word out and write it in terms of its primitive exponents, the  $v_i$ . For example, we can expand the word  $(A^{-1}B^{n_1}) \cdot [B \cdot ((A^{-1} \cdot B^{n_1})^{n_2})^{n_3}] \dots$  in the form  $B^{v_0}A^{-1}B^{v_1}A^{-1}B^{v_2}A^{-1}B^{v_3} \dots B^{v_w}$  for some integer  $w$ . The primitive exponents can be determined by the F-sequence and vice-versa ([16]).

**4.2. Continued fraction expansions and the left-right sequence.** A Farey word,  $W_{p/q} = W(A, B)$ , corresponds to a geodesic connecting the point at infinity to the rational point  $p/q$  on the boundary of the upper-half-plane. It will cross successive triangles in the  $PSL(2, \mathbb{Z})$  tessellation. After replacing the tessellation by that of an appropriate subgroup of  $PSL(2, \mathbb{Z})$ , such a geodesic when exiting a given triangle, either enters the next left triangle or the next right triangle and thus corresponds to a sequence of lefts and rights, denoted by  $L^{m_0}R^{m_1}L^{m_2} \dots R^{m_j}$ . The sequence  $m_0, \dots, m_j$  is the same as the sequence of the  $a_i$ 's in the continued fraction expansion ([32]).

**4.3. Connections: continued fraction expansions, Farey words and algorithmic words.** In [16] it was established that the primitive exponents, the F-sequence, the continued fraction expansion and the left-right sequences are all related. The F-sequence, the left-right sequence and the continued fraction expansion are all essentially the same. Let  $G = \langle A, B \rangle$  and  $(C, D)$  be a pair of primitive words in  $A$  and  $B$ . Then  $C$  and  $D$  are *primitive associates* if  $G = \langle C, D \rangle$ .

**THEOREM 4.1 (Gilman-Keen).** *If the F-sequence of the algorithm is  $[n_1, \dots, n_k]$ , let  $\frac{p}{q}$  be the rational number with continued fraction expansion  $[n_1, \dots, n_k]$  and  $\frac{r}{s}$  the rational number with continued fraction expansion  $[n_1, \dots, n_{k-1}]$ . Let  $(C, D) = (A_k, B_k)$  be the pair of generators at which the algorithm stops. Then  $C$  is the Farey word  $W_{\frac{p}{q}}(A, B)$  and  $D$  is the Farey word  $W_{\frac{r}{s}}(A, B)$ .*

**4.4. Connections: computational complexity of the algorithm and short geodesics.** The computational complexity of the  $PSL(2, \mathbb{R})$  algorithm depends upon the F-sequence [13, 18]. The size of the input is measured by the trace of the initial generators. The polynomial bound on complexity of the algorithm was established by YC Jiang [18] and the proof depended upon showing the fact that while Fibonacci steps made the length of the words considered by the algorithm grow exponentially, in that case the traces of the words considered decreased logarithmically (as opposed to linearly).

When the group is discrete, words in the  $PSL(2, \mathbb{R})$  algorithm correspond to geodesics on the surface. Steps in the algorithm can be thought of as *unwinding one curve about another*. A *simple* geodesic has no self-intersections. At step  $k$ , the words  $(A_k, B_k)$  represent curves on the surface with fewer self-intersections than the words  $(A_{k-1}, B_{k-1})$ . The F-sequence of the algorithm gives its computational complexity and it also can be used to assign a *complexity* to the curves on the surface that the algorithm visits. The *complexity* of a curve is the number of

essential self-intersections and roughly speaking, as these decrease the length of the geodesic decreases.

The axes of  $A$  and  $B$  are disjoint when the trace of the multiplicative commutator of  $A$  and  $B$  is positive. For this case

**THEOREM 4.2 (Gilman-Keen).** *When the algorithm stops saying that  $G$  is discrete, the algorithm has found the three shortest simple geodesics on the surface. These are unique.*

This has potential applications to the CARS program where symbolic computation is fastest for short geodesics. The length of a geodesic is computed through the absolute value of the trace of the corresponding matrix, which is why the concept of trace minimizing has an important role.

**4.5. Connection: good words, algorithm words and Farey words.** We outline another connection between the semi-group of good words under composition of good words  $*_{\mathcal{G}}$  and addition of Farey words  $+_{\mathcal{F}}$ . More details about this will appear in [14]. First of all we observe that for any two words in  $A$  and  $B$ ,  $W_1(A, B)$  and  $W_2(A, B)$ , the operation  $*_{\mathcal{G}}$  is well defined:  $W(A, B) = W_2(W_1(A, B), B) = W_2 *_{\mathcal{G}} W_1$ . We, therefore, refer to this operation as the *good product* as opposed to the *good word product*. Any particular family of words may or may not be closed under this product.

Assume that  $W_1(X, Y)$  and  $W_2(X, Y)$  are Farey words in the generators  $X$  and  $Y$  with  $W_1(X, Y) = W_{\frac{p}{q}}(X, Y)$  and  $W_2(X, Y) = W_{\frac{r}{s}}(X, Y)$  where  $\frac{p}{q} = [n_1, \dots, n_j]$  and  $\frac{r}{s} = [m_1, \dots, m_i]$ , so that  $W_1$  has  $F$ -sequence  $[n_1, \dots, n_j]$  and  $W_2$  has  $F$ -sequence  $[m_1, \dots, m_i]$ . Then  $W_1 = W_{\frac{p}{q}}$  and  $W_2 = W_{\frac{r}{s}}$  will not be Farey neighbors unless  $|ps - qr| = 1$ . However, the following is true.

**PROPOSITION 4.3.** *Let  $W(A, B)$  be the word in  $A$  and  $B$  with  $F$ -sequence  $[n_1, \dots, n_j, m_1, \dots, m_i]$ . Then  $W$  is the good product  $W_2 *_{\mathcal{G}} W_1$ .*

**PROOF.** We see from corollary 2.8 and proposition 2.9 of [16] that if the algorithm words  $W_1(A, B)$  and  $W_2(A, B)$ , respectively, have  $F$ -sequences  $[n_1, \dots, n_j]$  and  $[m_1, \dots, m_i]$ , respectively, then the word with  $F$  sequence  $[n_1, \dots, n_j, m_1, \dots, m_i]$  is precisely  $W_2(W_1(A, B), B)$ . Alternately, to see this, we begin with  $W_1$  and locate its first occurrence in the Farey diagram. We then behave as though  $W_1(A, B)$  and  $B$  were the initial generators and follow the  $F$  sequence for  $W_2$  down in the Farey diagram for an initial generating pair  $(W_1(A, B), B)$ . The word we end at is  $W$  and one can check the Farey expansion and the good word multiplication formula.  $\square$

We also note that algorithm or Farey words in  $ABA^{-1}$  and  $B$  correspond to good words in  $A$  and  $B$  as follows:

**PROPOSITION 4.4.** *Let  $C = ABA^{-1}$  and  $D = B$ . Let  $W_F$  be the algorithm/Farey word in  $C$  and  $D$  with primitive exponents  $(v_1, \dots, v_t)$  so that  $W_F(C, D) = AB^{-1}A^{-1} \cdot B^{v_1} \dots AB^{-1}A^{-1} \cdot B^{v_t}$ . Then  $W_F \cdot A$  will be a good word in  $A$  and  $B$ .*

**PROOF.** Look at the exponents.  $\square$

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