

A REMARK ABOUT COMPONENTS OF RELATIVE TEICHMÜLLER SPACES

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Our aim is to compute for all $n > 2$, $\psi(n, h)$, the number of components of a certain quotient of the fixed point set of an involution in the “mod- n ” Teichmüller space. This answers part of a question raised by Earle [2] and corrects and extends the answer due to Zarrow (See Theorem 2 of [6]).

NOTATION. Let X be a smooth surface of genus $g \geq 2$ and $M(X)$ the space of smooth complex structures with the C^∞ topology. Let $\text{Diff}(X)$ (respectively $\text{Diff}^+(X)$) be the group of diffeomorphisms (respectively orientation preserving) of X , and $\text{Diff}_n^+(X)$ those elements of $\text{Diff}^+(X)$ which induce the identity on homology modulo n . Let $\sigma: \text{Diff}(X) \rightarrow \text{Diff}(X)/\text{Diff}_0(X)$ (here $\text{Diff}_0(X) = \{f \in \text{Diff}(X) \mid f \text{ is homotopic to the identity}\}$), and let $\sigma_n: \text{Diff}(X) \rightarrow \text{Diff}(X)/\text{Diff}_n^+(X) = \Gamma_n(X)$. $T_n(X) = M(X)/\text{Diff}_n^+(X)$ is the mod- n Teichmüller space of X . Let h be an involution in $\text{Diff}^+(X)$, $T_n(X)^{\sigma_n(h)}$ the fixed point set of $\sigma_n(h)$ acting on $T_n(X)$, and $\Gamma_n(h)$ the normalizer of $\sigma_n(h)$ in $\sigma_n(\text{Diff}^+(X))$.

Finally we let $\psi(n, h)$ be the number of components of $T_n(X)^{\sigma_n(h)}/\Gamma_n(h)$. We use Earle’s result (Theorem 4b of [2]) that $\psi(n, h)$ is equal to the number of $\text{Diff}^+(X)$ conjugacy classes of involutions p in $\text{Diff}^+(X)$ with $\sigma_n(p) = \sigma_n(h)$.

PROPOSITION. *If $n > 2$, $\psi(n, h) = 1$.*

Proof. We define some matrices. I_q will be the $q \times q$ identity matrix;

$$L(0, t, g') = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \quad \text{and} \quad L(1, 0, k) = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$$

where

$$S = \begin{pmatrix} 0 & I_{g'} & 0 \\ I_{g'} & 0 & 0 \\ 0 & 0 & -I_t \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{Here } g = 2g' + t = 2k + 1. \quad (1)$$

Here q, t, g, g' and k are integers.

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The results of Theorem 3 for $\mu(2, g, 0, T)$, of [4], together with Proposition 1 of [3] and Theorems 1(ii), 1(iii), and 2 of [5] imply that $\sigma(p)$ is conjugate to an involution whose action on a canonical homology basis is given by $L(r, s, m)$ where either $r=0$ or $r=1$ and $s=0$. Let $h(r, s, m)$ in $\text{Diff}^+(X)$ induce the matrix $L(r, s, m)$. A conjugate of $\sigma(p)$ equals $\sigma(h(r, s, m))$ for some triple (r, s, m) . By the remark in Section 5G of [1], we can conclude that p and $h(r, s, m)$ are already conjugate in $\text{Diff}^+(X)$. Therefore, it suffices to show that there is no symplectic matrix K with $KL(r, s, m) \equiv L(u, v, w)K$ modulo n unless $r=u$, $s=v$, and $m=w$.

Assume that $L(r, s, m)$ and $L(u, v, w)$ are conjugate modulo n . Then so are $\tau = I_{2g} + L(r, s, m)$ and $\rho = I_{2g} + L(u, v, w)$. τ and ρ are endomorphisms of the abelian group $G = \mathbb{Z}/n\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n\mathbb{Z}$, $2g$ copies. If τ is any group homomorphism, let $R(\tau)$ be the minimal number of generators of $\tau(G)$. Since τ and ρ are homomorphisms which are conjugate by an automorphism of G , $\tau(G)$ and $\rho(G)$ are isomorphic, so that $R(\tau) = R(\rho)$. In our case $\tau(G)$ is generated by the rows of $I_{2g} + L(r, s, m)$. We can compute $R(I_{2g} + L(0, t, g')) = 2g'$ and $R(I_{2g} + L(1, 0, k)) = 2k + 2$. If $r=0$ and $u=0$, this forces $m=w$ and $s=v$. If $r=0$ but $u=1$ and $v=0$, this forces $2m = 2w + 2$. But by (1) $g = 2m + s = 2w + 1$. This contradicts the fact that s is nonnegative.

REMARK. The referee has pointed out that the result of this paper provides a new proof that $R(X, H)$, the relative Riemann space, is a complex algebraic variety when $H \subset \text{Diff}^+(X)$ is generated by a sense preserving involution [see 2].

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