

# ON CONJUGACY CLASSES IN THE TEICHMÜLLER MODULAR GROUP

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Let  $S$  be a compact Riemann surface of genus  $g$  with  $n$  punctures, and let  $M(g, n)$  be the Teichmüller modular group of  $S$ . Let  $\lambda(p, g, n)$  denote the number of conjugacy classes of elements of prime order  $p$  in  $M(g, n)$ . The purpose of this paper is to obtain an explicit formula for  $\lambda(p, g, n)$  when  $g \geq 2$ . (W. J. Harvey has considered this problem [1]. He obtained a generating function for  $\lambda(p, g, 0)$  in [1], but he has recently pointed out that his function actually gives the number of conjugacy classes of subgroups of  $M(g, 0)$  of order  $p$  and not  $\lambda(p, g, 0)$ .)

Let  $|M_{T,p}^x|$  be the number of distinct  $(p-1)$ -tuples of nonnegative integers  $(n_1, \dots, n_{p-1})$  with

$$\sum_{i=1}^{p-1} n_i = T \quad \text{and} \quad \sum_{i=1}^{p-1} in_i \equiv x \pmod{p},$$

where  $T$  is a fixed integer. In Section 1 we obtain a set of invariants for each conjugacy class; these invariants make it clear that to compute  $\lambda(p, g, n)$  we need to know  $|M_{T,p}^x|$ . In Section 2 we compute  $|M_{T,p}^x|$ , and in Section 3 we compute  $\lambda(p, g, n)$ .

## 1. INVARIANTS FOR A CONJUGACY CLASS

Let  $S$  and  $S'$  be compact surfaces of genus  $g$  with  $n$  punctures, so that  $S = \bar{S} - Q$  and  $S' = \bar{S}' - Q'$ , where  $Q$  and  $Q'$  are sets of  $n$  points on the compact surfaces  $\bar{S}$  and  $\bar{S}'$ . Let  $h$  and  $h'$  be homeomorphisms of  $S$  and  $S'$ , respectively, whose  $p$ th powers are homotopic to the identity.

*Definition.* The pairs  $(S, h)$  and  $(S', h')$  are *topologically equivalent* if there is a homeomorphism  $f$  of  $S$  onto  $S'$  with  $fhf^{-1} \simeq h'$ , where  $\simeq$  denotes homotopy (see [3]). This is clearly an equivalence relation.

**LEMMA 1.** (i) *The number of conjugacy classes of elements of order  $p$  in  $M(g, n)$  is equal to the number of topological equivalence classes of pairs  $(S, h)$ , where  $S$  is of genus  $g$  with  $n$  punctures and  $h^p \simeq \text{identity}$ .*

(ii) *For each pair  $(S, h)$  there is an equivalent pair  $(S', h')$  in which  $h'$  is conformal.*

*Proof.* Fix a surface  $S$ . Given any pair  $(S', h')$ , let  $f$  be a homeomorphism of  $S'$  onto  $S$ . Then  $(S', h')$  is topologically equivalent to  $(S, fh'f^{-1})$ . Thus, to count topological equivalence classes we need only consider pairs where the surface is fixed. But  $(S, h)$  is equivalent to  $(S, h')$  if and only if  $h$  and  $h'$  are conjugate in  $M(g, n)$ .

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- Let  $[(p, \alpha)]$  denote the equivalence class of  $(p, \alpha)$ . The homomorphism  $\phi$  induces a homeomorphism  $\tilde{h}_\phi$  of  $\tilde{S}$  defined by  $\tilde{h}_\phi[(p, \alpha)] = [(p, \alpha + x)]$ , where  $x$  is any element of  $F_0$  for which  $\phi(x) = 1$ . The group of cover transformations is generated by  $\tilde{h}_\phi$  and is isomorphic to  $Z_p$ .

Since  $\phi$  satisfies (1), (2), and (3), there exist a compact surface  $\bar{S}$  and sets  $Q$  and  $P$  of (respectively)  $n$  and  $T$  points on  $\bar{S}$  such that  $P \cap Q$  contains  $n_0$  points and  $\tilde{S} = \bar{S} - (P \cup Q)$ . The homeomorphism  $\tilde{h}_\phi$  of  $\tilde{S}$  is actually conformal, and it extends to a conformal homeomorphism  $\bar{h}_\phi$  of  $\bar{S}$  with  $\bar{h}_\phi(Q) = Q$ . By construction,  $\bar{h}_\phi$  has  $T$  fixed points. We denote the restriction of  $\bar{h}_\phi$  to  $\bar{S} - Q$  by  $h_\phi$ , and we let  $S_\phi = \bar{S} - Q$ .

**LEMMA 3.** *Let  $(S, h)$  be a pair consisting of a compact surface  $S$  of genus  $g$  with  $n$  punctures and a conformal homeomorphism  $h$  of  $S$  of order  $p$ . Then there is a unique homomorphism  $\phi$  of  $F_0$  onto  $Z_p$  such that  $h = h_\phi$  and  $S = S_\phi$ .*

*Proof.* Using the same notation as before, we have a covering

$$\pi: \bar{S} - (P \cup Q) \rightarrow S_0.$$

We let  $F$  be the defining subgroup of this covering, and we assume that  $F = \ker \phi$ , where  $\phi$  is a homomorphism of  $F_0$  onto  $Z_p$ . By Lemma 2,  $F = \ker r\phi$  for all integers  $r$  between 0 and  $p$ , but  $F$  is not the kernel of any other homeomorphism.  $\bar{S} - (P \cup Q) = \tilde{S}$  and  $S = S_\phi = S_{r\phi}$  for all relevant integers  $r$ . The  $(p - 1)$  covering transformations  $h_\phi, h_{2\phi}, \dots, h_{(p-1)\phi}$  are all distinct. Exactly one of them,  $h_\phi$  say, must be the restriction of  $h$  to  $\bar{S} - (P \cup Q)$ , since this restriction is a covering transformation and there are only  $(p - 1)$  covering transformations that are not the identity. Since  $h$  and  $h_\phi$  agree on  $\tilde{S}$ , they must also agree on  $S_\phi = S$ .

*Remark.* Let  $r$  and  $t$  be integers between 0 and  $p$ . Assume  $rt \equiv 1 \pmod{p}$ . Then  $h_{r\phi} = (h_\phi)^t$ .

In what follows,  $\phi$  will always be a homomorphism of  $F_0$  onto  $Z_p$ , and  $F_0$  will have presentation (\*).

*Notation.* Denoting by  $|E|$  the cardinality of the set  $E$ , we associate with  $\phi$  the two sets of integers

$$\begin{aligned} N(\phi) &= (n_1, \dots, n_{p-1}), & \text{where } n_i &= |\phi^{-1}(i) \cap X|, \\ S(\phi) &= (s_0, \dots, s_{p-1}), & \text{where } s_i &= |\phi^{-1}(i) \cap Y|. \end{aligned}$$

*Remark.* Since  $\phi$  must preserve the defining relation of  $F_0$ , we see that

$$(4) \quad \sum_{i=1}^{p-1} i(s_i + n_i) \equiv 0 \pmod{p}.$$

*Notation.* Let  $(S, h)$  be any pair satisfying the hypotheses of Lemma 3. We associate with  $h$  the two sets of integers

$$N(h) = N(\phi) \quad \text{and} \quad S(h) = S(\phi),$$

where  $\phi$  is the homomorphism of Lemma 3 with  $h = h_\phi$ .

generators to 0 except  $b_1$ . If  $T \neq 0$ , using the notation of Theorem 14 of [1], we replace  $\sigma$  by  $\sigma \circ \mathcal{A}_4^t$ , where  $t$  is the integer satisfying  $t\phi(y_{T-n_0}) \equiv -\phi(b_1) \pmod{p}$ .

If  $T = 0$ , let  $\phi \circ \sigma(b_1) = r$ . Let  $x$  satisfy  $rx + 1 \equiv 0 \pmod{p}$ . Let  $\psi(b_1) = 1$ , and let  $\psi$  agree with  $\phi \circ \sigma$  on all other generators of  $F_0$ . Let  $\tau_x, \sigma_r$ , and  $\delta$  be automorphisms of  $F_0$  that fix all generators except  $a_1$  and  $b_1$ . Let  $\tau_x(a_1) = a_1$  and  $\tau_x(b_1) = b_1 a_1^x$ . Let  $\sigma_r(a_1) = a_1 b_1^r$  and  $\sigma_r(b_1) = b_1$ . Let  $\delta(a_1) = a_1 b_1 a_1^{-1}$  and  $\delta(b_1) = a_1^{-1}$ . Then  $\psi \circ \delta \circ \sigma_r \circ \tau_x = \phi \circ \sigma$ .

**THEOREM 1.** *The conjugacy class of an element  $h$  in  $M(g, n)$  of prime order  $p$  is completely determined by  $S(h)$  and  $N(h)$ .*

*Proof.* Let  $\{(S_i, h_i)\}$  ( $i = 1, \dots, r$ ) be as in the paragraphs preceding Lemma 4. It suffices to show that  $(S_i, h_i)$  is topologically equivalent to  $(S_j, h_j)$  if and only if  $N(h_i) = N(h_j)$ ,  $S(h_i) = S(h_j)$ , and  $\text{tr } h_i = \text{tr } h_j$ . But by construction, all of these coverings have  $\text{tr } h_i = 2 - T$ , and two conformal maps can only be conjugate if they have the same number of fixed points.

By Lemma 5, if  $T \neq 0$ , we may assume  $\phi_i$  ( $i = 1, \dots, r$ ) sends all hyperbolic generators to 0. If  $T = 0$ , the theorem is just Lemma 5 (ii).

Assume  $(S_i, h_i)$  and  $(S_j, h_j)$  are topologically equivalent, and let  $\tau: S_i \rightarrow S_j$ , with  $\tau h_i \tau^{-1} \simeq h_j$ . Since  $h_i$  and  $h_j$  are conformal, we may assume, replacing  $\tau$  by the minimal quasiconformal map in its homotopy class, that  $\tau h_i \tau^{-1} = h_j$ , so that  $\tau$  induces a homeomorphism  $\tilde{\tau}$  of  $S_0$  and thus induces an automorphism  $\tau_*$  of  $F_0$  with  $\tau_*(F_i) = F_j$ . If  $\psi = \phi_j \circ \tau_*$ , one can verify that  $F_i = \ker \psi$ . By Lemma 2,  $\phi_j \circ \tau_* = k\phi_i$  for some integer  $k$ . We want to show  $k = 1$ . Let  $x$  and  $y$  be elements of  $F_0$  with  $\phi_i(x) = 1$  and  $\phi_j(y) = 1$ . We compute

$$\begin{aligned} \tau h_i[(p, \alpha)] &= \tau[(p, \alpha + x)] = [(\tilde{\tau}(p), \tau_*(\alpha) + \tau_*(x))], \\ h_j \tau[(p, \alpha)] &= h_j[(\tilde{\tau}(p), \tau_*(\alpha))] = [(\tilde{\tau}(p), \tau_*(\alpha) + y)], \end{aligned}$$

where  $p$  is any point on  $S_0$  and  $\alpha$  any curve from  $q_0$ , the base point, to  $p$ . Since  $h_j \tau[(p, \alpha)] = \tau h_i[(p, \alpha)]$ ,  $\tau_*(x) - y$  is in the kernel of  $\phi_j$ . Since  $\phi_j(y) = 1$ ,  $\phi_j \circ \tau_*(x) = 1$ . But  $\phi_j \circ \tau_*(x) = k\phi_i(x) = 1$ . Thus  $k = 1$  and  $\phi_j \circ \tau_* = \phi_i$ . Since  $\tau(Q_i) = Q_j$ , we see that  $\tau(P_i) = P_j$ . Thus  $\tau_*$  maps the curves in  $X$  onto conjugates of curves in  $X$  and does the same to curves in  $Y$ . Since  $Z_p$  is an abelian group, it follows that

$$\begin{aligned} |\phi_j^{-1}(t) \cap X| &= |(\phi_j \circ \tau_*)^{-1}(t) \cap X| \quad \text{for } t = 0, \dots, p-1, \\ |\phi_j^{-1}(t) \cap Y| &= |(\phi_j \circ \tau_*)^{-1}(t) \cap Y| \quad \text{for } t = 1, \dots, p-1. \end{aligned}$$

Thus  $N(h_i) = N(h_j)$  and  $S(h_i) = S(h_j)$ .

Conversely, assume  $\phi_j$  and  $\phi_i$  are two mappings such that

$$|\phi_i^{-1}(t) \cap X| = |\phi_j^{-1}(t) \cap X| \quad \text{for } t = 0, \dots, p-1$$

and  $|\phi_i^{-1}(t) \cap Y| = |\phi_j^{-1}(t) \cap Y|$  for  $t = 1, \dots, p-1$ . Corresponding to each permutation  $\sigma$  of the set  $\{1, \dots, s_0 + n_0\}$ , there is an automorphism of  $F_0$  that maps  $x_i$  onto a conjugate of  $x_{\sigma(i)}$  and fixes the elements of  $Y$  and the hyperbolic generators. This is clear when we look at the automorphisms of  $F_0$  generated by the permutations

$$H(p, T, x) = \begin{cases} 0 & \text{if } T \not\equiv 0 \pmod{p} \text{ and } T \not\equiv 1 \pmod{p}, \\ -\frac{1}{p} & \text{if } T \equiv 0 \pmod{p} \text{ and } x \not\equiv 0 \pmod{p}, \\ -\frac{1}{p} + 1 & \text{if } T \equiv 0 \pmod{p} \text{ and } x \equiv 0 \pmod{p}, \\ \frac{1}{p} & \text{if } T \equiv 1 \pmod{p} \text{ and } x \not\equiv 0 \pmod{p}, \\ \frac{1}{p} - 1 & \text{if } T \equiv 1 \pmod{p} \text{ and } x \equiv 0 \pmod{p}. \end{cases}$$

*Proof.* We want to compute  $|M_{T,p}^x|$  for arbitrary  $x$ . We can identify  $N_{T,p}^x$  with  $\bigcup_{y=0}^T M_{T-y,p}^x$  by identifying  $(n_0, \dots, n_{p-1})$  in  $N_{T,p}^x$  with  $(n_1, \dots, n_{p-1})$  in  $M_{T-n_0,p}^x$ . We identify  $N_{T-1,p}^x$  with

$$\bigcup_{z=0}^{T-1} M_{T-1-z,p}^x = \bigcup_{y=1}^T M_{T-y,p}^x.$$

Thus  $|M_{T,p}^x| = |N_{T,p}^x| - |N_{T-1,p}^x|$ . Therefore, to compute  $|M_{T,p}^x|$ , we need to compute  $|N_{T,p}^x|$  for all  $x$  and  $T$ .

We shall show that

$$|N_{T,p}^x| = \begin{cases} \frac{1}{p} \binom{p+T-1}{T} & \text{if } T \not\equiv 0 \pmod{p}, \\ \frac{1}{p} \binom{p+T-1}{T} - \frac{1}{p} & \text{if } T \equiv 0 \pmod{p} \text{ and } x \not\equiv 0 \pmod{p}, \\ \frac{1}{p} \binom{p+T-1}{T} - \frac{1}{p} + 1 & \text{if } T \equiv 0 \pmod{p} \text{ and } x \equiv 0 \pmod{p}. \end{cases}$$

The theorem follows from this.

Assume first that  $T \not\equiv 0 \pmod{p}$ . We shall show that  $|N_{T,p}^x| = |N_{T,p}^y|$  for all  $x$  and  $y$ . For any integers  $a$  and  $b$ , we let  $f_{a,b}$  be the mapping of  $Z_p$  to  $Z_p$  given by sending  $i$  into the remainder when  $ai + b$  is divided by  $p$ . Assume that  $a \not\equiv 0 \pmod{p}$ , so that  $a^{-1}$  makes sense. Let  $F$  map  $N_{T,p}^x$  to itself according to the formula

$$F(n_0, \dots, n_{p-1}) = (n_{f(0)}, \dots, n_{f(p-1)}).$$

It is easy to verify that  $F$  maps  $N_{T,p}^x$  to itself and that  $F$  in fact maps  $N_{T,p}^x$  to  $N_{T,p}^{a^{-1}(x-bT)}$ .

If  $T \not\equiv 0 \pmod{p}$ , then for each  $x$  we can find  $a$  and  $b$  such that  $a^{-1}(x-bT) \equiv 0 \pmod{p}$ . Therefore  $F$  is a one-to-one map of  $N_{T,p}^x$  onto  $N_{T,p}^0$ . Thus  $|N_{T,p}^x| = |N_{T,p}^0|$  for all  $p, T$ , and  $x$  if  $T \not\equiv 0 \pmod{p}$ .

$$r = \left\lfloor \frac{2g - (k_0 - 2)(p - 1)}{p(p - 1)} \right\rfloor,$$

where  $\lfloor \cdot \rfloor$  denotes the greatest-integer function. If  $p = 2$ , we let  $k_0 = 0$  when  $g \equiv 1 \pmod{2}$  and  $k_0 = 2$  when  $g \equiv 0 \pmod{2}$ . Let  $\tilde{n}_0$  be the remainder when  $n$  is divided by  $p$ , and let  $b$  be the minimum of  $\lfloor (T - \tilde{n}_0)/p \rfloor$  and  $(n - \tilde{n}_0)/p$  when it is positive, and zero otherwise.

$$\mu(p, g, n, T) = \sum_{s=0}^b \sum_{x=0}^{p-1} |M_{n_0+sp,p}^x| \cdot |M_{T-(\tilde{n}_0+sp),p}^{p-x}|$$

when  $p \neq 2$ ,  $T \equiv k_0 \pmod{p}$ , and  $0 \leq T \leq k_0 + rp$ .

$$\mu(2, g, n, T) = b + 1$$

when  $p = 2$ ,  $\frac{T}{2} \equiv \frac{k_0}{2} \pmod{2}$ , and  $0 \leq T \leq 2g + 2$ .

In all other cases,  $\mu(p, g, n, T) = 0$ . The value of  $|M_{Y,p}^x|$  is given in Theorem 2.

*Proof.* Assume first that  $p \neq 2$ . Let  $S, \bar{S}, Q, h, P$  and other notation be defined as in Section 1. Recall that  $g_0$  is the genus of the factor surfaces  $S_0$  and  $\bar{S}/\langle h \rangle$ , and that  $g$  is the genus of  $S$ . The Riemann-Hurwitz formula implies that  $g = pg_0 + (p - 1)(T - 2)/2$ .

If  $S$  has an automorphism  $h$  of order  $p$ , then  $g_0$  must be an integer. Thus  $g - (p - 1)(T - 2)/2$  is divisible by  $p$ , and this implies that  $T \equiv 2 - 2g \pmod{p}$ . An automorphism of order  $p$  on  $S$  can have  $k_0 + tp$  fixed points, where

$$0 \leq t \leq \left\lfloor \frac{2g - (k_0 - 2)(p - 1)}{p(p - 1)} \right\rfloor.$$

The latter inequality follows from the requirement that  $g_0$  be nonnegative. Thus  $\mu(p, g, n, T) = 0$  if  $T \not\equiv 2 - 2g \pmod{p}$  or  $T \geq k_0 + (r + 1)p$ .

Since  $n - n_0$  must be divisible by  $p$ , the possible values for  $n_0$  are  $\tilde{n}_0, \tilde{n}_0 + p, \dots, \tilde{n}_0 + bp$ , where  $b$  is the minimum of  $\lfloor (T - \tilde{n}_0)/p \rfloor$  and  $(n - \tilde{n}_0)/p$ . This follows from the fact that  $\tilde{n}_0 + bp \leq n$  and  $\tilde{n}_0 + bp \leq T$ . Thus if  $T < \tilde{n}_0$ ,  $\mu(p, g, n, T) = 0$ .

Fix  $T$  and  $n_0$ , and suppose  $n - n_0$  is divisible by  $p$ . Assume  $T = k_0 + mp$ , where  $0 \leq m \leq r$  and  $n_0 \leq T$  and  $n_0 \leq n$  with  $n_0 \equiv \tilde{n}_0 \pmod{p}$ . We want to show that for these numbers and for each  $(p - 1)$ -tuple  $(n_1, \dots, n_{p-1})$  and each  $p$ -tuple  $(s_0, \dots, s_{p-1})$ , where  $s_0 = (n - n_0)/p$  and

$$\sum_{i=0}^{p-1} s_i = n, \quad \sum_{i=1}^{p-1} (n_i + s_i) = T, \quad \sum_{i=1}^{p-1} i(n_i + s_i) \equiv 0 \pmod{p},$$

we can construct a surface  $S$  of genus  $g$  with  $n$  punctures, and a homeomorphism  $h$  with  $2 - \text{tr } h = T$ ,  $N(h) = (n_1, \dots, n_{p-1})$ , and  $S(h) = (s_0, \dots, s_{p-1})$ .

The Riemann-Hurwitz formula shows that  $2g - 2 + T = p(2g_0 - 2 + T)$ . Thus, if  $g \geq 2$ , then  $2g_0 - 2 + T > 0$ , so that  $2g_0 - 2 + T + s_0 > 0$ . This means that we can

The value of  $|M_{T,p}^x|$  is given in Theorem 2.

If  $p = 2$ , let  $b_y$  be the minimum of  $[(k_0 + 4y - \tilde{n}_0)/2]$  and  $(n - \tilde{n}_0)/2$  when both numbers are positive, and zero otherwise; then

$$\lambda(2, g, n) = \sum_{y=0}^r (b_y + 1).$$

*Proof.*  $\lambda(p, g, n) = \sum \mu(p, g, n, T)$ , the sum being taken over all possible values of  $T$ .

#### REFERENCES

1. W. J. Harvey, *On branch loci in Teichmüller space*. Trans. Amer. Math. Soc. 153 (1971), 387-399.
2. C. Maclachlan and W. J. Harvey, *On mapping-class groups and Teichmüller spaces*. Proc. London Math. Soc. (3) 30 (1975), 496-512.
3. J. Nielsen, *Die Struktur periodischer Transformationen von Flächen*. Danske Vid. Selsk. Mat.-Fys. Medd. 15, No. 1 (1937), 1-77.
4. Chih-han Sah, *Groups related to compact Riemann surfaces*. Acta Math. 123 (1969), 13-42.

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