ENUMERATING PALINDROMES AND PRIMITIVES IN RANK TWO FREE GROUPS

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Abstract. Let $F = \langle a, b \rangle$ be a rank two free group. A word $W(a, b)$ in F is *primitive* if it, along with another group element, generates the group. It is a *palindrome* (with respect to a and b) if it reads the same forwards and backwards. It is known that in a rank two free group any primitive element is conjugate either to a palindrome or to the product of two palindromes, but known iteration schemes for all primitive words give only a representative for the conjugacy class. Here we derive a new iteration scheme that gives either the unique palindrome in the conjugacy class or expresses the word as a unique product of two unique palindromes. We denote these words by $E_{p/q}$ where p/q is rational number expressed in lowest terms. We prove that $E_{p/q}$ is a palindrome if pq is even and the unique product of two unique palindromes if pq is odd. We prove that the pairs $(E_{p/q}, E_{r/s})$ generate the group when $|ps - rq| = 1$. This improves the previously known result that held only for pq and rs both even. The derivation of the enumeration scheme also gives a new proof of the known results about primitives.

1. Introduction

It is well known that up to conjugacy primitive generators in a rank two free group can be indexed by the rational numbers and that pairs of primitives that generate the group can be obtained by the number theory of the Farey tessellation of the hyperbolic plane. It is also well known that up to conjugacy a primitive word can always be written as either a palindrome or a product of two palindromes and that certain pairs of palindromes will generate the group. [1, 19]

In this paper we give new proofs of the above results. The proofs yield a new enumerative scheme for conjugacy classes of primitive words, still indexed by the rationals (Theorem 2.1 and Theorem 6.1). We denote the words representing each conjugacy class by $E_{p/q}$. In addition to proving that the enumeration scheme gives a unique representative for each conjugacy class containing a primitive, we prove that the words in this scheme are either palindromes or the canonically defined product of a pair of palindromes that have already appeared in the scheme and thus give a new proof of the palindrome/product result. Further we show that pairs of these words $(E_{p/q}, E_{r/s})$ are primitive pairs that generate the group if and only of $|pq - rs| = 1$ (Theorem 6.2). This improves the previous known result that held only for pairs of palindromes.

Pairs of primitive generators arise in the discreteness algorithm for $PSL(2, \mathbb{R})$ representations of two generator groups [4, 5, 11]. This enumerative scheme will be useful in extending discreteness criteria to $PSL(2, \mathbb{C})$ representations where the

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hyperbolic geometry of palindromes plays an important role [10]. Here we use the discreteness algorithm and its relation to continued fractions as described in [8], and its relation to the Farey tessellation of the hyperbolic plane, to find the enumeration scheme and to prove that it actually enumerates all primitives and all primitive pairs.

2. The Main Result

We are able to state and use our main result with very little notation, only the definition of continued fractions. Namely, we let p and q be relative prime integers positive integers. Write

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_k}}}} = [a_0; a_1, \ldots, a_k]$$

where the a_i are integers with $a_j > 0$, $j = 1 \ldots k$, $a_0 \geq 0$.

Enumeration Scheme for positive rationals

Set

$$E_{0/1} = A^{-1}, E_{1/0} = B, \text{ and } E_{1/1} = BA^{-1}.$$

Suppose p/q has continued fraction expansion $[a_0; a_1, ..., a_{k-1}, a_k]$. Consider the two rationals defined by the continued fractions $[a_0;_1, ..., a_{k-1}]$ and $[a_0, ..., a_{k-1}, a_k - 1]$. One is smaller than p/q and the other is larger; call the smaller one m/n and the larger one r/s so that $m/n < p/q < r/s$. The induction step in the scheme is given by

Case 1 pq - odd:

$$E_{p/q} = E_{r/s} E_{m/n}.$$

Case 2 pq - even:

$$E_{p/q} = E_{m/n} E_{r/s}.$$

We have a similar scheme for negative rationals described in section 6. With both schemes we can state our main result as

Theorem 2.1. (Enumeration of Primitives by Rationals) *Up to inverses, the primitive elements of a two generator free group can be enumerated by the rationals using continued fraction expansions. The resulting words are denoted by $E_{p/q}$. In the enumeration scheme, when pq is even, $E_{p/q}$ is a palindrome, and when pq is odd, $E_{p/q}$ is a product of palindromes that have already appeared in the scheme. Moreover,*

- *For pq even, $E_{p/q}$ is a palindrome. It is cyclically reduced and the unique palindrome in its conjugacy class.*
- *For pq odd, when $E_{p/q} = E_{m/n} E_{r/s}$ both $E_{m/n}$ and $E_{r/s}$ are palindromes; $E_{p/q}$ is cyclically reduced.*

Remark 2.1. *Note that although there are several ways a word in the pq odd conjugacy class can be factored as products of palindromes, in this theorem we specifically choose the unique factorization for $E_{p/q}$ that makes the enumeration scheme work.*

In addition we have,

Theorem 2.2. *Let $\{E_{p/q}\}$ denote the words in the enumeration scheme for rationals. Then if $(p/q, p'/q')$ satisfies $|pq' - qp'| = 1$, the pair $(E_{p/q}, E_{p'/q'})$ generates the group.*

These theorems will be proved in section 6. In order to prove the theorems and the related results we need to review some terminology and background.

3. Preliminaries

The main object here is a two generator free group which we denote by $F = \langle a, b\rangle$. A word $W = W(a, b) \in F$ is, of course, an expression of the form

$$a^{m_1} b^{n_1} a^{m_2} \cdots b^{n_r} \tag{1}$$

for some set of $2r$ integers $m_1, ..., m_r, n_1, ..., n_r$ with $m_2, ..., m_r, n_1, ..., n_{r-1}$ nonzero. The expression $W(c, d)$ denotes the word W with a and b replaced by c and d. The expressions $W(b, a)$, $W(a^{-1}, b), W(a^{-1}, b^{-1})$ and $W(a, b^{-1})$ have the obvious meaning.

Definition 1. *A word $W = W(a, b) \in F$ is* primitive *if there is another word $V = V(a, b) \in F$ such that W and V generate F. V is called a* primitive associate *of W and the unordered pair W and V is called a* pair of primitive associates *or a primitive pair for short.*

In the next subsections we summarize terminology and facts about the Farey tessellation and continued fraction expansions for rational numbers. Details and proofs can be found in [20, 21, 22]. See also [23].

3.1. Preliminaries: The Farey Tessellation.

In what follows when we use r/s to denote a rational number, we assume that r and s are integers with $s > 0$, and that r and s are relatively prime, that is, that $(r, s) = 1$. We let $\mathbb{Q}$ denote the rational numbers, but we identify the rationals with points on the extended real axis on the Riemann sphere. We use the notation $1/0$ to denote the point at infinity.

We need the concept of Farey addition for fractions.

Definition 2. *If $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ with $|ps - qr| = 1$, the* Farey sum *is*

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$$

Two fractions are called Farey neighbors *if $|ps - qr| = 1$.*

When we write $\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$ we tacitly assume the fractions are Farey neighbors.

Remark 3.1. *If $\frac{p}{q} < \frac{r}{s}$ then it is a simple computation to see that*

$$\frac{p}{q} < \frac{p}{q} \oplus \frac{r}{s} < \frac{r}{s}$$

and that both pairs of fractions

$$\left(\frac{p}{q}, \frac{p}{q} \oplus \frac{r}{s}\right) \text{ and } \left(\frac{p}{q} \oplus \frac{r}{s}, \frac{r}{s}\right)$$

are Farey neighbors if $(p/q, r/s)$ are.

It is easy to calculate that the Euclidean distance between finite Farey neighbors is strictly less than one unless they are adjacent integers. This implies that unless one of the fractions is $0/1$, both neighbors have the same sign.

One creates the Farey diagram in the upper half-plane by marking each fraction by a point on the real line and joining each pair of Farey neighbors by a semi-circle orthogonal to the real line. The point here is that because of the above properties none of the semi-circles intersect in the upper half plane. This gives a tessellation of the hyperbolic plane where the semi-circles joining a pair of neighbors, together with the semi-circles joining each member of that pair to the Farey sum of the pair, form an ideal hyperbolic triangle. The tessellation is called the Farey tessellation and the vertices are precisely the points that correspond to rational numbers. See Figure 1

Figure 1. The Farey Tesselation with the curve γ

The Farey tessellation is invariant under the semi-group generated by $z \mapsto z+1$ and $z \mapsto 1/z$.

Fix any point ζ on the positive imaginary axis. Given a fraction, $\frac{p}{q}$, there is an oriented hyperbolic geodesic γ connecting ζ to $\frac{p}{q}$. We assume γ is oriented so that moving from ζ to a positive rational is the positive direction. This geodesic γ will intersect some number of triangles.

Definition 3. *The* Farey level *or the* level *of p/q, denoted by $Lev(p/q)$, is the number of triangles traversed by γ*

Note that our definition implies $Lev(p/q) = Lev(-p/q)$.

The geodesic γ will enter any given triangle along an edge. This edge will connect two vertices and γ will exit the triangle along an edge connecting one of these two vertices and the third vertex of the triangle, the *new* vertex. Since γ is oriented, the edge through which γ exits a triangle is either a left edge or a right edge depending upon whether the edge γ cuts is to the right or left of the new vertex.

Definition 4. *We determine a* Farey sequence for $\frac{p}{q}$ *inductively by choosing the new vertex of next triangle in the sequence of triangles traversed by γ. The sequence ends at p/q.*

Given p/q, we can find the smallest rational m/n and the largest rational r/s that are neighbors of p/q. These neighbors have the property that they are the only neighbors with lower Farey level. That is, $m/n < p/q < r/s$ and $Lev(p/q) > Lev(m/n)$, $Lev(p/q) > Lev(r/s)$, and if u/v is any other neighbor $Lev(p/q) < Lev(u/v)$.

Definition 5. *We call the smallest and the largest neighbors of the rational p/q the* distinguished neighbors *or the parents of p/q.*

Note that we can tell whether a distinguished neighbor r/s is smaller or larger than p/q by the sign of $rq - ps$.

We emphasize that we have two different and independent orderings of the rational numbers: the ordering as rational numbers and the partial ordering by level. Our proofs will often use induction on the level of the rational numbers involved as well as the rational order relations among parents and grandparents.

Remark 3.2. *It follows from remark 3.1 that if m/n and r/s are the parents of p/q, then any other neighbor of p/q is of the form $\frac{m+pt}{n+tp}$ or $\frac{r+pt}{s+tp}$ for some positive integer t. The neighbors of ∞ are precisely the set of integers.*

Finally we note that we can describe the Farey sequence of p/q by listing the number of successive left or right edges of the triangles that γ crosses where a left edge means that there is one vertex of the triangle on the left of γ and two on the right, and a right edge means there is only one vertex of γ on the right. This will be a sequence of integers, the *left-right sequence* $\pm(n_0; n_1, ..., n_t)$ where the integers $n_i, i > 0$ are all positive and n_0 is positive or zero. The sign in front of the sequence is positive or negative depending on the orientation of γ.

3.2. **Preliminaries: Continued Fractions.** Farey sequences are related to continued fraction expansions of positive fractions; they can also be related to expansions of negative fractions. We review the connection in part to fix our notation precisely. We do not use the classical notation of [12] for negative fractions, but instead use the notation of [17, 21, 20] which is standardly used by mathematicians working in Kleinian groups and three manifolds. This notation reflects the symmetry about the imaginary axis in the Farey tessellation which plays a role in our applications. This symmetry is built in to the semi group action on the tessellation.

For, for $p/q \geq 0$ write

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \frac{1}{a_k}}}} = [a_0; a_1, \ldots, a_k]$$

where $a_j > 0$, $j = 1 \ldots k$, $a_0 \geq 0$. For $0 \leq n \leq k$ set

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n].$$

Remark 3.3. *The continued fraction of a rational is ambiguous; the continued fractions $[a_0; a_1, ..., a_n]$ and $[a_0; a_1, ..., a_n - 1, 1]$ both represent the same rational. Therefore, if we have $[a_0; a_1, ..., a_{n-1}, 1]$ we may replace it with $[a_0; a_1, ..., a_{n-1} + 1]$.*

Remark 3.4. *Note that if $\frac{p}{q} \geq 1$ has continued fraction expansion $[a_0; a_1, ..., a_n]$, then $\frac{q}{p}$ has expansion $[0; a_0, ..., a_n]$ while if $\frac{p}{q} < 1$ has continued fraction expansion $[0; a_1, ..., a_n]$, then $\frac{q}{p}$ has expansion $[a_1; a_2, ..., a_n]$.*

The approximating fractions, $\frac{p_n}{q_n}$, are also known as the *approximants*. They can be computed recursively from the continued fraction for p/q as follows:

$$p_0 = a_0, q_0 = 1 \text{ and } p_1 = a_0 a_1 + 1, q_1 = a_1$$

$$p_j = a_j p_{j-1} + p_{j-2} \ , q_j = a_j q_{j-1} + q_{j-2} \ j = 2, \ldots, k.$$

One can calculate from these recursion formulas that the approximants are alternately to the right and left of p/q.

The Farey level can be expressed in terms of the continued fraction $p/q = [a_0; a_1, \ldots a_k]$ by the formula

$$Lev(p/q) = \sum_{j=0}^{k} a_j.$$

The distinguished neighbors or parents of p/q have continued fractions

$$[a_0; a_1 \ldots, a_{k-1}] \text{ and } [a_0; a_1, \ldots, a_{k-1}, a_k - 1].$$

The Farey sequence contains the approximating fractions as a subsequence. The points of the Farey sequence between $\frac{p_j}{q_j}$ and $\frac{p_{j+1}}{q_{j+1}}$ have continued fraction expansions

$$[a_0; a_1, \dots a_j + 1], [a_0; a_1, \dots, a_j + 2], \dots, [a_0; a_1, \dots a_j + a_{j+1} - 1].$$

They are all on the same side of p/q.

We extend the continued fraction notion to negative fractions by defining the continued fraction of $\frac{p}{q} < 0$ to be the negative of that for $|\frac{p}{q}|$. That is, by setting

$$\frac{p}{q} = -[a_0; a_1 \dots, a_k] = [-a_0; -a_1 \dots, -a_k] \text{ where } |\frac{p}{q}| = [a_0; a_1, \dots a_k].$$

We also set $Lev(p/q) = Lev(|p/q|)$.

In [12] the continued fraction $[a_0; a_1, \dots, a_k]$ of $p/q < 0$ is defined so that $a_0 < 0$ is the largest integer in p/q and $[a_1, \dots a_k] = p/q - a_0$ is the continued fraction of a positive rational. With this notation the symmetry about the origin which plays a role in our applications is lost.

We note that for any pair of neighbors, unless one of them is $0/1$ or $1/0$, they both have the same sign and thus have equal a_0 entries. Since we almost always work with neighbors the difference between our notation and the classical one does not play a role.

4. Preliminaries: Lifting results from $PSL(2, \mathbb{R})$ to the free group F.

In addition to the free group on two generators, $F = \langle a, b \rangle$, we also consider a non-elementary representation of F into $PSL(2, \mathbb{R})$ where $\rho : F \to PSL(2, \mathbb{R})$ with $\rho(a) = A, \rho(b) = B$ and $\rho(F) = G = \langle A, B \rangle$. In [11] it was shown that if the representation were discrete and free, then up to taking inverses as necessary, any pair of primitive words could be obtained from (A, B) by applying a sequence of Nielsen transformations to the generators. This sequence is described by an ordered set of integers, $[a_0, \dots a_k]$ and is used in computing the computational complexity of the algorithm [5, 13]. It was termed the Fibonacci or $\mathcal{F}-$sequence in [8]. The words obtained by applying the algorithm are known as algorithmic words; here we call them the $\mathcal{F}$-words. We give precise definitions below (section 4.2).

Our aim is to lift these results for the generators of the representation groups to pairs of primitives in the free group. To do this we review the algorithm and other prior results. In the free group F there is no concept equivalent to a geometric orientation. Therefore in lifting statements from $PSL(2, \mathbb{R})$ to F we need to carefully analyze the role of geometric orientation in the algorithm.

4.1. Coherent Orientation, algorithmic words and stopping generators.

The group $PSL(2, \mathbb{R})$ consists of isometries in the hyperbolic metric on the upper half plane $\mathbb{H}$. It is conjugate in $PSL(2, \mathbb{C})$ to the group of isometries of the unit disk $\mathbb{D}$ with its hyperbolic metric. By abuse of notation, we identify these groups and use whichever model is easier at the time. All of the results below are independent of the model we use. An isometry is called *hyperbolic* if it has two fixed points on the boundary of the half-plane or the disk, and leaves the hyperbolic geodesic joining them invariant. This geodesic is called the axis of the element. One of the fixed points is attracting and the other is repelling. This gives a natural orientation to the axis since points are moved along the axis toward the attracting fixed point. This natural orientation does not exist in the free group.

The result that we will apply from $PSL(2,\mathbb{R})$ uses the orientation of an axis of an element to define the notion of a coherently oriented pair of elements or axes. In what follows we need to lift this concept to the free group.

Definition 6. *Let A and B be any pair of hyperbolic generators of the group G acting as isometries on the unit disk. Assume that they are given by representatives in $SL(2,\mathbb{R})$ with $trA \geq trB > 2$. Suppose the axes of A and B are disjoint. Let L be the common perpendicular geodesic to these axes oriented from the axis of A to the axis of B. We may assume that the attracting fixed point of A is to the left of L, replacing A by A^{-1} if necessary. We say A and B are* **coherently oriented** *if the attracting fixed point of B is also to the left of L and* **incoherently oriented** *otherwise.*

If (A,B) are coherently oriented, then (A,B^{-1}) are incoherently oriented.

If $G = \langle A,B\rangle$ is discrete and free, and the axes of A and B are disjoint, the quotient Riemann surface $\mathbb{D}/G$ is a sphere with three holes; that is a pair of pants. The axes of hyperbolic group elements project to closed geodesics on S. The length of the geodesic is determined by the trace of the element.

4.1.1. *Stopping generators.* If G is a discrete free subgroup of $PSL(2,\mathbb{R})$, the Gilman-Maskit algorithm [11] goes through finitely many steps and at the last, or $k+1^{th}$ step, it determines that the group is discrete and stops. At each step, $t = 0,\ldots,k$, it determines an integer a_t and a new pair of generators (A_{t+1},B_{t+1}); these integers form an $\mathcal{F}-$sequence and the pairs, (A_t,B_t), of *algorithmic words* are the $\mathcal{F}-$ words in the process above. The final pair of generators $(C,D) = (A_{k+1},B_{k+1})$ are called the *stopping generators*. It is shown in [8] that if the axes of the original generators are disjoint, the stopping generators have the geometric property that their axes, together with the axis of $A_{k+1}^{-1}B_{k+1}$, project to the three shortest geodesics on the quotient Riemann surface and these geodesics are disjoint and simple [8].

Lemma 4.1. *If the pair (A,B) is coherently oriented, then either the pair (C,D) is coherently oriented or one of the pairs (D,C^{-1}) or (C,D^{-1}) is.*

Proof. We assume without loss of generality that the pair (A,B) is coherently oriented because if it is not, one of the pairs (A,B^{-1}) or (B,A^{-1}) or $B^{-1},A^{-1})$ is coherently oriented and we can replace it with that one. We can analyze the steps in the algorithm and the orientations of the intermediate generators carefully and see that, if we start with a coherently oriented pair, at each step, up to the next to last, $t=k$, the pair we arrive at, (B_{t-1}^{-1},B_t), is coherently oriented. We therefore need to check whether the last pair,

$$(C,D) = (A_{k+1},B_{k+1}) = (B_k^{-1},B_{k+1})$$

is coherently oriented.

The stopping condition is that the last word B_{k+1} have negative trace. We know $trB_{k-1} > trB_k$; we don't know the relation of $|trB_{k+1}|$ to these traces. We will have either

$$|trB_{k+1}| > trB_{k-1} > trB_k \text{ or}$$
$$trB_{k-1} > |trB_{k+1}| > trB_k \text{ or}$$
$$trB_{k-1} > trB_k > |trB_{k+1}|.$$

In the first case (B_{k-1}^{-1}, B_k) is coherently oriented. In the second case (B_k^{-1}, B_{k+1}) is incoherently oriented but $(D, C^{-1}) = (B_{k+1}^{-1}, B_k)$ is coherently oriented. In the third case, again (B_k^{-1}, B_{k+1}) is incoherently oriented but this time $(C, D^{-1}) = (B_k^{-1}, B_{k+1}^{-1})$ is coherently oriented. □

4.2. $\mathcal{F}$-sequences.

Definition 7. *An $\mathcal{F}-$sequence is an ordered set of integers $[a_0, \dots a_k]$ where all the a_i, $i = 0, \dots, k$ have the same sign and all but a_0 are required to be non-zero.*

Given an $\mathcal{F}$-sequence we define a sequence of words in the group G.

Definition 8. **$\mathcal{F}$-words.** *Let A and B generate the group G and let $\mathcal{F} = [a_0, ..., a_k]$ be an $\mathcal{F}-$sequence. We define the ordered pairs of words (A_t, B_t), $t = 0, \dots, k$ inductively, replacing the pair (X, Y) given at step t by the pair $(Y^{-1}, X^{-1}Y^{a_t})$ as follows: Set*

$$(A_0, B_0) = (A, B)$$

and

$$(A_1, B_1) = (B^{-1}, A^{-1}B^{a_0}).$$

Then for $t = 1, \dots, k$, set

$$(A_{t+1}, B_{t+1}) = (B_t^{-1}, A_t^{-1}B_t^{a_t}).$$

Note that $A_{t+1} = B_t^{-1}$ and $B_{t+1} = B_{t-1}B_t^{a_t}$ We call the words (A_t, B_t) the $\mathcal{F}$-words determined by the $\mathcal{F}$-sequence.

We use the notation $B_t = W_{[a_0,\dots a_t]}(A, B)$. With this notation the last pair is

$$A_{k+1} = (W_{[a_0,\dots a_{k-1}]}(A, B))^{-1} \text{ and } B_{k+1} = W_{[a_0,\dots a_k]}(A, B).$$

4.3. Winding and Unwinding.

In [8] we studied the relationship between a given pair of generators for a free discrete two generator subgroup of $PSL(2, \mathbb{R})$ with disjoint axes and the stopping generators produced by the Gilman-Maskit algorithm. We found we could interpret the algorithm as an unwinding process, a process that at each step reduces the number of self-intersections of the corresponding curves on the quotient surface and unwinding the way in which stopping generators had been wound around each other to obtain the original primitive pair.

Here is an example where we denote the original given pair of generators by (A, B) and the stopping generators by (C, D).

Example 1. *We begin with the (unwinding) $\mathcal{F}-$sequence $[3, 2, 4]$ and obtain the words*

$$(A_0, B_0) = (A, B)$$
$$(A_1, B_1) = (B^{-1}, A^{-1}B^3)$$
$$(A_2, B_2) = (B^{-3}A, BA^{-1}B^3A^{-1}B^3)$$

and

$$(A_3, B_3) = (B^{-3}AB^{-3}AB^{-1}, A^{-1}B^3 \cdot (BA^{-1}B^3A^{-1}B^3)^4) = (C, D).$$

Going backwards

$$(C_0, D_0) = (C, D)$$
$$(C_1, D_1) = (C_0^{-4}D_0^{-1}, C_0^{-1}) = (B^{-3}A, BA^{-1}B^3A^{-1}B^3)$$

$$(C_2, D_2) = (C_1^{-2} D_1^{-1}, C_1^{-1}) =$$
$$((B^{-3}A)^{-2} \cdot (BA^{-1}B^3A^{-1}B^3)^{-1}, A^{-1}B^3) = (B^{-1}, A^{-1}B^3)$$
$$(C_3, D_3) = (C_2^{-3} D_2^{-1}, C_2^{-1}) = (B^3B^{-3}A, B) = (A, B)$$

We can think of this as the (winding) sequence given by $[-4,-2,-3]$ and write

$$A = W_{[-4,-2,-3]}(C, D) \text{ and } B = W^{-1}_{[-4,-2]}(C, D).$$

Definition 9. *Let q be a positive integer. A winding step labeled by the integer $-q$ will send the pair (U, V) to the pair $(U^{-q}V^{-1}, U^{-1})$ and an unwinding step labeled by the integer q the will send the pair (M, N) to the pair $(N^{-1}, M^{-1}N^q)$.*

Theorem 4.2. [8] *If $G = \langle A, B\rangle$ is a non-elementary, discrete, free subgroup of $PSL(2,\mathbb{R})$ where A and B are hyperbolic isometries with disjoint axes, then there exists an* unwinding *$\mathcal{F}$-sequence $[a_0, ..., a_k]$ such that the stopping generators (C, D) are obtained from the pair (A, B) by applying this $\mathcal{F}$-sequence. There is also an unwinding $\mathcal{F}$-sequence $[b_0, ..., b_k]$ such that the pair (A, B) is the final pair in the set of $\mathcal{F}$-words obtained by applying the winding $\mathcal{F}$-sequence to the pair (C, D).*

The sequences are related by $[b_0, ..., b_k] = [-a_k, ..., -a_0]$

This motivates the following definition.

Definition 10. *(1) We call the $\mathcal{F}$-sequence $[a_0, a_1, \ldots, a_k]$, determined by the discreteness algorithm that finds the stopping generators when the group is discrete, the* **unwinding** *$\mathcal{F}$-sequence.*

(2)We call the $\mathcal{F}$-sequence $[b_0, b_1, \ldots, b_k]$, that determines the original generators (A, B) from the stopping generators (C, D), the **winding** *$\mathcal{F}-$sequence.*

4.4. **$\mathcal{F}$-sequences and rational numbers.** We have been using a notation for our $\mathcal{F}$-sequences that looks very much like the continued fraction notation. We justify this by identifying the rational p/q with continued fraction $[a_0; \ldots, a_k]$ with the $\mathcal{F}-$sequence $[a_0, \ldots, a_k]$. This justifies our modifying the classical definition of continued fractions for negative numbers in section 3.2. Moreover, the ambiguity in the definition of stopping generators corresponds exactly to the ambiguity in the definition of a continued fraction.

5. Primitive exponents

It follows from Theorem 4.2 that the stopping generators are independent of the given set of generators. This means that every primitive word in the group G is the last word in a winding $\mathcal{F}$-sequence. Using the rules for winding and unwinding and the identification of the $\mathcal{F}$-sequence with the rational p/q it is easy to show that if we expand the $\mathcal{F}$-words into the form of (1) we have

Theorem 5.1. *Let (C, D) be stopping generators for G and assume they are labeled so that they are coherently oriented. Every primitive word $W(C, D)$ in G has one of the following four forms where the $v_i > 0$, $i = 1, \ldots, j-1$, and $v_0, v_j \geq 0$.*

$$W(C, D) = \begin{cases} C^{\epsilon v_0} D^{-\epsilon} C^{\epsilon v_1} D^{-\epsilon} C^{\epsilon v_2} \cdots D^{-\epsilon} C^{v_j} & \text{for } p/q > 1 \text{ or} \\ C^{-\epsilon v_0} D^{\epsilon} C^{\epsilon v_1} D C^{-v_2} \cdots D^{\epsilon} C^{-\epsilon v_j} & \text{for } 0 < p/q \leq 1 \text{ or} \\ D^{\epsilon v_0} C^{\epsilon} D^{\epsilon v_1} C^{\epsilon} D^{\epsilon v_2} \cdots C^{\epsilon} D^{\epsilon v_j} & \text{for } p/q < -1 \text{ or} \\ C^{\epsilon v_0} D^{\epsilon} C^{v_1} D^{\epsilon} C^{\epsilon v_2} \cdots D^{\epsilon} C^{\epsilon v_j} & \text{for } -1 \leq p/q < 0 \end{cases}$$

where $p/q = [a_0; a_1, \ldots, a_k]$ is the rational corresponding to the winding $\mathcal{F}$-sequence and $\epsilon = \pm 1$ as k is even or odd.

Proof. Starting with coherently oriented generators (C, D), and an $\mathcal{F}$-sequence with non-negative entries, the exponents of C and D in the $\mathcal{F}$-words always have opposite signs. If $a_0 > 0$, and $p/q \geq 1$, we see that $D_1 = C^{-1}D^{a_0}$ and, as we go through the $\mathcal{F}$-words, the exponent of C will always have absolute value 1 as in the first line. If, on the other hand, $a_0 = 0$, and $0 < p/q < 1$, we see that $(C_1, D_1) = (D^{-1}, C^{-1})$ and the roles of C and D^{-1} and D and C^{-1} are interchanged as in the second line.

If we begin with an $\mathcal{F}$-sequence with non-positive entries, the negative entries cause the exponents of C and D in the $\mathcal{F}$-words always to have the same sign. Again, if $a_0 < 0$, we see that $D_1 = C^{-1}D^{a_0}$ and as we go through the $\mathcal{F}$-words, the exponent of C will always have absolute value 1 as in the third line. Similarly, if $a_0 = 0$ we get the form of the last line.

In either case, as we step from $t-1$ to t, we have $D_t = C_{t-1}^{-1}$ so that the signs of all the exponents change. This accounts for the appearance of ϵ in the exponents. □

Remark 5.1. (No Cancellation) *We see from the above theorem that in any primitive word the exponents of the C generator are all of the same sign as are those of the D generator. Moreover, by Theorem 4.2 and the identification of the $\mathcal{F}$-words with the algorithmic words, we see that if we have a primitive pair, the $\mathcal{F}$-sequences agree in their first k entries so that both words correspond to fractions in the same interval of Theorem 5.1. This implies that there is no cancellation when we form products. Thus we do not need to distinguish between concatenation and free reduction.*

We call the exponents v_i the *primitive exponents* of the word $W(C, D)$. They have the property that two adjacent primitive exponents differ by at most 1. There are formulas for writing the primitive exponents in terms of the entries in the $(\mathcal{F})-$ sequence which can be found in [8] and [9] but we will not need them here.

The identification of continued fractions for rationals to $\mathcal{F}$-sequences, together with Remark 5.1, immediately imply

Corollary 5.2. *There is a one-to-one map, τ from pairs of rationals $(p/q, r/s)$, with $|ps - rq| = 1$ to coherently oriented primitive pairs defined by $\tau : (p/q, r/s) \mapsto (A, B)$ where p/q is the rational with continued fraction expansion $[a_0; .a_1, \dots, a_k]$ and r/s is the rational with continued fraction expansion $[a_0; a_1, \dots, a_{k-1}]$.*

Corollary 5.3. *Up to replacing a primitive word by its inverse, there is a one-to-one map from the set of primitive elements to the set of all rationals.*

Proof. In the map τ above, to each rational we either obtain a word in either (C, D) or (D^{-1}, C^{-1}) or (C, D^{-1}) or (D, C^{-1}). No word and its inverse both appear. □

In the unwinding example, Example 1, the $\mathcal{F}$-sequence is $[3, 2, 4]$, the rational is $31/9$ and the $\mathcal{F}$-word is $A^{-1}B^3 \cdot (BA^{-1}B^3A^{-1}B^3)^4$.

5.1. **Lifting to the free group.** We can now achieve our first goal which is to extend these results from a two-generator non-elementary discrete free subgroup G of $PSL(2, \mathbb{R})$ with oriented generators to the free group on two generators. To do this we take a faithful representation of $F = (a, b)$ into such a group but map the pair of generators (a, b) to the coherently oriented stopping generators (C, D)

We have

Corollary 5.4. *Every pair of primitive associates in $F = \langle a, b\rangle$ the free group on two generators can be written in the form $(W(a,b), V(a,b))$ where*

$$W(a,b) = W_{[a_0;\ldots,a_k]}(a,b) \text{ and } V(a,b) = W^{-1}_{[a_0;\ldots,a_{k-1}]}(a,b)$$

and is thus associated to a pair of rationals that are Farey neighbors.

Proof. Take a faithful representation of F into $PSL(2,\mathbb{R})$ this time mapping the ordered pair (a,b) to the ordered pair (C,D) where C and D are the stopping generators for the group they generate. □

5.2. **Concatenation vs. Free Reduction.** Because the words that we obtain from the algorithm are freely reduced and in a form where there is never any reduction with the words we work with, we do not distinguish between freely reduced products and the concatenation of two words.

6. Enumerating primitives: palindromes and products

We first work with positive rationals. We do this merely for ease of exposition and to simplify the notation. We then indicate the minor changes needed for negative rationals.

Enumeration Scheme for positive rationals

Set

$$E_{0/1} = A^{-1}, E_{1/0} = B, \text{ and } E_{1/1} = A^{-1}B.$$

If p/q has continued fraction expansion $[a_0; a_1, ..., a_{k-1}, a_k]$, consider the parent fractions $[a_0;_1, ..., a_{k-1}]$ and $[a_0, ..., a_{k-1}, a_k - 1]$. Choose labels m/n and r/s so that $m/n < p/q < r/s$. Set

Case 1 pq - odd:

$$E_{p/q} = E_{r/s}E_{m/n}$$

Case 2 pq - even:

$$E_{p/q} = E_{m/n}E_{r/s}.$$

Note that in Case 1 the word indexed by the larger fraction is on the left and in Case 2 it is on the right.

Enumeration Scheme for negative rationals

Now assume $p/q < 0$. We use the reflection in the imaginary axis to obtain the enumeration scheme. The reflection sends A^{-1} to A. This reverses the order of the distinguished neighbors.

Set

$$E_{0/1} = A \text{ and } E_{1/0} = B$$

These are trivially palindromes. At the next level we have $E_{-1/1} = BA$.

To give the induction scheme: we assume $m/n, r/s$ are the distinguished neighbors of p/q and they satisfy $m/n > p/q > r/s$, $p/q = (m+r)/(n+s)$ and set

Case 1 pq - odd:

$$E_{p/q} = E_{r/s}E_{m/n}.$$

Case 2 pq - even:

$$E_{p/q} = E_{m/n}E_{r/s}.$$

Note that in Case 1 the word indexed by the larger fraction is on the right and in Case 2 it is on the left.

Theorem 6.1. (Enumeration by Rationals) *The primitive elements of a two generator free group can be enumerated by the rationals using Farey sequences. The resulting words are denoted by $E_{p/q}$. In the enumeration scheme, when pq is even, $E_{p/q}$ is a palindrome, and when pq is odd, $E_{p/q}$ is a product of palindromes that have already appeared in the scheme.*

- *For pq even, $E_{p/q}$ is a palindrome. It is cyclically reduced and the unique palindrome in its conjugacy class.*
- *For pq odd, $E_{p/q} = E_{m/n}E_{r/s}$ where m/n and r/s are the parents of p/q with m/n the smaller one. Both $E_{m/n}$ and $E_{r/s}$ are palindromes; $E_{p/q}$ is cyclically reduced.*

Remark 6.1. *Note that although there are several ways words in the conjugacy class $E_{p/q}$ can be factored as products of palindromes, in this theorem we specifically choose the one that makes the enumeration scheme work.*

Not only are the words in this enumeration scheme primitive, we have

Theorem 6.2. *Let $\{E_{p/q}\}$ denote the words in the enumeration scheme for positive rationals. Then if $(p/q, p'/q')$ are neighbors, the pair of words $(E_{p/q}, E_{p'/q'})$ is a pair of primitive associates.*

Before we give the proofs we note that given $p/q = [a_0; a_1, \dots a_k] \geq 0$ the $\mathcal{F}-$sequence word $W_{[a_0;a_1,\dots a_k]}(A,B)$ determines a specific word in its conjugacy class in G. The enumeration scheme also determines a word, $E_{p/q}$, in the same conjugacy class. In general these words, although conjugate, are different.

Theorem 6.2 tells us that words in the enumeration scheme labeled with neighboring Farey fractions give rise to primitive pairs. Note that although cyclic permutations are obtained by conjugation, we cannot necessarily simultaneously conjugate both elements of a primitive pair coming from the $\mathcal{F}$-sequence to get to the corresponding primitive pair coming from Theorem 6.2.

Proof. (**proof of Theorem 6.1**) The proof uses the connection between continued fractions and the Farey tessellation. We observe that in every Farey triangle with vertices $m/n, p/q, r/s$ one of the vertices is *odd* and the other two are *even*. To see this simply use the fact that $mq - np, ps - rq, ms - nr$ are all congruent to 1 modulo 2. (This also gives the equivalence of parity cases for pq and the $p + q$ used by other authors.) In a triangle where pq is even, it may be that the smaller distinguished neighbor is even and the larger odd or vice-versa and we take this into account in discussing the enumeration scheme. We note that in general if X and Y are palindromes, then so is $(XY)^t X$ for any positive integer t.

We give the proof assuming $p/q > 0$. The proof proceeds by induction on the Farey level. The idea behind the proof is that each rational has a pair of parents (distinguished neighbors) and each parent in turn has two parents so there are at most four *grandparents* to consider. The parents and grandparents may not all be distinct. The cases considered below correspond to the possible ordering of the grandparents as rational numbers and also the possible orders of their levels.

To deal with negative rationals we use the reflection in the imaginary axis. The reflection sends A^{-1} to A. We again have distinguished neighbors m/n and r/s, and using the reflection our assumption is $m/n > p/q > r/s$. In the statement of the theorem, m/n is now the larger neighbor. Using our definition of the Farey level of p/q as the Farey level of $|p/q|$, the proof is exactly the same as for positive rationals.

In case 1) by induction we get the product of distinguished neighbor palindromes.

In case 2) we need to show that we get palindromes.

The set up shows that we have palindromes for level 0, ($\{0/1, 1/0\}$) and the correct product for level 1, $\{1/1\}$.

Assume the scheme works for all rationals with level less than N and assume $Lev(p/q) = N$. Since $m/n, r/s$ are distinguished neighbors of p/q both their levels are less than N.

Suppose $p/q = [a_0, a_1, \ldots, a_k]$. Then $Lev(p/q) = \sum_0^k a_j = N$ and the continued fractions of the parents of p/q are

$$[a_0, a_1, \ldots, a_{k-1}] \text{ and } [a_0, a_1, \ldots, a_{k-1}, a_k - 1].$$

Assume we are in case 2 where pq is even.

Suppose first that so that we have

$$m/n = [a_0, a_1, \ldots, a_{k-1}] \text{ and } r/s = [a_0, a_1, \ldots, a_{k-1}, a_k - 1]. \tag{2}$$

Then the smaller distinguished neighbor of r/s is m/n and the larger distinguished neighbor is

$$w/z = [a_0, a_1, \ldots, a_{k-1} - 2]. \tag{3}$$

The smaller distinguished neighbor of m/n is

$$u/v = [a_0, \ldots, a_{k-2}, a_{k-1} - 1] \tag{4}$$

and the larger distinguished neighbor is

$$x/y = [a_0, \ldots, a_{k-2}]. \tag{5}$$

If rs is odd we have, by the induction hypothesis

$$E_{r/s} = E_{r/s} E_{w/z} E_{m/n}$$

and

$$E_{p/q} = E_{m/n}E_{r/s} = E_{m/n}(E_{w/z}E_{m/n})$$

which is a palindrome.

If mn is odd we have, by the induction hypothesis,

$$E_{m/n} = E_{x/y}E_{u/v}$$

and by equations (2), (3), (4))and (5)

$$E_{r/s} = E_{m/n}^{(a_k-1)}E_{x/y} = (E_{x/y}E_{u/v})^{(a_k-1)}E_{x/y}$$

so that

$$E_{p/q} = E_{m/n}E_{r/s} = (E_{x/y}E_{u/v})^{a_k}E_{x/y}$$

is a palindrome.

If

$$Lev(r/s) < Lev(m/n),$$

we have

(6) $$m/n = [a_0, a_1, \ldots, a_{k-1}, a_k - 1] \text{ and } r/s = [a_0, a_1, \ldots, a_{k-1}].$$

Then the larger distinguished neighbor of m/n is r/s and the smaller distinguished neighbor is

(7) $$w/z = [a_0, a_1, \ldots, a_{k-1} - 2].$$

The larger distinguished neighbor of r/s is

(8) $$x/y = [a_0, \ldots, a_{k-2}, a_{k-1} - 1]$$

and the smaller distinguished neighbor is

(9) $$u/v = [a_0, \ldots, a_{k-2}].$$

If mn is odd we have, by the induction hypothesis

$$E_{m/n} = E_{r/s}E_{w/z}$$

and

$$E_{p/q}E_{m/n}E_{r/s} = E_{r/s}(E_{w/z}E_{r/s})$$

which is a palindrome.

If rs is odd we have, by the induction hypothesis

$$E_{r/s} = E_{x/y}E_{u/v}$$

and by equations (6), (7), (8) and (9)

$$E_{m/n} = E_{u/v}E_{r/s}^{(a_k-1)} = E_{u/v}(E_{x/y}E_{u/v})^{(a_k-1)}$$

so that

$$E_{p/q} = E_{m/n}E_{r/s} = E_{u/v}(E_{x/y}E_{u/v})^{a_k}$$

is a palindrome.

We have yet to establish that the $E_{p/q}$ words are primitive but this follows from the proof of Theorem 6.2. □

6.1. **Proof of Theorem 6.2.**

Proof. The proof is by induction on the maximum of the levels of p/q and p'/q'. Again we proceed assuming $p/q > 0$; reflecting in the imaginary axis we obtain the proof for negative rationals.

At level 1, the theorem is clearly true: (A^{-1}, B), (A^{-1}, BA^{-1}) and (BA^{-1}, B) are all primitive pairs.

Assume now that the theorem holds for any pair both of whose levels are less than N.

(1) Let $(p/q, p'/q')$ be a pair of neighbors with $Lev(p/q) = N$.

(2) Let $m/n, r/s$ be the distinguished neighbors of p/q and assume $m/n < p/q < r/s$.

  (a) Then $m/n, r/s$ are neighbors and both are have level less than N so that by the induction hypothesis $(E_{m/n}, E_{r/s})$ is a pair of primitive associates.

  (b) It follows that all of the pairs
$$(E_{m/n}, E_{m/n}E_{r/s}),\ (E_{m/n}, E_{r/s}E_{m/n}),$$
$$(E_{m/n}E_{r/s}, E_{r/s}) \text{ and } (E_{r/s}E_{m/n}, E_{r/s})$$
  are pairs of primitive associates since we can retrieve the original pair $(E_{m/n}, E_{r/s})$ from any of them.

  (c) Since $E_{p/q} = E_{r/s}E_{m/n}$ or $E_{p/q} = E_{m/n}E_{r/s}$ we have proved the theorem if p'/q' is one of the distinguished neighbors.

(3) If p'/q' is not one of the distinguished neighbors, then either $p'/q' = (tp + m)/(tp + n)$ for some $t > 0$ or $p'/q' = (tp + r)/(tp + s)$ for some $t > 0$.

  (a) Assume for definiteness $p'/q' = (tp + m)/(tp + n)$; the argument is the same in the other case.

  (b) Note that the pairs $p/q, (jp + m)/(jp + n)$ are neighbors for all $j = 1, \dots t$.

  (c) We have already shown $E_{m/n}, E_{p/q}$ is a pair of primitive associates. The argument above applied to this pair shows that $E_{m/n}, E_{(p+m)/(q+m)}$ is also a pair of primitive associates. Applying the argument t times proves the theorem for the pair $E_{p'/q'}, E_{p/q}$.

□

An immediate corollary is

Corollary 6.3. *The scheme of Theorem 6.1 also gives a scheme for enumerating only primitive palindromes and a scheme for enumerating only primitives that are canonical palindromic products.*

7. Examples

We compute some examples:

Fraction	Parents	Parity	$E_{p/q}$	Parental Product	simplified
1/2	$0/1 \oplus 1/1$	even	$E_{1/2}$	$A^{-1} \cdot BA^{-1}$	$A^{-1}BA^{-1}$
2/1	$1/1 \oplus 1/0$	even	$E_{2/1}$	$BA^{-1} \cdot B$	$BA^{-1}B$
1/3	$1/2 \oplus 0/1$	odd	$E_{1/3}$	$A^{-1}BA^{-1} \cdot A^{-1}$	$A^{-1}BA^{-2}$
2/5	$1/3 \oplus 1/2$	even	$E_{2/5}$	$A^{-1}BA^{-2} \cdot A^{-1}BA^{-1}$	$A^{-1}BA^{-3}BA^{-1}$
1/4	$1/3 \oplus 0/1$	even	$E_{1/4}$	$A^{-1} \cdot A^{-1}BA^{-2}$	$A^{-2}BA^{-2}$
2/7	$1/4 \oplus 1/3$	even	$E_{2/7}$	$A^{-2}BA^{-2} \cdot A^{-1}BA^{-2}$	$A^{-2}BA^{-3}BA^{-2}$

Let us see how the word in the enumeration scheme compares with the word we get from the corresponding $\mathcal{F}-$ sequence. In section 4.3, the $\mathcal{F}-$word of $31/9 = [3, 2, 4]$ was $A^{-1}B^3 \cdot (BA^{-1}B^3A^{-1}B^3)^4$.

To find the word $E_{31/9}$ note that the distinguished Farey neighbors are $[3; 2] = 7/2$ and $[3; 2, 3] = 24/7$. We form the following words indicating $E_{31/9}$ and its neighbors in boldface.

$$E_{1/1} = A^{-1}B, E_{2/1} = BA^{-1}B.E_{3/1} = E_{0/1}E_{2/0} = B \cdot BA^{-1}B,$$

$$E_{4/1} = E_{3/1}E_{0/1} = BBA^{-1}B \cdot B$$

$$\mathbf{E_{7/2}} = E_{3/1}E_{4/1} = \mathbf{BBA^{-1}B \cdot BBA^{-1}BB},$$

$$E_{10/3} = E_{3/1}E_{7/2} = B^2A^{-1}B^3A^{-1}B^3A^{-1}B^2,$$

$$E_{17/5} = E_{7/2}E_{10/3} = B^2A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^3A^{-1}B^2,$$

$$\mathbf{E_{24/7}} = E_{17/5} \cdot E_{7/2} = \mathbf{B^2A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^2}$$

$$\mathbf{E_{31/9}} = E_{7/2}E_{24/7} =$$

$$\mathbf{B^2A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^2}$$

Conjugate by B^{-1} and regroup to get

$$BA^{-1}B^3A^{-1}B^3 \cdot BA^{-1}B^3A^{-1}B^3 \cdot BA^{-1}B^3A^{-1}B^3 \cdot A^{-1}B^3 \cdot BA^{-1}B^3A^{-1}B^3.$$

Finally conjugate by $(BA^{-1}B^3A^{-1}B^3)^{-3}$ to obtain the $\mathcal{F}-$word of $31/9 = [3, 2, 4]$.

8. Farey Diagram Visualization

We can visualize the relation between primitive pairs and neighboring rationals using the Farey diagram. Suppose the stopping generators (C, D) correspond to $(0/1, 1/0)$ as above, and the primitive pair (A, B) corresponds to $(r/s, p/q)$. Note that we have done this so that the Farey level of p/q is greater than that of r/s and r/s is the parent of p/q with lowest Farey level. (The other parent is $(r-p)/(q-s)$ and corresponds to $A^{-1}B$.) Draw the curve γ from a point on the imaginary axis to p/q. If p/q is positive, orient it toward p/q; if p/q is negative, orient it towards the imaginary axis.

The left-right sequence, the continued fraction for p/q and the $\mathcal{F}$ winding sequence whose last word is B are all the same. Traversing the curve in the other direction reverses left and right and gives the unwinding sequence. The symmetry about the imaginary axis is reflected in our definition of negative continued fractions.

Given two primitive pairs (A, B) an $(A'B')$ such that B corresponds to p/q and B' corresponds to p'/q' draw the curves γ and γ'. We can find the sequence to go from (A, B) to (A', B') by traversing γ from p/q to the imaginary axis and then traversing γ' to p'/q'. We can also draw an oriented curve from p/q to p'/q' and read off the left-right sequence along this curve to get the $\mathcal{F}-$ sequence that gives (A', B') as words in (A, B) directly. We have

Corollary 8.1. *Given any two sets of primitive pairs, (A, B) and (A', B') there is an $\mathcal{F}$-sequence containing either only positive or only negative integers that connects one pair to the other.*

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