Planar families of discrete groups

Jane Gilman and Linda Keen

Abstract. In this paper we study certain families of discrete groups built from a given group with the non-separating disjoint circle property, the NSDC property. A two generator NSDC group determines a unique planar family of such groups. We find real moduli for these families. Our constructions are geometric and do not use the techniques of quasi-conformal mappings or coverings.

1. Introduction

Determining the space of free discrete two generator groups of Möbius transformations is an old and difficult problem, as is the related problem of determining whether or not a given two generator group is discrete. One discreteness test introduced by Gilman is known as the non-separating disjoint circle test, the NSDC discreteness test for short [**2**]. Gilman and Waterman proved that the groups which satisfy this discreteness test determine a subspace of the space of free discrete groups, *NSDC space*, and found an explicit formula for the boundary of NSDC in the case of two parabolic generator groups [**3**].

In this paper we show how to construct families of marked NSDC groups. These are called planar families and we show how to move around within a given planar family. We then construct a map from a given planar family into the classical space of trace moduli and relate moving around in a given planar family to moving around in the trace moduli space.

The map into trace moduli space is not necessarily an embedding. Rather, at points where it is not an embedding, we may regard the double point as also lying in the image of a different planar family, a planar family that is the image of the given one under a Möbius transformation.

We begin with a marked discrete group of non-separating disjoint circle type, *NSDC group* (see [**2**] and section 2.2 for the definition). Such a group determines three disjoint or tangent planes. We prove that there is a whole family of discrete groups, which we term the *NSDC-planar family*, that share these planes.

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From specific information about the NSDC planes which depends on the marking data for a base point NSDC group, we find a set of six real numbers that serve as parameters for the NSDC family. We then construct a map of our parameters into a representation of the full space of conjugacy classes of free discrete groups as a subset of $\hat{\mathbb{C}}^3$ by triples of traces. Finally we show that if two points in the planar family are mapped to the same triple, and are therefore conjugate in the group of all Möbius transformations, the conjugating transformation does not preserve the triple of planes.

We remark that our construction does not use the usual theory of quasiconformal deformations of a given group nor does it depend simply on the use of coverings and quotients that give commensurable groups.

The paper is organized as follows. In section 2 we set notation and give some background. We recall the definition of NSDC groups from [**2**] and summarize the basic properties of such groups that were derived there. In section 3 we define a new object, the NSDC planar family. Sections 4 and 5 develop computational techniques and in section 6 we construct our new parameters. In section 7 we give a necessary and sufficient condition for a marked group to belong to a given planar family. Finally, in section 8 we show how our parameters are related to the classical parameters.

2. Notation, definitions and prior results

An element of $PSL(2,\mathbb{C})$ acts as a Möbius transformation on $\hat{\mathbb{C}}$, the complex sphere. The action extends in a natural way to hyperbolic three-space $\mathbb{H}^3$. When considered as the boundary of hyperbolic three-space, $\hat{\mathbb{C}}$ is referred to as the sphere at infinity.

Elements of $PSL(2,\mathbb{C})$ are classified as loxodromic, elliptic, parabolic according to the square of their traces. The classification of transformations can also be described by the action on $\hat{\mathbb{C}}$ or on $\hat{\mathbb{C}} \cup \mathbb{H}^3$.

We follow the notation of [**1**]. For x and $y \in (\hat{\mathbb{C}} \cup \mathbb{H}^3)$ with $x \neq y$ we let $[x, y]$ denote the oriented hyperbolic geodesic passing through x and y. The *ends* of the geodesic $[x, y]$ are by definition the points v and w with $\{v, w\} = [x, y] \cap \hat{\mathbb{C}}$. If v and w are the ends of $[x, y]$, then $[x, y] = [v, w]$. The notation $[x, y]$ also indicates a direction so that $[y, x]$ is the geodesic with the opposite orientation.

Following Fenchel [**1**], we include *improper lines*, in our considerations. A *proper line* in one whose ends are distinct. An improper line is one whose ends coincide.

A Möbius transformation fixes one or two points on $\hat{\mathbb{C}}$. A loxodromic transformation A has two distinct fixed points as does an elliptic transformation. These transformations fix the geodesic X in $\mathbb{H}^3$ whose ends are the fixed points of A on $\hat{\mathbb{C}}$. This geodesic is called the *axis* of A and is denoted by Ax_A.

A parabolic transformation has one fixed point. Using the terminology of improper lines, parabolic transformations also have axes. If A is a parabolic transformation with fixed point $u \in \hat{\mathbb{C}}$, we consider the improper line $[u, u]$ to be its axis.

With this terminology, every pair of distinct lines (proper or improper) in $\mathbb{H}^3$ has a unique common perpendicular (proper or improper) (III.2.6 of [**1**]). In the situation where A and B are elements of $PSL(2,\mathbb{C})$ with distinct axes, the group $\langle A, B\rangle$ is non-elementary. Moreover, every pair of elements $A, B \in PSL(2,\mathbb{C})$ with distinct axes determines a unique (proper or improper) hyperbolic line in $\mathbb{H}^3$, the common perpendicular of their axes; we denote this common perpendicular by L and denote the ends of L by n and n'. Note that L is improper if $n = n'$. In this paper, we always tacitly assume the axes are distinct.

2.1. Half-turns. For any hyperbolic line $[x, y]$ with $x \neq y$ we let $H_{[x,y]}$ be the half-turn about the line with ends x and y. We note that $H_{[x,y]}$ leaves invariant every hyperbolic plane $\mathbb{P}$ whose boundary $C_{\mathbb{P}}$, also called its *horizon*, passes through x and y ([**2**]). The half-turn will interchange the exterior and the interior of $\mathbb{P} \in \mathbb{H}^3$.

2.2. NSDC Groups. There is a natural way to attach a three generator group and six complex numbers to any non-elementary marked two generator group. This is described in [**2**] where the concept of a non-separating disjoint circle group, an NSDC group for short, was first defined. Motivating the formulation of this concept was the fact that having the NSDC property implied that a group was discrete. Thus it was seen as a discreteness test. The discreteness test was also applied to certain n generator groups, but these are not of interest here.

We recall definitions and properties derived in [**2**] which will be relevant here, but only as they apply to the two and three generator groups.

Let $G = \langle A, B\rangle$ and let L be the common perpendicular to A and B. If L is a proper line, then there are unique hyperbolic lines L_A and L_B such that $A = H_{L_A} \cdot H_L$ and $B = H_{L_B} \cdot H_L$. We let a and a' be the ends of L_A so that $L_A = [a, a']$ and let b and b' be those of L_B so that $L_B = [b, b']$. We refer to the lines L, L_A, and L_B as the *L-lines*. We define

Definition 2.1. [**2**] *The marked three generator group determined by $G = \langle A, B\rangle$ is denoted by $\mathbb{T}G$ and defined by $\mathbb{T}G = \langle H_{L_A}, H_L, H_{L_B}\rangle$.*

By construction G is a normal subgroup of $\mathbb{T}G$ of index at most two which immediately implies that G is discrete if and only if $\mathbb{T}G$ is discrete.

Letting n and n' be the ends of L, we also define

Definition 2.2. [**2**] *The ortho-end of the marked group $G = \langle A, B\rangle$ is the six-tuple of complex numbers (a, a', n, n', b, b').*

Definition 2.3. [**2**] *Six points in $(a, a', n, n', b, b') \in \hat{\mathbb{C}}^6$ such that $a \neq a'$, $b \neq b'$ and $n \neq n'$ have the* non-separating disjoint circle property *if there are pairwise disjoint or tangent circles on $\hat{\mathbb{C}}$, C_A, C_D and C_B (respectively) passing through a and a', n and n', and b and b' (respectively) such that no one circle separates the other two.*

Definition 2.4. [**2**] *A marked group $G = \langle A, B\rangle$ is a* marked non-separating disjoint circle group *or* NSDC group *if the ortho-end of A and B has the non-separating disjoint circle property. G is a* non-separating disjoint circle group *if some pair of generators for G has the non-separating disjoint circle property.*

Remark 2.5. *We note that if one of A or B is elliptic, then the ortho-end of the marked group is well defined, but it will not have the NSDC property. Note that the condition $n \neq n'$ in the definition of an NSDC group rules out the possibility of A and B sharing a fixed point. Such groups do not determine any three generator group and do not have an ortho-end and thus are not NSDC. However, we note that if $n = n'$, in the case one of the generators is loxodromic and the two generators* *share a fixed point, the group is not discrete; if one is elliptic, the group is not NSDC and if both are parabolic with distinct axes,* $n \neq n'$.

If G is NSDC, we also call the corresponding group $\mathbb{T}G$ an NSDC group.

The point of these definitions is

PROPOSITION 2.6. [**2**] *Let* G *be a two-generator subgroup of* $PSL(2,\mathbb{C})$. *If some ortho-end of* G *has the non-separating disjoint circle property, then* G *is discrete.*

REMARK 2.7. *The referee has pointed out an equivalent way to define an NSDC group. Let* $D_1, \ldots, D_n$ *be closed Euclidean discs, with disjoint interiors, in the extended complex plane. For* $i = 1, \ldots, n$, *let* x_i *and* y_i *be distinct points on* S_i, *the boundary of* D_i. *Let* A_i *be the half-turn with fixed points* x_i *and* y_i. *Then by the Klein-Maskit combination theorem, the group* Γ *generated by* $A_1, \ldots, A_n$ *is Kleinian of the second kind; it is the free product of the* n *cyclic groups generated by the* A_i. *Further,* Γ *contains parabolic elements if and only if some* A_i *and some* A_j *have a fixed point in common.*

The torsion-free subgroups of the groups obtained from the above theorem, in the case that $n = 3$, *are precisely the NSDC groups of this paper.*

3. The planar family of NSDC groups

Let G be an NSDC group with ortho-end (a, a', n, n', b, b'). Let the three circles C_A, C_D, C_B be a set of non-separating disjoint circles for this ortho-end. Let $\mathbb{P}_A$, $\mathbb{P}_D$ and $\mathbb{P}_B$ be planes in $\mathbb{H}^3$ whose respective horizons are C_A, C_D, and C_B. Let L'_A be any hyperbolic line lying on $\mathbb{P}_A$, let L'_B be any hyperbolic line lying on $\mathbb{P}_B$, and let L' be any hyperbolic line lying on $\mathbb{P}_D$. Set $\mathbb{T}G' = \langle H_{L'_A}, H_{L'}, H_{L'_B} \rangle$ and $G' = \langle H_{L'_A} \cdot H_{L'}, H_{L'_B} \cdot H_{L'} \rangle$. The point of this construction is that G' is an NSDC group and, therefore, G' and $\mathbb{T}G'$ are both discrete.

The family of groups $\mathbb{T}G'$ depends upon $\mathbb{P}_A$, $\mathbb{P}_D$ and $\mathbb{P}_B$. We write $\mathbb{P}$ for $\mathbb{P}_D$.

DEFINITION 3.1. *We call the triple of planes* $(\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)$ *a* non-separating disjoint planar triple *or an* NSDC-triple *if the horizons are a set of NSDC circles. We call the set of all NSDC groups with a fixed planar triple, a* planar family of NSDC groups. *The marked group* $G = \langle A, B \rangle$ *is a* base-point *for the family. We denote the planar family with base-point* $\langle A, B \rangle$ *either by the triple* $(\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)$ *or by* $\mathcal{P}_{A,B}$.

We want to explore all discrete groups corresponding to a given NSDC planar triple.

4. Ortho-ends and pull back circles

In this section we present notation needed to describe a triple of NSDC planes. We begin with a marked NSDC group, $G = \langle A, B \rangle$ with ortho-end (a, a', n, n', b, b') and NSDC-planes $\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B$. The planes each are determined by their *pull-back angles*, which can be described as follows:

Let k and k' be any two points in $\hat{\mathbb{C}}$. Any circle through k and k' has center lying on the perpendicular bisector of the line connecting k and k'. Thus its center is at $c_t = \frac{k+k'}{2} + it\frac{k-k'}{2}$ for some real number t. We think of the center as having been pulled back t units from $\frac{k+k'}{2}$. We let θ_t denote the angle from the radius connecting c_t to k to the Euclidean segment joining k and k' so that $t = \frac{|k-k'|}{2} \cdot \tan \theta_t$. We call θ_t the *pull-back angle* of the circle. Each circle passing through k and k'

corresponds to a unique pull-back angle θ_t with $-\pi/2 \leq \theta_t < \pi/2$. If $r(t)$ denotes the radius of the circle with pull back angle θ_t, we have $r(t) = \frac{|k-k'|}{2\cos\theta_t}$.

When we are discussing a fixed plane whose horizon passes through k and k', we write θ_K for the pull-back angle. We write r_{θ_K} for the radius of the horizon of the plane and c_{θ_K} for its center.

In particular, if we have a family of NSDC-planes with bas-point $G = \langle A, B\rangle$, we assume that $\mathbb{P}_A$ has pull-back angle θ_A, $\mathbb{P}_B$ has pull-back angle θ_B, and $\mathbb{P}$ has pull-back angle θ. Thus the NSDC planar family $\mathcal{P}_{A,B}$ is determined by the ortho-end of $\langle A, B\rangle$ and the triple of angles $(\theta_A, \theta, \theta_B)$.

Remark 4.1. *Note that the pull-back angles are not Möbius invariants.*

5. Computations and perpendiculars

In what follows we will be interested in being able to compute quantities such as points and planes directly from the ortho-end (a, a', n, n', b, b') and the three pull-back angles $(\theta_A, \theta, \theta_B)$.

We define three points $v \in \mathbb{P}$, $v_A \in \mathbb{P}_A$ and $v_B \in \mathbb{P}_B$ by $v = L \cap Ax_B$, $v_A = L_A \cap Ax_A$, $v_B = L_B \cap Ax_{AB^{-1}}$ and note that these are determined by the base point. That is, the three points are all computable from the ortho-end and the pull-back angles. We also define for future use $w \in \mathbb{P}$, $w_A \in \mathbb{P}_A$, $w_B \in \mathbb{P}_B$ by $w = L \cap Ax_A$, $w_A = L_A \cap Ax_{A^{-1}B}$, and $w_B = L_B \cap Ax_B$.

If $\mathbb{P}$ is any hyperbolic plane, $L^{\mathbb{P}}$ a hyperbolic line on $\mathbb{P}$ and x a point on $L^{\mathbb{P}}$, there is a line perpendicular to $\mathbb{P}$ passing through x which we denote by $V_x^{\mathbb{P}}$ and a unique line perpendicular to $L^{\mathbb{P}}$ and $V_x^{\mathbb{P}}$ lying on $\mathbb{P}$ and passing through x which we denote $M_x^{\mathbb{P}}$. If there is no confusion we will omit the superscript $\mathbb{P}$ from the notation for these lines. Let L_W be a line with ends $[w, w']$ and suppose it lies on a plane $\mathbb{P}_W$ with pull-back angle θ_W. Then we can compute that

$$c_{\theta_W} = \frac{w+w'}{2} + i\frac{(w-w')^2}{4}\tan\theta_W.$$

Thus for any x on L_W

$$P_x^{\mathbb{P}_W} = [\frac{w+w'}{2} + i\frac{(w-w')^2}{4}\tan\theta_W, x].$$

Also since one end of $M_x^{\mathbb{P}_W}$ must be $\frac{w+w'}{2} + i(r_{\theta_W})$, we have

$$M_x^{\mathbb{P}_W} = [\frac{w+w'}{2} + i(r_{\theta_W}), x].$$

The point is that this explicit calculation only depends upon θ_W. We note that a line perpendicular to $\mathbb{P}_W$ passing through a point x in $\mathbb{P}_W$ is the line $[c_W, x]$. We are interested in these perpendiculars when $x = v_A$.

6. Constructing the planar family

A loxodromic transformation that is a pure translation by distance d with zero rotation angle is called a *hyperbolic transformation with translation length d.*

Definition 6.1. *Let X be any oriented hyperbolic line and let $\tau \in (-\pi, \pi)$ be any angle measured so that τ is positive for a right-handed screw motion along X. We let $R_{X,\tau}$ be the elliptic transformation that is rotation through an angle τ*

about the line X. If d is any positive real number, we let $T_{X,d}$ be the hyperbolic transformation whose axis is X and whose translation length is d.

We note that given any NSDC planar family, these two types of moves, rotation about a line perpendicular to a plane of the family and parallel transport along a line lying in that plane of the family preserve that plane. It is also important to note that these moves will not simultaneously preserve all of the planes in the family, rather just one of the planes. Specifically we prove

LEMMA 6.2. *Assume $G = \langle A, B\rangle$ is of NSDC type and is the base-point of the planar family $\mathcal{P}_{A,B} = (\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)$. Let $L_{A'} = g(L_A)$ where $g = g_1, ..., g_n$ and for each i, either*

- *$g_i = T_{X_i,d_i}$ where $X_i \in \mathbb{P}_A$ and d_i is a real number or*
- *$g_i = R_{X_i,\tau_i}$ where X_i is perpendicular to $\mathbb{P}_A$ and τ_i is any angle $-\pi < \tau_i \leq \pi$.*

Then $\langle H_{L_{A'}}, H_L, H_{L_B}\rangle$ is a discrete group in the same family $\mathcal{P}_{A,B}$. If $A' = H_{L'_A} H_L$, then $G' = \langle A', B\rangle$ is also a discrete group of NSDC type and it also belongs to $\mathcal{P}_{A,B}$.

PROOF. By construction each of the g_i maps lines on $\mathbb{P}_A$ to lines on $\mathbb{P}_A$; $g_i H_{L_A} g_i^{-1} = H_{g_i(L_A)}$ and thus preserves the $\mathbb{P}_A$ plane of the planar family. □

Thus each of the moves in the lemma preserve a plane in the planar family. Next we see that every triple of geodesics $(L_{A'}, L', L_{B'})$ lying in the respective planes $\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B$ of the planar family can be so obtained working separately in each of the three planes. Note that the construction in the proof below depends on the base-point.

PROPOSITION 6.3. *Assume $G = \langle A, B\rangle$ is of NSDC type and is the base-point of the planar family $\mathcal{P}_{A,B} = (\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)$. Let $v_A = L_A \cap Ax_A$. Then $G' = \langle A', B\rangle$ is in the NSDC planar family $\mathcal{P}_{A,B}$ with $A' = H_{L_{A'}} \cdot H_L$, where $L_{A'}$ is obtained from L_A by the following. (The plane $\mathbb{P}_A$ is implied).*

1. *$X = R_{V_{v_A},\tau}(M_{v_A})$ where τ is an angle with $-\pi \leq \tau < \pi$,*
2. *$Y = R_{V_{v_A},\tau}(L_A)$, and*
3. *$L_{A'} = T_{X,d}(Y)$ for some positive number d.*

Conversely, given any $L_{A'} \in \mathbb{P}_A$ there exists an angle τ, $-\pi \leq \tau < \pi$ and a positive number d so that $L_{A'}$ can be obtained from L_A by the above construction.

PROOF. If $L_{A'}$ is any line on $\mathbb{P}_A$, we can drop a perpendicular from v_A to $L_{A'}$, whether or not the lines L_A and $L_{A'}$ are disjoint. Let X be the geodesic on which this perpendicular lies, oriented towards v_A, and let d be the distance from $L_{A'}$ to v_A along X. Then $T^{-1}_{X,d}(L_{A'}) = Y$ passes through v_A and makes some angle τ with L_A. We can rotate Y to L_A by applying $R_{V_{v_A},\tau^{-1}}$. (Alternatively, rotating by $\pi/2 + \tau$ sends Y to M_{v_A}.) Combining these moves we have

$$L_{A'} = (T_{R_{V_{v_A},\tau}(L_A),d} \circ R_{V_{v_A},\tau})(L_A) = (T_{R_{V_{v_A},\pi/2+\tau}(M_{v_A}),d} \circ R_{V_{v_A},\tau})(L_A)$$

Thus any line L'_A on $\mathbb{P}_A$ determines a positive number $d = d_A$ and an angle $\tau = \tau_A$ in $(-\pi, \pi)$. Conversely, given the initial data (θ_A, a and a') and the pair (d_A, τ_A) we can use this construction to find $L_{A'}$. □

DEFINITION 6.4. *For the fixed planar family $\mathcal{P}_{A,B}$ with base-point $\langle A, B\rangle$, we call the sequence of moves in the proposition the* (d_A, τ_A) *moves determining L'_A.*

Recalling that $v = L \cap Ax_B$ and $v_B = L_B \cap Ax_{AB^{-1}}$ we can apply this proposition to obtain any L' and L'_B and their (d, τ) and (d_B, τ_B).

Corollary 6.5. *All of these moves can be calculated directly from the data (a, a', n, n', b, b) and $(\theta_A, \theta, \theta_B)$ that determine $\mathcal{P}_{A,B}$.*

Proof. This is the content of section 5 on perpendiculars. □

In summary we have

Theorem 6.6. *Let $\mathcal{P}_{A,B}$ be an NSDC planar family with base-point $G = \langle A, B\rangle$ and let*

$$\mathcal{D}_{A,B} = \{((d_A, \tau_A), (d, \tau), (d_B, \tau_B)) \in ([0, \infty), (-\pi, \pi])^3\}.$$

Then there is a bijection

$$M : \mathcal{P}_{A,B} \to \mathcal{D}_{A,B}.$$

The bijection M defines a set of global coordinates for the planar family $\mathcal{P}_{A,B}$ and the triples in $D_{A,B}$ are moduli. That is, given the data $(a, a', n, n', b, b') \in \hat{\mathbb{C}}^6$, the ortho-end for the marked group, and pull-back angles $(\theta_A, \theta, \theta_B)$, for the planar family $\mathcal{P}_{A,B}$, every group $G' = \langle A', B'\rangle$ in the family determines, and is determined by the six real numbers $(d_A, \tau_A,\ d, \tau, d_B, \tau_B)$ that describe the moves determining the lines $L_{A'}$, L' and $L_{B'}$ respectively.

Proof. Proposition 6.3 shows that given each pair (d_A, τ_A), appropriate moves can be chosen to find $L_{A'}$; the same is true for L' and $L_{B'}$. We apply the moves in order: first to $\mathbb{P}_A$ to obtain the new $L_{A'}$, then to $\mathbb{P}$ to obtain the new L' and finally to $\mathbb{P}_B$ to obtain a new $L_{B'}$. Thus all marked groups G' with these NSDC planes are determined.

Conversely, given $G' \in \mathcal{P}_{A,B}$, the line $L_{A'}$ is determined. Thus the perpendicular from v_A to $L_{A'}$ is determined so that d_A and τ_A are determined and the moves needed to move L_A to $L_{A'}$ can be read off. The pairs for L' and $L_{B'}$ are obtained similarly.

The point is that given the ortho-end of the base-point and the pullback angles, every ortho-end that gives a group in this planar NSDC family can be obtained. □

We emphasize that the (v_A, v, v_B) are determined by the (L_A, L, L_B) and the corresponding (d, τ) coordinates (i.e. $(d_A, \tau_A, d, \tau, d_B, \tau_B)$) are defined in terms of this marking.

Remark 6.7. *For two given triples of L-lines, (L_A, L, L_B) and $(L_{A'}, L', L_{B'})$, however, the (d, τ) co-ordinates of $(L_{A'}, L', L_{B'})$ with respect to (L_A, L, L_B) are related in an obvious manner to those of (L_A, L, L_B) with respect to $(L_{A'}, L', L_{B'})$. The situation is much more complicated if there is a third triple $(L_{A''}, L'', L_{B''})$. The (d, τ) coordinates of $(L_{A''}, L'', L_{B''})$ with respect to $(L_{A'}, L', L_{B'})$ and the (d, τ) co-ordinates of $(L_{A'}, L', L_{B'})$ with respect to (L_A, L, L_B) do not combine in an obvious manner to give the (d, τ) coordinates of $(L_{A''}, L'', L_{B''})$ with respect to (L_A, L, L_B). A geometric relationship can be worked out, but it is not relevant or useful here.*

7. Ortho-ends in the planar family

Given a planar NSDC family with ortho-end (a, a', n, n', b, b') and planes $(\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)$ with pull back angles $\theta_A, \theta, \theta_B$, we can find necessary and sufficient conditions for a six-tuple $(\alpha, \alpha', \eta, \eta', \beta, \beta')$ to be the ortho-end of a group in the family.

As usual, we work one plane at a time. That is, we look at the six-tuple $(\alpha, \alpha', n, n', b, b')$. We have

Proposition 7.1. *$(\alpha, \alpha', n, n', b, b')$ lies in the NSDC planar family of (a, a', n, n', b, b') if and only if*

$$\frac{(a+a') - (\alpha+\alpha') - i\frac{(a-a')(|a-a'|)}{2} \cdot \tan\theta_A}{i(\alpha-\alpha')} \in \mathbb{R}.$$

Proof. We need to show that this condition is equivalent to a, a', α, and α' lying on the pull back circle θ_A. Recall that the pull-back angle, the pull-back distance, and the center of the pull-back circle are related by

$$t_{\theta_A} = \frac{|a-a'|}{2} \cdot \tan\theta_A \tag{1}$$

$$c_{\theta_A} = c_{t_{\theta_A}} = \frac{a+a'}{2} + i\frac{(a-a')(|a-a'|)}{4} \cdot \tan\theta_A = \frac{a+a'}{2} + i\frac{(a-a')}{2} t_{\theta_A} \tag{2}$$

Equations (1) and (2) must also hold for $\alpha, \alpha', \theta_{A'}$ and $t_{\theta_{A'}}$. The four points lie on the same circle precisely when $c_{\theta_A} = c_{\theta_{A'}}$. This happens if and only if there is a real $t_{\theta_{A'}}$ with

$$\frac{a+a'}{2} + i\frac{(a-a')(|a-a'|)}{4} \cdot \tan\theta_A = \frac{\alpha+\alpha'}{2} + i\frac{(\alpha-\alpha')}{2} t_{\theta_{A'}}$$

□

8. Relation to classical moduli

The marked nsdc groups are a subset of the space $\mathcal{S}$ of marked discrete free groups $G = \langle A, B\rangle$. Classically, $\mathcal{S}$ can be embedded into a subset of $\mathbb{C}^3$ by choosing as moduli $(Tr\ A, Tr\ B, Tr\ AB^{-1})$. Specifically, one picks a marked group G as base point and arbitrarily chooses elements in $SL(2, \hat{\mathbb{C}})$ (again called A and B) to represent the generators. The signs of the traces of these elements depends on this choice but once it is made, there is a unique matrix corresponding to every other group element. Under quasiconformal deformation the traces of group elements are analytic functions of the deformation. The space $\mathcal{S}$ is the space of conjugacy classes of groups under Möbius transformations.

The computation of the full boundary of $\mathcal{S}$ in this embedding is an open hard question.

In this section we show how these trace parameters are related to the nsdc moduli we found above.

8.1. Fenchel's Skew Hexagons. In this section we summarize definitions and results from Fenchel [**1**] as they apply to our situation. In particular, we use the conventions on signs, orientation, traces, cross-ratios and complex lengths, etc. developed there.

Let $G = \langle A, B\rangle$ be a fixed group of NSDC type in the planar family $(\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)$ with pull back-angles $(\theta_A, \theta, \theta_B)$ and let $\mathbb{T}G$ be the corresponding group. As we saw in section 2.2, we can write $A = H_{L_A} \cdot H_L$, $B = H_{L_B} \cdot H_L$ and $AB^{-1} = H_{L_A} \cdot H_{L_B}$. Here we assume the axes are oriented from their repelling fixed points to their attracting fixed points. We assume L is oriented from Ax_A to Ax_B, L_A is oriented from Ax_A to $Ax_{AB^{-1}}$ and L_B is oriented from $Ax_{AB^{-1}}$ to Ax_B.

We form the *skew right angled hexagon* associated to G, which we denote by $\mathcal{H}_G$ whose sides lie along the geodesics:

$$L_A, Ax_A, L, Ax_B, L_B, Ax_{AB^{-1}}$$

Here we adopt the convention that we label sides of a hexagon by the hyperbolic line upon which they lie taking the segment indicated by the order. That is, six ordered geodesics determine six vertices: if the geodesics are $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_6$, set $v_i = \tilde{s}_i \cap \tilde{s}_{i+1}$ for $i \leq i \leq 6$ with indices taken modulo 6 and let the hexagon side denoted by s_i be the segment of $\tilde{s}_i$ traversed from v_{i-1} to v_i where $v_0 = v_6$. With this convention it is clear which segment of $\tilde{s}$ we are talking about, and, therefore, we do not distinguish notationally between the segment s and the geodesic $\tilde{s}$ on which it lies. We write s for both. A *degenerate hexagon* is one where one or more sides reduce to points.

The labeling gives an orientation to the hexagon and its sides. Note that the orientation of a side within the hexagon may be opposite to the orientation of the geodesic containing the side.

Following [**1**], relative to the orientation of the hexagon we define the *complex length* $\delta_i = \delta(s_i)$ *of the side* s_i as follows: $t_i = \Re\delta(s_i)$ is the hyperbolic length of s_i; $\psi_i = \Im\delta(s_i)$ is the angle from $T_{s_i,t_i}(s_{i-1})$ to s_{i+1} with indices taken modulo 6. Here $\Re$ and $\Im$ denote the real and the imaginary parts of a complex number and the real part is always assumed non-negative. With this convention we have the following relation between the trace $Tr\ T$ and the complex translation length $\delta(T)$ of a matrix in $SL(2, \mathbb{C})$

$$Tr\ T = 2\cosh(\delta(T)/2). \tag{3}$$

The complex lengths satisfy the *cosine rule* and it follows that the complex lengths of any three alternating sides or any three adjacent sides determine the other three lengths. We also note that these complex lengths can be computed directly from the ortho-ends using the cross ratio. Namely, for $(x_1, x_2, x_3, x_4) \in \hat{\mathbb{C}}^4$, denote the (appropriately defined) cross-ratio by $\mathbb{CR}[x_1, x_2, x_3, x_4]$. To a given ortho-end, (a, a', n, n', b, b'), associate the quantity $M_A = \log \mathbb{CR}[a, a', n, n']$. Then, up to *pii*, M_A is the complex length of the side of the hexagon lying along the axis of A that is bounded by L_A and L. The relation between the trace of A and this complex length is determined by equation (3) and careful consideration of the orientation of the sides in the hexagon. Similar statements hold for B and $A^{-1}B$. The cross-ratio and complex lengths can be appropriately defined for all relevant cases of degenerate hexagons as well.

8.2. The map into $\mathcal{S}$. The classical representation into $\mathbb{C}^3$ of the space of conjugacy classes of marked free groups $G = \langle A, B\rangle$ is given by the map $\rho : G \mapsto (Tr\ A, Tr\ B, Tr\ AB^{-1})$. We denote by $\mathcal{S}$ the image in $\mathbb{C}^3$ of those (conjugacy classes) of free discrete groups. We want to identify the image of $\mathcal{D}_{A,B}$ (or equivalently $\mathcal{P}_{A,B}$) under ρ.

Thus if M identifies the point $(d_{A'}, \tau_{A'}, d', \tau', d_{B'}, \tau_{B'})$ in $\mathcal{D}_{A,B}$ with the group $G' = \langle A', B'\rangle$ in $\mathcal{P}_{A,B}$, then $\rho(G')$ is well defined.

THEOREM 8.1. *Given an NSDC family with base-point, $\mathcal{P}_{A,B}$, the mapping $\Psi = \rho \circ M^{-1} : \mathcal{D}_{A,B} \to \mathcal{S}$ is well-defined.*

PROOF. Given a point $G' = \langle A', B' \rangle \in \mathcal{P}_{A,B}$ we can read off the trace parameters $(Tr\ A', Tr\ B', Tr\ A'B'^{-1})$. Therefore, by theorem 6.6, since the map M is an isomorphism, the map $\Psi = \rho \circ M^{-1} : \mathcal{D}_{A,B} \to \mathcal{S}$ is well defined as claimed. □

The points in $\mathcal{S}$ correspond to conjugacy classes of groups and each group lives in many different planar families. The mapping Ψ is not an injection.

COROLLARY 8.2. *If two distinct points in $\mathcal{D}_{A,B}$ have the same image in $\mathcal{S}$, that is if*

$$\Psi(d_{A_1}, \tau_{A_1}, d_1, \tau_1, d_{B_1}, \tau_{B_1}) = \Psi(d_{A_2}, \tau_{A_2}, d_2, \tau_2, d_{B_2}, \tau_{B_2}),$$

then there is a different planar family $\mathcal{P}_{A',B'}$ with planar triple $(\mathbb{P}_{A'}, \mathbb{P}', \mathbb{P}_{B'})$ and base-point $\langle A', B' \rangle$, whose groups have ortho-ends in the planes $(\mathbb{P}_{A'}, \mathbb{P}', \mathbb{P}_{B'})$ and

$$\Psi(\mathcal{D}_{A,B}) \cap \Psi(\mathcal{D}_{A',B'}) \neq \emptyset.$$

PROOF. Assume that $(d_{A_1}, \tau_{A_1}, d_1, \tau_1, d_{B_1}, \tau_{B_1})$ and $(d_{A_2}, \tau_{A_2}, d_2, \tau_2, d_{B_2}, \tau_{B_2})$ are distinct points in the same planar family. Let let the groups they determine in the planar family $\mathcal{D}_{A,B}$ be denoted by $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$ respectively. If these points have the same image under Ψ then the groups $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$ are conjugate in the Möbius group by a Möbius transformation h because the conjugacy class of a group $\langle A, B \rangle$ is determined by the triple $(Tr\ A, Tr\ B, Tr\ AB^{-1})$. Since the complex traces determine the lengths of three alternating sides of the hexagon for the group, it must be that the hexagons of $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$ are congruent under h.

After replacing h by $h \circ g$ for some g in the Möbius group if necessary, we may assume that h leaves $\mathbb{P}_A$ invariant, although the pull-back angle, now defined in terms of $h(a), h(a')$, may have changed. We write $\theta_{h(A)}$, θ_h and $\theta_{h(B)}$ for the new angles. Since h maps circles into circles, it will map the planes $\mathbb{P}$ and $\mathbb{P}_B$ respectively to planes $\mathbb{P}'$ and $\mathbb{P}_{B'}$. The data for the new planar family will come from $(h(a), h(a'), h(n), h(n'), h(b), h(b'))$ and the new pull-back angles on the new planes. The assertion of the corollary is that even if it were true that either (1) $h(a) = a$ and $h(a') = a'$ or the weaker condition (2) $\theta_A = \theta_{h(A)}$ held, it would always be true that either $\mathbb{P} \neq \mathbb{P}'$ or $\mathbb{P}_B \neq \mathbb{P}_{B'}$.

Since we are working with an NSDC family, the circles that are the horizons of the planes $\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B$ have a common exterior. Their images under h thus bound a domain. It is easy to see that any Möbius transformation that leaves invariant three circles bounding a domain must be the identity. Since h is not the identity, the corollary follows. □

References

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DEPARTMENT OF MATHEMATICS, SMITH HALL, RUTGERS UNIVERSITY, NEWARK, NJ 07102,
E-mail address: gilman@andromeda.rutgers.edu

MATHEMATICS DEPARTMENT, CUNY LEHMAN COLLEGE AND GRADUATE CENTER, BRONX, NY 10468, USA
E-mail address: linda.keen@lehman.cuny.edu