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A Geometric Approach to Jørgensen's Inequality*

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1. INTRODUCTION

Jørgensen's inequality [J] gives a necessary condition for two elements of $PSL(2, \mathbb{C})$ to generate a non-elementary discrete group. If A and B are in $PSL(2, \mathbb{C})$, the inequality says that

(I) $|\mathrm{tr}^2 A - 4| + |\mathrm{tr}[A, B] - 2| \geqslant 1,$

where tr is the trace and [,] represents the commutator. This is one of the most useful and powerful tools available for determining non-discreteness. The precise geometric meaning of this inequality has been unclear.

Here we prove that purely hyperbolic subgroups of $PSL(2, \mathbb{R})$ obey a stronger inequality (see (III)) involving only the second term, $|\mathrm{tr}[A, B] - 2|$, and we show that the inequality is sharp. This is done by first giving a more geometric formulation of Jørgensen's inequality (see (II)) for hyperbolics and then using the more geometric formulation to derive the proof of the stronger inequality (see (III) or (IV)) for purely hyperbolic groups. For purely hyperbolic groups, (I) is trivially satisfied for all but the finite number of conjugacy classes of elements (or their inverses) with multiplier between 1 and $(3+\sqrt{5})/2$, whereas the stronger inequality is not.

While the geometric formulation of Inequality (IV) may seem awkward to those accustomed to working with traces of commutators, it has the advantage of showing in a clear and simple manner how discreteness is affected when one varies multipliers and certain cross-ratios. Thus it sheds light upon how one moves in and out of the space of discrete groups and makes the construction of examples of both discrete and non-discrete groups easy.

Groups of Möbius transformations in space are an increasingly important area of study. A Jørgensen type inequality would be significant there, and it is hoped that this formulation of Jørgensen's inequality might point the way toward the proper formulation in space.

Something like Inequality (III) is known, at least to Jørgensen, but it does not seem to be in print.

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2. The Inequalities

Let A and B represent hyperbolic transformations with multipliers R and K, respectively. Conjugate A and B by a Möbius transformation so that A fixes 0 and ∞ and B fixes 1 and C. Verify that A and B can thus be represented as matrices

$$A = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & \sqrt{R}^{-1} \end{pmatrix}$$

and

$$B = \frac{1}{(C-1)\cdot\sqrt{K}} \begin{pmatrix} C-K & (K-1)C \\ 1-K & KC-1 \end{pmatrix}.$$

Further, if V_A, W_A and V_B, W_B are the repelling and the attracting fixed points of A and B, respectively, then C or $1/C$ is the cross-ratio of $[V_A, V_B, W_A, W_B] = ((V_A - V_B)\cdot(W_A - W_B))/((V_A - W_B)\cdot(W_A - V_B))$. Note that C, R and K are conjugation invariant and thus independent of our original normalization. Elements of $PSL(2, \mathbb{R})$ are equivalence classes of elements of $SL(2, \mathbb{R})$. Since the square of the trace and the trace of the commutator are both independent of the choice of representatives in $SL(2, \mathbb{R})$, we do not distinguish notationally between an element of $SL(2, \mathbb{R})$ and its image in $PSL(2, \mathbb{R})$.

We let $f(x)$ be the function $f(x) = x/(x-1)^2$ and note that $f(x) = f(1/x)$ and that for $0 \leqslant x < 1$, $f(x)$ is increasing.

For any matrix L in $PSL(2, \mathbb{R})$, let M be its multiplier. Then $\operatorname{tr} L = \sqrt{M} + (1/\sqrt{M})$ and $\operatorname{tr}^2 L - 4 = 1/f(M)$. Next compute that $\operatorname{tr}[A, B] = 2 + (f(C)/(f(R)\,f(K))$ (see Lemma 3.3 of [G1]). Rewrite (I) as (II) below to obtain

The Geometric Formulation of Jørgensen's Inequality for Hyperbolics:

(II) $\quad 1/f(R) + |f(C)|/(f(R)\,f(K)) \geqslant 1.$

The main result of this paper is

Theorem 1. *If A and B generate a non-elementary purely hyperbolic subgroup of $PSL(2, \mathbb{R})$, then*

(III) (a) *if the axes of A and B are disjoint,*

$$|\operatorname{tr}[A, B] - 2| > 16$$

and

(b) *if the axes of A and B intersect,*

$$|\operatorname{tr}[A, B] - 2| > 4.$$

Equivalently

(IV) (a) *if the axes of A and B are disjoint,*

$$\frac{|f(C)|}{f(R)\,f(K)} > 16$$

and

(b) *if the axes of A and B intersect,*

$$\frac{|f(C)|}{f(R)\,f(K)} > 4.$$

In all cases, these results are sharp.

Proof. We prove (IV) and then observe that since $\operatorname{tr}[A, B] = 2 + f(C)/(f(R)\,f(K))$, (III) is equivalent.

Note that $C > 0$ or $C < 0$ as the axes are or are not disjoint.

Let

$$Q^2 = \left(\frac{\sqrt{\tilde{R}} + \sqrt{\tilde{K}}}{\sqrt{\tilde{R}\tilde{K}} + 1}\right)^2$$

and

$$S^2 = \left(\frac{\sqrt{\tilde{R}} - \sqrt{\tilde{K}}}{\sqrt{\tilde{R}\tilde{K}} - 1}\right)^2,$$

where $\tilde{R} > 1$ and $\tilde{R} = R$ or $1/R$, $\tilde{K} > 1$ and $\tilde{K} = K$ or $1/K$. Similarly choose $\tilde{C}$ so if $0 < C$, $\tilde{C} < 1$ and $\tilde{C} = C$ or $1/C$. If $C < 0$, $\tilde{C} = C$.

In an earlier paper [G1], we showed (using a proof independent of Jørgensen's inequality) that a non-elementary subgroup F of $PSL(2, \mathbb{R})$ is torsion free and discrete if and only if for each pair of hyperbolics A, B in F which generate a non-elementary subgroup one of the following inequalities holds:

(i) $\tilde{C} < 0$ and $f(\tilde{C}) \leqslant -4f(\tilde{K})\,f(\tilde{R})$,

(ii) $\tilde{C} > 0$ and $\tilde{C} \geqslant Q^2$,

(iii) $\tilde{C} > 0$ and $\tilde{C} \leqslant S^2$.

Further, it was shown (Theorem 7.1 of [G1]) that strict inequalities in (i), (ii), and (iii) hold for purely hyperbolic groups and that if a pair satisfies (i) or (ii), they generate a discrete free group (see the proof of Theorem 6.1 and Remark 6.5 of [G1]). Using [P] and [B, Chaps. 9 and 10], one can conclude that a pair that satisfies (i) or (ii) with strict inequality actually generates a purely hyperbolic group.

First note that since $f(x)=f(1/x)$, we may assume that $\tilde{R}=R>1$, $\tilde{K}=K>1$, and $\tilde{C}=C$. If (i) holds, $f(C)$ is negative and (IV) (b) follows.

If (ii) holds, $1>C\geqslant Q^2$ so that $f(C)\geqslant f(Q^2)$. Thus

$$f(C)\geqslant \frac{\left(\dfrac{\sqrt{R}+\sqrt{K}}{\sqrt{RK}+1}\right)^2}{\left(\left(\dfrac{\sqrt{R}+\sqrt{K}}{\sqrt{RK}+1}\right)^2-1\right)^2}=\frac{(\sqrt{R}+\sqrt{K})^2\,(\sqrt{RK}+1)^2}{((\sqrt{R}+\sqrt{K})^2-(\sqrt{RK}+1)^2)}$$

$$=\frac{(\sqrt{R}+\sqrt{K})^2\,(\sqrt{RK}+1)^2}{(R-1)^2\,(K-1)^2}\cdot\frac{RK}{RK}$$

$$=f(R)\,f(K)\left(\sqrt{R}+\frac{1}{\sqrt{R}}+\sqrt{K}+\frac{1}{\sqrt{K}}\right)^2.$$

Let $F(X, Y)=(X+1/X+Y+1/Y)^2$. Consider F in the region $X>1$, $Y>1$. In this region $\partial F/\partial X>0$, so the minimum occurs when $X=1$. Substituting $X=1$ yields an expression in Y whose derivative, for $Y>1$, is positive. Hence, the minimum of F occurs when $X=Y=1$ and is 16.

Finally, if (iii) holds, then by [P] or [R], the pair (A, B) is Nielsen equivalent to a pair (E, F) which satisfies (ii) and thus satisfies (IV)(a) or which satisfies (iii) with equality. In the first case since the trace of the commutator is preserved under a (finite number of) Nielsen transformation(s), the pair (A, B) satisfies (IV)(a), too. Theorem 7.1 of [G1] says that the requirement that A and B generate a purely hyperbolic group is equivalent to not allowing equality in (i), (ii), or (iii).

We see that the results are sharp as follows: Let N be any real number greater than 16. Let R be chosen so that $1<R<(N-8+\sqrt{N^2-16N})/8$. This inequality assures that $N>F(R, R)$. We can solve for C in the equation $f(C)=Nf(R)\,f(R)$. Now construct A and B with these values for R and C and with $K=R$. Since (IV)(a) is satisfied without equality, A and B generate a discrete purely hyperbolic group. This shows that (IV)(a) is sharp; (IV)(b) is similar but simpler.

Remark. If one is interested in a straightforward test for non-discreteness, (III) is an appropriate formulation. However, if one wants to understand what it is about the transformations that controls discreteness, (IV) has an advantage. For $R>1$, $K>1$, f is a decreasing function and for $0<C<1$, f is an increasing function. Thus the expression $|f(C)|/(f(K)\,f(R))$ is an increasing function of C, R, and K. The inequality thus demonstrates that to preserve discreteness, a decrease in R, for example, must be off set by an increase in either C or K. Further, since one can construct parameters for Teichmüller space using C, Q^2, S^2, and R or K, (IV) could have useful applications.

Note that $C > 0$ if and only if the axes of A and B are disjoint. In that case $C = [V_A, V_B, W_A, W_B] = 1/(\tanh^2(D/2))$ where D is the length of the common perpendicular joining the two axes (see [B, p. 166], but note that one must compensate for the two different definitions of cross-ratio). Also $C < 0$ if and only if the axes intersect. If Θ is their angle of intersection, then $C = -\tan^2 \Theta/2$.

Also note that $e^{T_A} = R$ and $e^{T_B} = K$ where T_A and T_B are the translation lengths of A and B. From this it is an easy calculation to see that the theorem implies:

COROLLARY. *Let A and B generate a non-elementary purely hyperbolic subgroup of $PSL(2, \mathbb{R})$.*

(V) *If the axes of A and B are disjoint and D is the length of their common perpendicular, then*

$$(\sinh^2 D)(\sinh^2(T_A/2))(\sinh^2(T_B/2)) > 4.$$

(b) *If the axes of A and B meet at an angle Θ, then*

$$(\sin^2 \Theta)(\sinh^2(T_A/2))(\sinh^2(T_B/2)) > 1.$$

Remark. If we wish to allow parabolics in the group generated by A and B, the Theorem and its Corollary remain true if the hypothesis becomes "If the hyperbolics A and B generate a discrete free group..." and the strict inequalities in (III), (IV), and (V) are allowed to be $\geqslant$.

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