PRIME ORDER AUTOMORPHISMS OF RIEMANN SURFACES

JANE GILMAN

Abstract. Recently there has been renewed interest in the mapping-class group of a compact surface of genus $g \geq 2$ and also in its finite order elements. A finite order element of the mapping-class group will be a conformal automorphism on some Riemann surface of genus g. Here we give the details of the proof that there is an *adapted basis* for any conformal automorphism of prime order on a surface of genus g and extend the original result to apply to fixed point free automorphisms. An adapted basis is one that reflects the action of the automorphism in the optimal manner described below. The proof uses the Schreier-Reidemeister rewriting process. We find some new consequences of the existence of an adapted basis. We also construct an explicit example of such a basis and compute its intersection matrix.

1. Introduction

For a conformal automorphism of a compact Riemann surface the notion of an *adapted* homology basis was developed as part of a proof that the Riemann space (also known as the Moduli Space) of a punctured surface had the structure of a quasi-projective variety [6, 8]. An adapted basis is one which reflects the action of the conformal automorphism in an optimal way. Such an action would be reflected in the structure of the period matrix of the surface in a useful manner. More recently Rodriguez, Riera, Gonzalez and others have used the notion of a basis adapted to a group of automorphisms to obtain information about abelian varieties and especially, the Prym variety. (See [23], [14] and the references given there.)

In this paper we survey earlier results about the matrix representation of a prime order automorphism with respect to an adapted basis and the corresponding intersection matrix for such a basis. The proof of the existence of an adapted basis uses the Schreier-Reidemeister rewriting process. Here we give full details of the application of the Schreier-Reidemeister rewriting process used in [10] to construct the

This work was partially supported by grants from the NSA and the Rutgers Research Council and by the Yale University Mathematics Department.

adapted basis. We have been told that the application in [10] was too sketchy for some readers to follow. We find some new consequences of the existence of an adapted basis and extend the result to the case of a fixed point free automorphism. We also construct an explicit example of such a basis and compute its intersection matrix.

The paper is organized as follows. In section 2 we fix notation and terminology and we review the conjugacy invariants for an element of the mapping-class group of prime order and basic facts about homology. Section 3 introduces the notion of an adapted homology basis, section 4 discusses the existence of such bases and section 5 the intersection numbers of elements in an adapted basis. Section 6 fixes some matrix notation. In section 7 the Schreier-Reidemeister rewriting process is explained and the calculation is carried out in detail (section 7.2). The case for a fixed point free automorphism is carried out in section 8 and some corollaries are drawn in section 10.

Contents

2. Preliminaries

2.1. **Notation and Terminology.** We let h be a conformal automorphism of a compact Riemann surface S of genus $g \geq 2$. Then h will have a finite number, t, of fixed points. We let S_0 be the quotient of S under the action of the cyclic group generated by h so that $S_0 = S/\langle h \rangle$ and let g_0 be its genus. If h is of prime order p with $p \geq 2$, then the Riemann-Hurwitz relation shows that $2g = 2pg_0 + (p-1)(t-2)$. If $p = 2$, of course, this implies that t will be even.

2.2. **Equivalent Languages.** We emphasize that h can be thought of in a number of equivalent ways using different terminology. For a compact Riemann surface of genus $g \geq 2$, homotopy classes of homeomorphisms of surfaces are the same as isotopy classes. Therefore, h can be thought of as a representative of a homotopy class or an isotopy class. Further, every isotopy class of finite order contains an element of finite order so that h can be thought of as a homeomorphism of finite order. For every finite order homeomorphism of a surface there is a Riemann surface on which its action is conformal. A conformal homeomorphism of finite order up to homotopy is finite. We use the language of conformal maps, but observe that all of our results can be formulated using these other classes of homeomorphisms.

We remind the reader that the Mapping-class group of a compact surface of genus g is also known as the Teichmüller Modular group or the Modular group, for short. We write $MCG(S)$ or $MCG(S_g)$ for the mapping class group of the surface S using the g when we need to emphasize that S is a compact surface of genus g. The Torelli Modular group or the Torelli group for short is homeomorphisms of S modulo those that induce the identity on homology and the homology of a surface is the abelianized homotopy. There is surjective map π from the mapping-class group onto $Sp(2g, \mathbb{Z})$ that assigns to a homeomorphism the matrix of its action on a canonical homology basis (see 2.4).

Since h can be thought of as a finite representative of a finite order mapping-class, we will always treat it as finite. For ease of exposition we use the language of conformal maps and do not distinguish between a homeomorphism that is of finite order or that is of finite order up to homotopy or isotopy, a finite order representative for the homotopy class, a conformal representative for the class or the class itself. That is, between the topological map, its homotopy class or a finite order representative or a conformal representative.

For ease of exposition in what follows we first assume that $t > 0$. We treat the case $t = 0$ separately in section 8.

2.3. **Conjugacy Invariants for prime order mapping classes or conformal automorphisms.** In determining the conjugacy class of a prime order element of the mapping class group, we may first assume that the element h is actually conformal and of finite order. If h has t fixed points $p_1, ..., p_t$ then in a neighborhood of p_i it is a (counter-clockwise) rotation through an angle $s_i \frac{2\pi}{p}$ for some s_i with $0 < s_i < p$. Nieslen showed topologically that the conjugacy class of h is completely determined by the rotation angles (see page 53 of [22]). The numbers s_i are called *the rotation numbers*. The set of rotation numbers determine the conjugacy class.

One can also consider *the complementary rotation numbers* n_i where $n_i \cdot s_i \equiv 1(p)$. Here $\equiv\ (p)$ denotes equivalence modulo p and we let $\mathbb{Z}_p$ denote the integers modulo p.

Alternately we can uniformize S and S_0 by a pair of Fuchsian groups F and F_0 so that $S = U/F$, $S_0 = U/F_0$ and the action of the the conformal map h is isomorphic to the action of F_0/F on S. In this setting, h is determined by a surjective homomorphism $\phi : F_0 \rightarrow \mathbb{Z}_p$. If F is the kernel of ϕ, then F_0/F acts in a natural way on $S = U/F$. The homomorphism ϕ will take t elliptic elements of F_0 onto non-trivial elements n_i of $\mathbb{Z}_p$ and these elements will satisfy $\Sigma_{i=1}^{t} n_i \equiv 0(p)$. One uses Harvey's list [15] of automorphisms of F_0 to assure that the conjugacy class of h depends only upon the number of n_i taking on a possible value and not the order in which the n_i appear in any list of complementary rotation numbers. This is the proof given in [9] where it is also shown that every such t-tuple of integers determines a conformal map. We will discuss this further in section 7.2.

Since the order in which the complementary rotation numbers are listed does not matter, we can instead count m_j, the number of n_i equal to j. Then we have $\Sigma_{i=1}^{p-1} i \cdot m_i \equiv 0\ (p)$ and the conjugacy class is also determined by the $(p-1)$-tuple, $(m_1, ..., m_{p-1})$.

2.4. **Homology.** We recall the following facts about Riemann surfaces.

The homology group of a compact Riemann surface of genus g is the abelianized homotopy. Therefore, a homology basis for S will contain $2g$ homologously independent curves. Every surface has a *canonical homology basis*, a set of $2g$ simple closed curves, $a_1, ..., a_g; b_1, ..., b_g$ with the property that for all i and j *the intersections numbers* satisfy $a_i \times a_j = 0$, $b_i \times b_j = 0$ and $a_i \times b_j = \delta_{ij} = -b_j \times a_i$ where δ_{ij} is the Kronecker delta.

3. Adapted homology bases

Roughly speaking a homology basis for S is *adapted to h* if it reflects the action of h in a simple manner: for each curve γ in the basis either all of the images of γ under powers of h are also in the basis or the basis contains all but one of the images of γ under powers of h and the omitted curve is homologous to the negative of the sum of the images of γ under the other powers of h.

To be more precise

Definition 3.1. *A homology basis for S is* adapted to h *if for each γ_0 in the basis there is a curve γ with $\gamma_0 = h^k(\gamma)$ for some integer k and either*

(1) *$\gamma, h(\gamma), ...h^{p-1}(\gamma)$ are all in the basis, or*
(2) *$\gamma, h(\gamma), ...h^{p-2}(\gamma)$ are all in the basis and*
$h^{p-1}(\gamma) \approx^h -(h(\gamma) + h(\gamma) + ... + h^{p-2}(\gamma))$.
Here $\approx^h$ denotes is homologous to.

If $\mathcal{A}$ is an adapted homology basis, we define the adapted matrix *to be the matrix of h with respect to the adapted basis and denote it by $M_{\mathcal{A}}(h)$. We shorten $M_{\mathcal{A}}(h)$ to $M_{\mathcal{A}}$ when h is clearly understood.*

4. Existence of adapted Homology Bases

It is known that

Theorem 4.1. (Gilman 1977) [10] *There is a homology basis adapted to h. In particular, if $g \geq 2$, $t \geq 2$, g_0 are as above, then the adapted basis has $2p \times g_0$ elements of type (1) above and $(p-1)(t-2)$ elements of type (2).*

and thus it follows that

Corollary 4.2. [10] *The adapted matrix $M_{\mathcal{A}}(h)$ will be composed of diagonal blocks, $2g_0$ of which are $p \times p$ permutation matrices with 1's along the super diagonal and 1 in the leftmost entry of the last row and t of which are $(p-1) \times (p-1)$ matrices with 1's along the super diagonal and all entries in the last row -1.*

A proof of theorem 4.1 is given in section 7.2.

Remark 4.3. *We adopt the following convention. When we pass from homotopy to homology, we use the same notation for the homology class of the curve as for the curve or its homotopy class, but write $\approx^h$ instead of $=$. It will be clear from the context which we mean.*

5. Intersection Matrix for an Adapted Homology Basis

So far information about $M_{\mathcal{A}}(h)$ seems to depend only on t and not upon the $(p-1)$-tuple $(m_1, ..., m_{p-1})$ or equivalently, upon the set of integers $\{n_1, ..., n_t\}$) which determines the conjugacy class of h in the mapping-class group. However, while the $2pg_0$ curves can be extended to a canonical homology basis for h, the rest of the basis cannot and its intersection matrix, $I_{\mathcal{A}}$ depends upon these integers.

In [12] the intersection matrix for the adapted basis was computed.

Remark 5.1. *For any one reading the original paper* [12] *note that m and n the letters denoting the rotation numbers and the complementary rotation numbers have been interchanged. To avoid confusion, we use u and v in this section when presenting result from* [12].

The adapted basis consisted of the curves of type (1):

$$\{A_w, B_w, w = 1, ..., g_0\} \cup \{h^j(A_w), h^j(B_w), j = 1, ..., p-1\}$$

and (some of) the curves of type (2):

$$X_{i,v_i}, h^j(X_{i,v_i}),\ i = 1, ..., (p-1),\ j = 1, ..., p-2,\ v_i = 1, ..., u_i.$$

A lexicographical order is placed on X_{i,v_i} so that $(r, v_r) < (s, v_s)$ if and only if $r < s$ or $r = s$ and $v_r < v_s$. The $t-2$ curves X_{s,v_s} with the largest subscript pairs are to be included in the homology basis. Let $\hat{s}$ be the smallest integer s such that $u_s \neq 0$ and let $\hat{q}$ be chosen so that $\hat{q}\hat{s} \equiv 1(p)$. For any integer v let $[v]$ denote the least non-negative residue of $\hat{q}v$ modulo p. Thus the integer $[v]$ satisfies $0 \leq [v] \leq p-1$ and $\hat{s} \cdot [v] \equiv v$ mod(p).

Theorem 5.2. (Gilman-Patterson, 1981) [12] *If $(u_1, ..., u_{p-1})$ determines the conjugacy class of h in the mapping-class group, then the surface S has a homology basis consisting of:*

1. $h^j(A_w), h^j(B_w)$ *where* $1 \leq w \leq g_0, 0 \leq j \leq p-1$.
2. $h^k(X_{s,v_s})$ *where* $0 \leq k \leq p-2$ *and for all pairs* (s, v_s) *with* $1 \leq s \leq p-1, 1 \leq v_s \leq u_s$ *except that the two smallest pairs are omitted.*

   The intersection numbers for the elements of the adapted basis are given by

   (a) $h^j(A_w) \times h^j(B_w) = 1$

   (b) *If* $(r, v_r) < (s, v_s)$, *then*

$$h^0(X_{r,v_r}) \times h^k(X_{s,v_s}) = \begin{cases} 1 & \textit{if } [k] < [r] \leq [k+s] \\ -1 & \textit{if } [k+s] < [r] \leq [k] \end{cases}$$

$$h^0(X_{s,v_s}) \times h^k(X_{s,v_s}) = \begin{cases} 1 & \textit{if } [k] \leq [s] < [k+s] \\ -1 & \textit{if } [k+s] < [s] < [k] \end{cases}$$

(3) *All other intersection numbers are* 0 *except for those following from the above by applying the identities below to arbitrary homology classes* C *and* D*.*
$C \times D = -D \times C$
$h^j(C) \times h^k(D) = h^0(C) \times h^{k-j}(D)$*, (*$k-j$ *reduced modulo* p*).*

Proof. For details we refer the reader to [12]. Basically, the proof of this theorem comes from a careful interpretation of the isomorphism between covering groups, fundamental groups, and defining subgroups of coverings and their relation to words corresponding to closed curves on the quotient surface that lift to closed curves. □

6. Matrix forms

We can write the results of theorems 4.1 and 5.2 and corollary 4.2 in an explicit matrix form. To do so we fix notation for some matrices. We will use the various explicit forms in subsequent sections.

We let $M_{\tilde{\mathcal{A}}}$ denote the matrix of the action of h on an adapted basis and $I_{\tilde{\mathcal{A}}}$ be the corresponding intersection matrix. Further, we let $M_{h_{CAN}}$ be the matrix of the action of h on a canonical homology basis. The corresponding intersection matrix is denoted by $I_{h_{CAN}}$. If we let I_k denote the $k \times k$ identity matrix, then $I_{h_{CAN}}$ is (conjugate to) the $2g \times 2g$ matrix $= \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

However, we prefer to replace it by the following block matrix where $q = \frac{(p-1)(t-2)}{2}$

$$I_{h_{CAN}} = \begin{pmatrix} 0 & I_{pg_0} & 0 & 0 \\ -I_{pg_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \\ 0 & 0 & -I_q & 0 \end{pmatrix}.$$

We denote the $p \times p$ permutation matrix by

$$M_{p\times p} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix};$$

the $(p-1) \times (p-1)$ non-permutation matrix of corollary 4.2 by

$$N_{(p-1)\times(p-1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 \end{pmatrix}.$$

Thus we have the $2g_0p \times 2g_0p$ block matrix

$$M_{\mathcal{A}_{2g_0,p\times p}} = \begin{pmatrix} M_{p\times p} & 0 & 0 & \dots & 0 & 0 \\ 0 & M_{p\times p} & 0 & \dots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & M_{p\times p} & 0 \\ 0 & 0 & 0 & \dots & 0 & M_{p\times p} \end{pmatrix}$$

and the $(t-2)(p-1) \times (t-1)(p-1)$ block matrix

$$N_{\mathcal{A}_{(t-2),(p-1)\times(p-1)}} = \begin{pmatrix} N_{(p-1)\times(p-1)} & 0 & 0 & \dots & 0 & 0 \\ 0 & N_{(p-1)\times(p-1)} & 0 & \dots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & N_{(p-1)\times(p-1)} & 0 \\ 0 & 0 & 0 & \dots & 0 & N_{(p-1)\times(p-1)} \end{pmatrix}$$

so that the $2g \times 2g$ matrix $M_{\mathcal{A}}$ breaks into blocks and can be written as

$$M_{\mathcal{A}} = \begin{pmatrix} M_{\mathcal{A}_{2g_0,p\times p}} & 0 \\ 0 & N_{\mathcal{A}_{(t-2),(p-1)\times(p-1)}} \end{pmatrix}$$

where the blocks are of appropriate size. The basis can be rearranged so that $2g \times 2g$ matrix $M_{\tilde{\mathcal{A}}}$ corresponding to the rearranged basis breaks into blocks

$$M_{\tilde{\mathcal{A}}} = \begin{pmatrix} M_{\mathcal{A}_{g_0,p\times p}} & 0 & 0 \\ 0 & M_{\mathcal{A}_{g_0,p\times p}} & 0 \\ 0 & 0 & N_{\mathcal{A}_{(t-2),(p-1)\times(p-1)}} \end{pmatrix}.$$

Here the submatrix

$$\begin{pmatrix} M_{\mathcal{A}_{g_0,p\times p}} & 0 \\ 0 & M_{\mathcal{A}_{g_0,p\times p}} \end{pmatrix}$$

is a symplectic matrix.

Remark 6.1. *We note that if $p = 2$, $M_{p\times p}$ reduces to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $N_{(p-1)\times(p-1)}$ to the 1×1 matrix -1.*

We obtain the corollary

Corollary 6.2. *Let S be a compact Riemann surface of genus g and assume that S has a conformal automorphism h of prime order $p \geq 2$. Assume that h has t fixed points where $t \geq 2$. Let S_0 be the quotient surface $S_0 = S/\langle h \rangle$ where $\langle h \rangle$ denotes the cyclic group generated by h and let g_0 be the genus of S_0 so that $2g = 2pg_0 + (t-2)(p-1)$.*

There is a homology bases on which the action of h is given by the $2g \times 2g$ matrix $M_{\tilde{\mathcal{A}}}$.

The matrix $M_{\tilde{\mathcal{A}}}$ contains a $2g_0p \times 2g_0p$ symplectic submatrix, but is not a symplectic matrix except in the special case $t = 2$.

The point here is that while two automorphisms with the same number of fixed points will have the *same* matrix representation with respect to an adapted basis, the intersection matrices will not be the same and, therefore, the corresponding two matrix representations in the symplectic group will not be conjugate.

There is an algorithm to replace $M_{\tilde{\mathcal{A}}}$ by the symplectic matrix $M_{h_{CAN}}$ by replacing the submatrix $N_{(t-2),(p-1)\times(p-1)}$ by a symplectic matrix of the same size (see [11]). We will call this matrix $N_{symp\tilde{\mathcal{A}}}$ and give an example in section 9.

We note that $I_{\tilde{\mathcal{A}}}$ is of the form

$$\begin{pmatrix} 0 & I_{pg_0} & 0 & 0 \\ -I_{pg_0} & 0 & 0 & 0 \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \end{pmatrix}$$

where the blocks B_i are of the appropriate dimension and we let B denote the matrix

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

7. Schreier-Reidemeister Rewriting

If we begin with an arbitrary finitely presented group G_0 and a subgroup G, the Schreier-Reidemeister rewriting process tells one how to obtain a presentation for G_0 from the presentation for G. In our case the larger group G_0 will correspond to the group uniformizing S_0 and the subgroup G corresponds to the group uniformizing S.

In particular, one chooses a special set of coset representatives for G modulo G_0, called Schreier representatives, and uses these to find a set of generators for G. These generators are labeled by the original generators of the group and the coset representative.

7.1. **The relation between the action of the homeomorphism and the surface kernel subgroup.** We may assume that $S_0 = U/F_0$ where F_0 is the Fuchsian group with presentation

$$(1) \quad \langle a_1, ..., a_{g_0}, b_1, ..., b_{g_0}, x_1, ...x_t |\; x_1 \cdots x_t(\Pi_{i=1}^{g}[a_i, b_i]) = 1; x_i^p = 1\rangle.$$

We summarize the result of [9] repeating some facts about the n_i and m_j. We let $\phi : F_0 \to \mathbb{Z}_p$ be given by

$$\phi(a_i) = \phi(b_i) = 0 \quad \forall i = 1, ..., g_0 \text{ and } \phi(x_j) = n_j \neq 0 \quad \forall j = 1, ..., t.$$

The n_i satisfy $\Sigma_{i=1}^{t} n_i \equiv 0\ (p)$. If $F = Ker\ \phi$, then $S = U/F$. Moreover, F_0/F acts on S with quotient S_0. Conjugation by x_1 acts on F and if h is the induced conformal map on S, $\langle h\rangle$ is isomorphic to the action induced by this conjugation and the conjugacy class of h in the mapping-class group is determined by the set of n_i. Replacing h by a conjugate we may assume that $0 < n_i \leq n_j < p$ if $i < j$.

Since m_i is the number of j such that $\phi(x_j) = i$, we have $\Sigma_{i=1}^{p-1} i m_i \equiv 0\ (p)$ and the conjugacy class of h is also completely determined by $(m_1, ..., m_{p-1})$.

When we need to emphasize the relation of h to ϕ, we write h_ϕ to mean the automorphism determined by conjugation by x_1. The conjugacy class of h^2, would then be determined by the homomorphism ψ with $\psi(x_j) \equiv 2\phi(x_j)\ (p)$ or by conjugation by x_1^2.

We note that in [9] results are written in greatest possible generality so that S_0 has punctures, some of which are fixed by the automorphism and others of which are not. Here, we use the results of [9] for compact surfaces. F_0 is sometimes called surface kernel and ϕ the surface kernel homomorphism [13, 15, 24].

Any other map from F_0 onto $\mathbb{Z}_p$ with the same $(m_1, ..., m_{p-1})$ and with $\phi(x_j) \neq 0\ \ \forall j$ will yield an automorphism conjugate to h.

7.2. **The rewriting.** We want to apply the rewriting process to words in the generators of this presentation for F_0 to obtain a presentation for F. We choose coset representatives for $H = \langle h\rangle$ as $x_1, x_1^2, x_1^3, ..., x_1^p$ and observe that these are a *Schreier* systems. (see page 93 of [21]). That is, every initial segment of a representative is again a representative.

The Schreier right coset function assigns to a word W in the generators of F_0, its coset representative $\overline{W}$ and $\overline{W} = x_1^q$ if $\phi(W) = \phi(x_1^q)$.

If $\mathfrak{a}$ is a generator of F_0, set $S_{K,\,\mathfrak{a}} = K\mathfrak{a}\overline{K\mathfrak{a}}^{\,-1}$. The rewriting process τ assigns to a word that is in the kernel of the map ϕ, a word written in the specific generators, $S_{K,\,\mathfrak{a}}$ for F. Namely, if $\mathfrak{a}_w$, $w = 1, ..., r$ are generators for F_0 and

$$U = \mathfrak{a}_{v_1}^{\epsilon_1}\mathfrak{a}_{v_2}^{\epsilon_2} \cdots \mathfrak{a}_{v_r}^{\epsilon_r} \qquad (\epsilon_i = \pm 1),$$

defines an element of F, then (corollary 2.7.2 page 90 of [21])

$$\tau(U) = S^{\epsilon_1}_{K_1,\, \mathfrak{a}_{v_1}} S^{\epsilon_2}_{K_2,\, \mathfrak{a}_{v_2}} \cdots S^{\epsilon_r}_{K_r,\, \mathfrak{a}_{v_r}}$$

where K_j is the representative of the initial segment of U preceding $\mathfrak{a}_{v_j}$ if $\epsilon_j = 1$ and K_j is the coset representative of U up to and including $\mathfrak{a}^{-1}_{v_j}$ if $\epsilon_j = -1$.

In our case each $\mathfrak{a}_v$ stands for some generator of F_0, that is one of the a_i or b_i or x_j.

We apply theorem 2.8 of [21] to see

Theorem 7.1. *Let F_0 have the presentation given by equation (1). Then F has presentation*

$$\langle S_{K,\, a_i},\ S_{K,\, b_i},\ i = 1, ..., g_0;\ S_{K, x_j},\ j = 1, ...t| \tag{2}$$

$$\tau(K \cdot x_1 \cdots x_t(\Pi^{g_0}_{i=1}[a_i, b_i]) \cdot K^{-1}) = 1,\ \tau(K x^p_j K^{-1}) = 1\rangle. \tag{3}$$

Proof. Let K run over a complete set of coset representatives for $\phi : F \to H$. Then F_0 has generators

$$\begin{aligned} S_{K,\, a_i}, S_{K,\, b_i} \qquad & i = 1, ..., g_0 \\ S_{K,\, x_j}, \qquad & j = 1, ...t \end{aligned}$$

and relations

$$\tau(K(x_1 \cdots x_t \Pi^{g_0}_{i=1}[a_i, b_i])K^{-1}) = 1 \tag{4}$$

$$\tau(K x^p_j K^{-1}) = 1 \tag{5}$$

□

We want to simplify this presentation and eliminate generators and relations so that there is a single defining relation for the subgroup. We first assume that $\phi(x_1) = h$. We note that if we can find a homology basis adapted to h, we can easily find a homology basis adapted to any power of h and, therefore, this assumption will not be significant.

Definition 7.2. *A relation is evenly worded if for each generator A that occurs in the relation A^{-1} also occurs. The generators A and B occurring in a relation are linked if the relation is of the form $W_0 A W_1 B W_2 A^{-1} W_3 B^{-1} W_4$ where the $W_i, i = 0, ..., 4$ are words in the generators not involving $A^{\pm 1}$ or $B^{\pm 1}$. The relation is fully linked if each generator A occurring in the relation is linked to a unique distinct generator.*

We will show:

Theorem 7.3. *Let F_0 have the presentation given by equation (1). Then F has presentation*

$$\langle h^j(A_i), h^j(B_i), i = 1, ..., g_0, j = 0, ..., p-1 : h^j(X_i), i = 3, ..., t, j = 0, ..., p-2 | \hat{\hat{R}} = 1\rangle.$$

The relation $\hat{\hat{R}}$ is the single defining relation for the group F. The single defining relation $\hat{\hat{R}}$ is evenly worded and fully linked.

and

Corollary 7.4. *The homology basis obtained by abelianizing the basis in theorem 7.3 gives a homology basis adapted to h.*

We note that an explicit formula for $\hat{\hat{R}}$ is given in [11].

Proof. If we let $\phi(x_1) = h$ and $\phi(K) = \phi(x_1)^r$, then we have $S_{K,X_j} = x_1^{r\phi(x_1)} \cdot x_j \cdot \overline{K \cdot x_j}^{-1}$. Thus if $X_j = x_j \cdot \overline{x_j}^{-1}$, then $S_{K,x_j} = h^r(X_j)$.

We begin by rewriting the generators and relations using this notation.

First we find $\tau(\overbrace{x_1 x_1 \cdots x_1}^{p-\text{factors}}) = 1$. Setting $X_1 = S_{x_1^p, x_1}$ since $\overline{1} = x_1^p$, we have

$$\tau(x_1^p) = X_1 \cdot h(X_1) \cdot h^2(X_1) \cdots h^{p-2}(X_1) h^{p-1}(X_1) = 1. \tag{6}$$

Similarly, if $\phi(x_j) = n_j$, and we set $X_j = S_{\overline{1}, x_j}$, then if $\overline{K} = x_1^s$, then we can write $S_{K,\, x_j} = h^{s \cdot n_j}(X_j)$.

This tells us that

$$\tau(x_j^p) = X_j \cdot h^{n_j}(X_j) \cdot h^{2n_j}(X_j) \cdots h^{(p-2)n_j}(X_j) h^{(p-1)n_j}(X_j) = 1. \tag{7}$$

In particular, we will make special use of this when $j = 2$

$$\tau(x_2^p) = X_2 \cdot h^{n_2}(X_j) \cdot h^{2n_2}(X_2) \cdots h^{(p-2)n_2}(X_2) h^{(p-1)n_2}(X_2) = 1. \tag{8}$$

Note that in deriving all equations we are free to make use of the fact (see [21]) that

$$S_{M,\, x_1} \approx 1 \forall \text{ Schreier representatives} M \tag{9}$$

where $\approx$ denotes *freely equal to.* This eliminates the p generators, $S_{x_1^j, x_1}, j = 1, ..., p$.

Recall that a word is freely equal to another word if one word can be obtained from the other by inserting and deleting expressions of the form $x \cdot x^{-1}$. This means in particular that these words represent the same element in every group.

Now equation (7) is a relation in the fundamental group. We remind the reader that for a compact Riemann surface, homology is abelianized homotopy so that when abelianized, it reduces to

(10) $$h^{p-1}(X_j) \approx^h -X_j - h(X_j) - \cdots - h^{p-2}(X_j)$$

where $\approx^h$ denotes *is homologous to.*

We also note that the $\tau(Kx_j^pK^{-1}) = 1$ do not give us any additional relations for $\bar{K} \neq 1$, but merely a conjugate relation already implied by $\tau(x_j^p) = 1$.

Next we set $A_i = S_{\bar{1},\, a_i}$ and $h^r(A_i) = S_{x_1^r,\, a_i}$, $B_i = S_{1,\, b_i}$ and $h^r(B_i) = S_{x_1^r,\, b_i}$, then the relation $R = 1$, where $R = x_1 \cdots x_t(\Pi_{i=1}^{g_0}[a_i, b_i])$, yields $\tau(R) = 1$ and we obtain

(11)
$$X_1 h^{n_1}(X_2) h^{n_1 \cdot n_2}(X_2) \cdots h^{n_1 \cdot n_2 \cdots n_{t-2}}(X_{t-1}) \cdot h^{n_1 \cdot n_2 \cdots n_{t-1}}(X_t)(\Pi_{i=1}^{g_0}[A_i, B_i]) = 1$$

and using the fact that $X_1 \approx 1$, we have

(12)
$$h^{n_1}(X_2) h^{n_1 \cdot n_2}(X_3) \cdots h^{n_1 \cdot n_2 \cdots n_{t-2}}(X_{t-1}) \cdot h^{n_1 \cdot n_2 \cdots n_{t-1}}(X_t)(\Pi_{i=1}^{g_0}[A_i, B_i] = 1$$

Similarly, we obtain the relations $\tau(KRK^{-1}) = 1$.

We can solve equation (12) for $(h(X_2))^{-n_1}$ to obtain

(13)
$$(h(X_2))^{-n_1} = h^{n_1 \cdot n_2}(X_3) \cdots h^{n_1 \cdot n_2 \cdots n_{t-2}}(X_{t-1}) \cdot h^{n_1 \cdot n_2 \cdots n_{t-1}}(X_t)(\Pi_{i=1}^{g_0}[A_i, B_i])$$

Now each of the relations $\tau(KRK^{-1})$ allows us to solve for $h^q(X_2)$ for some q and for each K we obtain a different q. Therefore, we can substitute equation (13) and its images under powers of h into equation (8) (i.e. the relation $\tau(x_2^p) = 1$). We thus eliminate all of the generators of the form S_{M,x_2} for each coset representative M and all of the relations $\tau(KRK^{-1})$. We obtain one new relation $\hat{R}$ from equation (8). This relations involves $h^w(X_j)$ for every $w = 0, 1, ..., p-2$ and every $j = 3, ..., t$. We also note that for each q the sequence $\Pi_{i=1}^{g_0}[h^q(A_i), h^q(B_i)]$ occurs in $\hat{R}$. Using equation (7) to replace the generator $h^{p-1}(X_j)$ by a word in the $h^{-v}(X_j)$ with $v = 0, ..., p-1$ we eliminate those relations and we obtain a single defining relation $\hat{\hat{R}}$ involving each of the following generators below and their inverses exactly once. The generators are

$$h^d(X_j) \;\; j = 3, ..., t \text{ and } d = 0, 1, ..., p-2$$

and

$$h^d(A_i), h^d(B_i) \;\; i = 1, ..., g_0 \text{ and } d = 0, 1, ..., p-1$$

We obtain $2g_0p + (t-2)(p-1)$ generators and a single defining relation. It is fairly straight forward to check that the relation has the last two properties of the theorem.

□

We recapitulate. The idea of this proof is that for each K, we can solve $\tau(KRK^{-1}) = 1$ for an appropriate image S_{K',x_2} where K' depends upon K. The appropriate image of S_{K',x_2} is placed on the left of the equation, and we then substitute the right hand side of the solution into the relation $\tau(x_2^p) = 1$. This yields one relation $\hat{\mathcal{R}}$ which involves each A_i, B_i generator and their inverses and all of their images under powers of h and each S_{K,x_j} and the images under powers of h but no inverses. We still have the finite order relations $\tau(x_j^p)$, $j = 3, ..., t$. The $\tau(Kx_j^pK^{-1}) = 1$ are merely permutations of the relation $\tau(x_j^p) = 1$ so we can eliminate all but one of these. For each $j = 3, ..., t$, $\tau(x_j^p)$ can be solved for $h^{p-1}(S_{\bar{1},x_j})$. It will be a word in the inverses of all of the other S_{K,x_j}. We substitute these into $\hat{\mathcal{R}}$ and obtain a single defining relation $\hat{\hat{R}}$ in which every generator and its inverse occurs exactly once.

8. The Case $t = 0$

If the number of fixed points is zero, we can still find an adapted basis. The calculations are slightly different. We have $2g = 2p(g_0 - 1) + 2$. The presentation given by (1) for the group F_0 becomes

$$\langle a_1, ..., a_{g_0}, b_1, ..., b_{g_0} | \ (\Pi_{i=1}^{g_0}[a_i, b_i]) = 1; \rangle. \tag{14}$$

Again, replacing h by a conjugate if necessary, the map $\phi : F_0 \to \mathbb{Z}_p$ can be taken to be

$$\phi(a_i) = \phi(b_i) = 0 \quad \forall i = 2, ..., g_0 \text{ and } \phi(a_1) = 1 \text{ and } \phi(b_1) = 0$$

Using the rewriting with coset representatives $1, a_1, ..., a_1^{p-1}$, note that $S_{a_1^k,a_1} \approx 1$ for $k = 0, ..., p-2$. Let $A = S_{a_1^{p-1},a_1}$. Then h acts on $Ker\ \phi$ via conjugation by a_1. We have $h(A) = A$

We let $\{h^k(A_j), h^k(B_j), j = 2, ..., g_0, k = 0, ..., p-1\}$ be as in the proof of Theorem 7.3 and let $B = S_{1,b_1}$.

We let $P = \Pi_{i=2}^{g_0}[A_i, B_i]$.

Then we can compute that

$$\tau(R) = 1 \implies h(B)B^{-1}T = 1 \tag{15}$$

$$\tau(a_1^kRa_1^{-k}) = 1 \implies h^k(B)(h^{k-1}(B))^{-1}h^k(P) = 1 \forall k = 1, ...p-2$$

$$\text{and } \tau(a_1^{p-1}Ra_1^{-(p-1)}) = 1 \implies ABA^{-1}(h^{p-1}(B))^{-1}h^{p-1}(P) = 1$$

We eliminate generators using (15) and let $\alpha = A$ and $\beta = h^{p-1}(B)$ to obtain generators

$$\{\alpha, \beta\} \cup \{h^k(A_j), h^k(B_j), j = 2, ..., g_0, k = 0, ..., p-1\}$$

and the single defining relation $\beta\alpha\beta^{-1} = h^{p-1}(P)\alpha\Pi_{i=k}^{p-2} h^k(P)$. Further we calculate that $h(\alpha) \approx^h \alpha$ and $h(\beta) \approx^h \beta$. Note that P is a product of commutators. Thus the matrix representation for h on $S = U/F$ where $F = \mathrm{Ker}\phi$ is given by $2(g_0 - 1)$ permutation matrices $M_{p\times p}$ and one two by two identity matrix.

The basis is a canonical homology basis. We have $\alpha \times \beta = 1$. Further α and β are disjoint from any other lifted curves in the basis. Each lift of a generator other than a_1 or b_1 is a simple closed curve and is disjoint from all other curves except for the corresponding $h^k(B_j)$. That is, $h^k(A_i) \times h^r(B_j) = \delta_{ij} \cdot \delta_{kr}$ for all integers $i, j \in \{2, ..., p-1\}$ and $k, r \in \{0, ..., p-1\}$. We replace α or β by an appropriate conjugate if necessary. We note that the curves α and β are by default of type (1) in definition 3.1

We have obtained the following version of Theorem 5.2 when $t = 0$

Theorem 8.1. *If $t = 0$, then the surface S has a canonical homology basis consisting of:*

(1) $h^j(A_w), h^j(B_w)$ *where* $2 \le w \le g_0, 0 \le j \le p-1$.
(2) α, β *where* $h^k(\alpha) \approx^h \alpha, h^k(\beta) \approx^h \beta, 0 \le k \le p$
(3) *The intersection numbers for the elements of the adapted basis are given by*
   (a) $h^j(A_w) \times h^j(B_w) = 1$
   (b) $\alpha \times \beta = 1$
   (c) *All other intersection numbers are 0 except for those that following from the above by applying the identities below to arbitrary homology classes C and D.*
   $C \times D = -D \times C$
   $h^j(C) \times h^k(D) = h^0(C) \times h^{k-j}(D)$, *($k-j$ reduced modulo p.)*

9. Example, $p = 3$, $t = 5$, $(1, 1, 2, 1, 1)$

In this section we work out the specific example with $p = 3$ and $t = 5$. Assume $\phi(x_1) = 1, \phi(x_2) = 1, \phi(x_3) = 2, \phi(x_4) = 1, \phi(x_5) = 1$ so that $(n_1, ..., n_5) = (1, 1, 2, 1, 1)$ and $(m_1, m_2) = (4, 1)$.

First replacing h by a conjugate, we may assume that $\phi(x_1) = 1, \phi(x_2) = 1, \phi(x_3) = 1, \phi(x_4) = 1, \phi(x_5) = 2$. We choose as coset representatives x_1, x_1^2 and x_1^3.

For any g_0, we have generators

$$h^q(A_i), h^q(B_i), q = 0, \dots, p-1 = 2, i = 1 \dots, g_0.$$

and

$$S_{\overline{x_1^r}, x_j}, \quad r = 1,2,3, j = 2,3,4.$$

We also have by equation (9)

$$S_{\overline{x_1^r}, x_1} \approx 1, r = 1,2,3.$$

and, therefore, these generators and the relation $\tau(x_1^3)$ drops out of the set of generators and relations to be considered.

We have

$$\tau(x_j^3) = S_{\overline{1}, x_j} \cdot S_{\overline{x_j}, x_j} \cdot S_{\overline{x_j^2}, x_j} \tag{16}$$

Set $S_{\overline{1}, x_1} = Y_1$. Then $h(Y_1) = S_{x_1, x_1}$ and $h^2(X_j) = S_{x_1^2, x_j}$;
Set $S_{\overline{x_1}, x_2} = Y_2$ Then $h(Y_2) = S_{x_1^2, x_2}$ and $h^2(Y_2) = S_{x_1^3, x_2}$;
Similarly, set $S_{\overline{x_1 x_2}, x_3} = Y_3$. Then $h(Y_3) = S_{x_1^3, x_3}$ and $h^2(Y_3) = S_{x_1, x_3}$; If $S_{\overline{x_1 x_2 x_3}, x_4} = Y_4$ Then $h(Y_4) = S_{x_1, x_4}$ and $h^2(Y_4) = S_{x_1^2, x_4}$;and finally if $S_{\overline{x_1 x_2 x_3 x_4}, x_5} = Y_5$, then $h(Y_5) = S_{x_1^2, x_5}$ and $h^2(Y_5) = S_{x_1^3, x_j}$;

Use this notation and use $\tau(x_j^3) = 1$ to see that

$$\begin{aligned} h^2(Y_2) \cdot Y_2 \cdot h(Y_2) &= 1 \\ h(Y_3) \cdot h^2(Y_3) \cdot Y_3 &= 1 \\ Y_4 \cdot h(Y_4) \cdot h^2(Y_4) &= 1 \\ h^2(Y_5) \cdot h(Y_5) \cdot Y_5 &= 1 \end{aligned} \tag{17}$$

We compute

(18)

$$\tau(R) = S_{\overline{1}, x_1} \cdot S_{\overline{x_1}, x_2} \cdot S_{\overline{x_1 x_2}, x_3} \cdot S_{\overline{x_1 x_2 x_3}, x_4} \cdot S_{\overline{x_1 x_2 x_3 x_4}, x_5} \cdot (\Pi_{i=1}^{g_0}[A_i, B_i]) = 1.$$

Using $S_{\overline{x_1^r}, x_1} \approx 1, r = 1,2,3$ and solving for $(Y_2)^{-1}$ in equation (18), we have

$$(S_{x_1, x_2})^{-1} = Y_2^{-1} = Y_3 \cdot Y_4 \cdot Y_5 \cdot (\Pi_{i=1}^{g_0}[A_i, B_i]) = 1. \tag{19}$$

and

$$(h(Y_2))^{-1} = h(Y_3) \cdot h(Y_4) \cdot h(Y_5) \cdot (\Pi_{i=1}^{g_0}[h(A_i), h(B_i)]) = 1. \tag{20}$$

$$(h^2(Y_2))^{-1} = h^2(Y_3) \cdot h^2(Y_4) \cdot h^2(Y_5) \cdot (\Pi_{i=1}^{g_0}[h^2(A_i), h^2(B_i)]) = 1. \tag{21}$$

Using $h^2(Y_2)Y_2h(Y_2) = 1$ and letting $P = (\Pi_{i=1}^{g_0}[A_i, B_i])$, we have

(22)

$$h(Y_3) \cdot h(Y_4) \cdot h(Y_5) \cdot h(P) \cdot Y_3 \cdot Y_4 \cdot Y_5 \cdot P) \cdot h^2(Y_3) \cdot h^2(Y_4) \cdot h^2(Y_5) \cdot h^2(P) = 1$$

We use equation (17) to replace the $h^2(Y_j)$ and obtain the relation

(23)
$$h(Y_3) \cdot h(Y_4) \cdot h(Y_5) \cdot h(P) \cdot Y_3 \cdot Y_4 \cdot Y_5 \cdot P) \cdot (h(Y_3))^{-1} \cdot (Y_3)^{-1} \cdot (h(Y_4))^{-1} \cdot (Y_4)^{-1} \cdot (Y_5)^{-1} \cdot (h(Y_5))^{-1} \cdot h^2(P) = 1.$$

Equation (23) is the relation $\hat{\hat{R}} = 1$.

We use the notation of section 6, in particular the definition of the matrix B given at the end of that section. We can compute from the formulas for intersection numbers in Theorem 5.2 that the relevant part of the intersection matrix, B, is the 6×6 submatrix that gives the intersection matrix for the curves in the basis given in the order

$$X_{1,3}, h(X_{1,3}), X_{1,4}, h(X_{1,4}), X_{2,1}, h(X_{2,1})$$

is

$$B = I_{\hat{\hat{R}}} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

That is, the matrix $I_{\tilde{\mathcal{A}}}$ breaks up into blocks $\begin{pmatrix} 0 & I_{pg_0} & 0 \\ -I_{pg_0} & 0 & 0 \\ 0 & 0 & I_{\hat{\hat{R}}} \end{pmatrix}$

We now rearrange the relation. To simply the notation we let $a = Y_3$, $b = Y_4$ and $c = Y_5$. So that the relation becomes

(24)
$$h(a)\cdot h(b)\cdot h(c)\cdot h(P)\cdot a\cdot b\cdot c\cdot P\cdot (h(a))^{-1}a^{-1}(h(b))^{-1}\cdot b^{-1}\cdot c^{-1}\cdot (h(c))^{-1}\cdot h^2(P) = 1.$$

We can also make the simplifying assumption, replacing the elements that occur in P, $h(P)$ and $h^2(P)$ by conjugates, that we are merely working with the symbol

$$h(a)\cdot h(b)\cdot h(c)\cdot a\cdot b\cdot c\cdot (h(a))^{-1}a^{-1}(h(b))^{-1}\cdot b^{-1}\cdot c^{-1}\cdot (h(c))^{-1} = 1. \tag{25}$$

We replace generators and relations using the algorithm of [11] as follows:

Let $M = h(a)\cdot W_1 \cdot h(b) W_2 \cdot (h(a))^{-1}$ where $W_1 = \emptyset$, $W_2 = h(c)\cdot a\cdot b\cdot c$. Set $W_3 = a^{-1}$ and $W_4 = b^{-1} \cdot c^{-1} \cdot (h(c))^{-1}$.

Let $N = W_3 W_2 (h(a))^{-1}$. Then

(26)
$$\begin{aligned} h(a) \cdot h(b) \cdot h(c) \cdot a \cdot b \cdot c \cdot (h(a))^{-1} a^{-1} (h(b))^{-1} \cdot b^{-1} \cdot c^{-1} \cdot (h(c))^{-1} \\ = [M, N] W_3 W_2 W_1 W_4 \\ = [M, N] \cdot a^{-1} \cdot h(c) abcb^{-1} \cdot c^{-1} \cdot (h(c))^{-1} \\ = 1. \end{aligned}$$

At this point one can proceed by inspection and let $[b, c]^*$ denote the conjugate of $[b, c]$ by $a^{-1} \cdot h(c) \cdot a$ to obtain

$$[M, N] \cdot [b, c]^* \cdot [a^{-1}, h(c)] = 1.$$

However, to follow the algorithm carefully, we would set $\tilde{M} = a^{-1} \cdot h(c) \cdot a$ and $\tilde{N} = b \cdot c \cdot b^{-1} \cdot c^{-1} \cdot a$.
Then equation (26) becomes

$$[M, N] \cdot [\tilde{M}, \tilde{N}] \cdot [b, c] = 1.$$

Thus the canonical homology basis is given by

$$\{h^j(A_i), h^j(B_i)\}, i = 1, ..., g_0, j = 0, ..., p-1 \cup \{M, N, \tilde{M}, \tilde{N}, b, c\}.$$

The reordered basis $\{M, \tilde{M}, b, N, \tilde{N}, c\}$ has intersection matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

We can compute the action of h on these last six elements of the homology basis. First we note that $h(b) \approx^h M - c - b - a - h(c)$ and $h(a) \approx^h -N + h(c) + b + c$. Therefore,
$c \mapsto h(c) \approx^h \tilde{M}$
$h(c) \mapsto -c - h(c) \approx^h -c - \tilde{M}$
$a \mapsto h(a) \approx^h -N + h(c) + b + c \approx^h -N + \tilde{M} + b + c$
$b \mapsto h(b) \approx^h M - c - b - a - h(c) \approx^h M - c - b - \tilde{N} - \tilde{M}$
$M \mapsto h(M) \approx^h -\tilde{M} - N$, and
$N \mapsto h(N) \approx^h -c + M - N$.
Thus the matrix of the action of h with respect to the ordered basis $M, \tilde{M}, b, N, \tilde{N}, c$ is the submatrix we have been seeking. Namely,

$$N_{symp\tilde{A}} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

One can verify that this 6×6 matrix really is a submatrix of a symplectic matrix, as it should be.

10. Remarks

Recall (section 2.2) that there is a surjective map $\pi : MCG(S_g) \to SP(2g, \mathbb{Z})$. It is well known that the restriction of π to elements (mapping-classes) of finite order is an isomorphism. It is shown in [10] that Theorem 4.1 implies a stronger result than this which we note for completeness.

Corollary 10.1. [10] *If h is a conformal automorphism of S of genus $g \geq 2$ and if there are two pairs of curves C_1, D_1 and C_2, D_2 with $C_i \times C_j = D_i \times D_j = 0$ for $i = 1, 2$ and $j = 1, 2$ and $C_i \times D_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta, and $h(C_i) \approx^h C_i$ and $h(D_i) \approx^h D_i$ for $i = 1, 2$, then h is the identity.*

Proof. If h is of prime order, simply write each of the four curves as a sum of the curves in the adapted homology basis, apply h and equate coefficients. If h is not of prime order, apply this to every power that is of prime order to see that each must be the identity. □

We obtain immediately,

Corollary 10.2. *If two mapping-classes of finite order have the same action on homology, then they are equal. Equivalently, the restriction of π to elements of finite order is an isomorphism.*

The idea of an adapted homology basis predates Thurston's notation of a reducible mapping-class. However, it is clear that the notions are related. A homeomorphism h is *reducible* if h fixes a *a partition* on the surface, that is, a set of disjoint simple closed curves on the surface. A mapping-class is reducible if it contains a reducible representative.

Corollary 10.3. *If h represents a mapping-class of prime order, $p \geq 2$, and $g_0 \neq 0$, then h is a reducible mapping-class.*

Proof. Replace h by a conformal representative if necessary. An element of type (1) (definition 3.1) in an adapted basis for h taken along with its images gives a set of closed curves on the surface, fixed by the homeomorphism of the surface. Once we have shown that these are simple closed curves (as we do in theorem 5.2), is clear that the element h is reducible, that is, h fixes a *a partition*, a set of disjoint simple closed curves on the surface. □

11. Acknowledgements

The author thanks Yair Minsky and the Yale Mathematics Department for their hospitality and support while some of this work was carried out.

We give a long, but by no means exhaustive bibliography.

References

[1] P. Buser, Peter G. Courtois, *Finite parts of the spectrum of a Riemann surface. Math. Ann. 287 (1990), no. 3, 523–530. MR 92c:58146*

[2] *P. Buser M. Seppälä, Computing on Rimeann Surfaces, Topology and Teichmüller Spaces, World Scientific, (1996) 5-30.*

[3] *P. Buser and M. Seppälä, Symmetric pants decompositions of Riemann surfaces, Duke Math. J.* **67** *(1992), no. 1, 39-55.*

[4] *P. Buser and R. Silhol, Geodesics, periods, and equations of real hyperelliptic curves, Duke Math. J.* **108** *(2001), no. 2, 211-250.*

[5] *M. D. E. Conder and R. S. Kulkarni, Ravi S., Infinite families of automorphism groups of Riemann surfaces in Discrete groups and geometry (Birmingham, 1991), 47–56, London Math. Soc. Lecture Note Ser.,* **173**, *Cambridge Univ. Press, Cambridge, (1992).*

[6] *J. Gilman, Relative Modular Groups in Teichmüller Spaces, PhD thesis, Columbia University, (1971).*

[7] *J. Gilman, Compact Riemann Surfaces with Conformal Involutions, Proc. Amer. Math. Soc.,* **37** *(1973), 105-107.*

[8] *J. Gilman, On the Moduli of Compact Riemann Surfaces with a Finite Number of Punctures Discontinuous Groups and Riemann Surfaces, Annals of Math. Studies,* **79** *(1974), 181-205.*

[9] *J. Gilman, On Conjugacy Classes in the Teichmüller Modular Group, Mich. Math. J.,* **23** *(1976) 53-63.*

[10] *J. Gilman, A Matrix Representation for Automorphisms of Riemann Surfaces, Lin. Alg. and its Applications,* **17** *(1977), 139-147.*

[11] *Gilman, J. Canonical Forms for Conjugacy Classes of Prime Order Symplctic Matrices in* $Sp(2g,\mathbb{Z})$*, submitted.*

[12] *J. Gilman, and D. Patterson, Intersection Matrices for Adapted Bases, Proc. 1978 Stony Brook Conference, Annals of Math. Studies* **97** *(1981), 149-166.*

[13] *J. Gilman, Structures of Elliptic Irreducible Subroups of the Mapping-class Group, Proc. London Math Soc.* **47 (3)** *(1983), 27-42.*

[14] *V. Gonzalez-Aguilera R. E. Rodriguez, On principally polarized abelian varieties and Riemann surfaces associated to prisms and pyramids in Lipa's legacy, Contemp. Math.* **211** AMS (1997), 269-284,

[15] W. J. Harvey, W.J. *On branch loci in Teichmüller space*, Trans. Amer. Math. Soc. **153** (1971), 387-399.
[16] R. S. Kulkarni, *Riemann surfaces admitting large automorphism groups* (1995), 63–79, Contemp. Math., 201, Amer. Math. Soc., Providence, RI.
[17] R. S. Kulkarni, *Infinite families of surface symmetries* Israel J. Math. **76** (1991), no. 3, 337–343.
[18] R. S. Kulkarni, *A note on Wiman and Accola-Maclachlan surfaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16 (1)** (1991), 83–94.
[19] R. S. Kulkarni and C. Maclachlan, *Cyclic p-groups of symmetries of surfaces*, Glasgow Math. J. **33(2)** (1991) 213–221.
[20] A.M. MacBeath, *Action of automorphisms of a compact Riemann surface on the First Homology Group*, Bull Lon. Math Soc., **5**, (1973), 103-108.
[21] Magnus, Karrass, and Solitar, sl Combinbatorial Group Theory, J. Wiley, (1966). 253-274.
[22] J. Nielsen, *Die Structur periodsiher Transformationen von Flachen*, D.K. Dan. Vidensk. Selsk. Math-fys. Medd **XV** (1937), 1-77.
[23] G. Riera, Gonzalo and R. E. Rodrguez, *Riemann surfaces and abelian varieties with an automorphism of prime order*. Duke Math. J. 69 (1993), no. 1, 199–217. MR 93k:14059
[24] D. Singerman, *Finitely maximal Fuchsian groups*, J. London Math. Soc. **(2) 6** (1972), 29–38.
[25] G. Springer, *Introduction to Riemann Surfaces* Addison-Wesley, (1957).
[26] A. Weaver, *Hyperelliptic surfaces and their moduli*, Geom. Dedicata **103** (2004), 69–87.

Mathematics Department
Rutgers University
Newark, NJ 07102
e-mail: gilman@andromeda.rutgers.edu; jane.gilman@yale.edu