RESEARCH STATEMENT

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APRIL 30, 2009

1. Introduction

My research is in Teichmüller theory and Kleinian groups. These two areas are both related to the classical problem of classifying Riemann surfaces up to conformal equivalence. The space of conformal equivalence classes of Riemann surfaces of a fixed topological type (i.e., the *moduli space*) is best approached by way of a covering space, called *Teichmüller space*, whose points are equivalence classes of marked surfaces. A marked surface is, roughly speaking, a surface with a fixed choice of generators for its fundamental group.

Riemann surfaces (with a few exceptions) are quotients of the upper half-plane by *Fuchsian groups*, that is, discrete subgroups of $PSL(2,\mathbb{R})$. (For ease of exposition all groups discussed here are assumed non-elementary.) One can study the Teichmüller space of a Fuchsian group, and more generally of a *Kleinian group* which is a non-elementary discrete subgroup of $PSL(2,\mathbb{C})$. For a Kleinian group one studies the quotient of hyperbolic three-space under the action of the group, that is, a hyperbolic three-manifold whose boundary is the union of Riemann surfaces of finite type. Because groups conjugate in $PSL(2,\mathbb{C})$ acting on the complex sphere yield conformally equivalent quotients, it is more natural to consider the representations of the group in the space $PSL(2,\mathbb{C})/\sim$, where $\sim$ denotes conjugacy. For surfaces there is not a substantial difference between studying the Teichmüller space of the surface and studying the Teichmüller space of the uniformizing group. With three-manifolds, technicalities make studying the Teichmüller space of the three-manifold different from studying the Teichmüller space of the Kleinian group. Other deformation spaces are often studied, some more tractable than others. Of particular interest is the space of (discrete and non-discrete, faithful and non-faithful) representations of a finitely generated Kleinian group modulo conjugacy.

The group of self-homeomorphisms (equivalently, of diffeomorphisms) of a surface onto itself (modulo isotopy) is called the *mapping class group* of the surface. Each representative of a mapping class acts on the surface and each mapping class induces an action on the corresponding Teichmüller space. The quotient under the action of the mapping class group is *moduli space*. In different times and settings the mapping class group is also refereed to as the “Teichmüller modular group” or simply “*the* modular group”. It has been known since the time of Nielsen that finite order elements of the mapping class group have representatives that act conformally on some Riemann surface. Such mapping classes (or elements of such mapping classes) are also referred to as automorphisms of surfaces (or simply as automorphisms).

Teichmüller theory can be approached from different perspectives. Teichmüller space has both a complex analytic structure and a real analytic structure and can be studied from both points of view. It can be studied using an algebraic approach based on group theory. It can also be approached from the direction of topology and hyperbolic geometry because of the three manifold topology of the quotient spaces and because elements of $PSL(2,\mathbb{C})$ act as isometries with respect to the hyperbolic metric on upper half three-space.

Recently big conjectures in three-manifold topology, including the tameness conjecture and the ending lamination conjecture, have been solved and many questions about moduli space and the mapping-class group have been answered using the complex of curves. There are still many open questions. One important deformation space that is not yet fully understand is the full space of representations (discrete and non-discrete) modulo conjugacy.

My papers fall into three main categories: Teichmüller space and moduli space; the Nielsen-Thurston classification, mapping-class groups and automorphisms; and discreteness criteria for Kleinian groups. In what follows I summarize my contributions in these areas and place them in context. Citations refer to the abstracts that appear in Section 6 (beginning in the middle of page 7). More details are given there and the papers that contain the major ideas and results from which other results flow are indicated there by asterisks. Section 5 (see pages 6 and 7) addresses current work and Sections 7 and 8 other publications.

2. Teichmüller space and Moduli space

[2, 7, 23, 29, 27, 28, 32]

In my dissertation I extended results of Bailey for compact surfaces to surfaces of finite type by showing that the moduli space for a Riemann surfaces of finite type is a quasi-projective variety (i.e., it can be embedded in complex projective space of sufficiently high dimension so as to be the difference of two projective algebraic varieties). This result uses the Satake compactification of the Siegel Upper-Half-Space and the techniques of factoring through period matrices and extending holomorphic maps. The Deligne-Mumford compactification and Bers's approach came later.

Since then I have studied the boundaries of representation spaces for different types of Kleinian groups. The relevant papers here are [7, 23, 29, 27, 28, 32]. I have also studied discreteness criteria for individual groups. I view these as two different approaches to Teichmüller theory that are complementary to each other. Papers relevant to the latter approach are summarized in Section 4 (see page 4).

In considering boundaries or components, I have used a variety of techniques. In early work I used results on automorphisms of Riemann surfaces [1, 4] to find the number of components of a relative Teichmüller space [7]. My results on discreteness criteria for two-generator subgroups of $PSL(2,\mathbb{R})$ generated by pairs of hyperbolic transformations with intersecting axes [19] came into play when Gehring, Martin, and I found Kleinian groups with real parameters [23]. After I introduced the concept of non-separating disjoint circle groups (nsdc groups for short) [21], Waterman and I found the boundary of the classical T-Schottky space generated by two parabolics by generalizing a technique initially applied to the boundary of nsdc space [29] (see also [27]). In [32] using iteration I extended the results to the more general class of nth-Schottky groups and *parabolic dust* groups. Keen and I studied the full space of nsdc groups, [28], and my student, Karan Puri, is extending some of these techniques and concepts to four dimensional hyperbolic space.

3. The Nieslen-Thurston Classification, Mapping-class groups and Automorphisms

[1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 18, 31, 33]

Thurston announced a classification of elements of the mapping-class group in the late 1970's. In 1978 at an AMS special session and in Bers's complex analysis seminar I described a connection between Thurston's theory and Nielsen's classification of elements of the mapping-class group. This connection was developed by me in [8], and also by others including Bleiler-Casson and Miller. My papers on the mapping-class group and Nielsen-Thurston theory that followed [8] include [9, 10, 11, 12, 18, 33].

My thesis introduced the concept of *adapted basis*, and it led to a wider interest in conformal automorphisms, culminating in [4]. Adapted bases have an obvious connection with Thurston's reducible elements of the mapping-class group. Adapted bases have been used by Rodriguez, her collaborators, and her students to study Jacobian and Prym varieties. Recently there has been renewed interest and activity in the area of conformal automorphism groups, prompting me to revisit this topic. Papers on automorphisms include [1, 3, 4, 5, 6, 7, 31, 33].

4. Discreteness Criteria

[13, 14, 15, 16, 17, 19, 20, 21, 22, 24, 25, 28, 29, 30, 34, 35, 36]

A number of authors have proposed criteria for discreteness of finitely generated non-elementary subgroups of $PSL(2,\mathbb{R})$. This is a technically difficult area. Maskit and I gave the first correct geometric characterization for subgroups generated by two hyperbolic elements of $PSL(2,\mathbb{R})$ with disjoint axes and their derivative cases [17]. Further work yielded a characterization for an arbitrary pair of generators [19].

A novel aspect of the Gilman-Maskit characterization is that it employs an algorithm (which we refer to as the G-M algorithm) to decide if a given two generator subgroup of $PSL(2,\mathbb{R})$ is discrete or not. The algorithm is geometric in the sense that atomic steps (i.e., the individual steps) are geometric operations, (e.g., deciding whether the axes of two transformations intersect or are disjoint). In [20] the time complexity of the G-M algorithm is shown to be polynomial in the size of the input when the algorithm is implemented as a BSS (Blum-Shub-Smale) machine and the conventional time complexity for the case that the input consists of matrices whose entries lie in some fixed finite extension of the rationals is estimated. Later my student Yicheng Jiang improved this estimate by showing that the latter form of the algorithm also has polynomial time complexity. The use of geometric algorithms are now quite popular (see, for example, the recent paper of Manning).

Developing a framework for thinking about the computational complexity of the G-M algorithm led back to new geometric results and discreteness criteria. For example (except in rare instances) an implementation of the algorithm will include steps that replace one ordered set of generators by another and one of two types of replacement steps are possible. The complexity analysis needs to track the types of successive steps as this affects the lengths of words in the initial generators that the algorithm must consider. The lengths of words in the initial generators is one obvious factor that a complexity analysis must consider [20]. Linda Keen and I studied the disjoint axes case when the algorithm outputs "group is discrete" [25]. In such a case, there is a quotient surface which is a three-holed sphere. We made a connection between the list that tracks the number of successive types of steps and a formula that counts the number of (essential) self-intersections of curves on the surface that are the projections of the axes of primitive words in the group. Another outgrowth of the complexity analysis is the connection that Keen and I have made between the geometry of palindromic generators of a subgroup of $PSL(2, \mathbb{C})$ and discreteness [34].

Families of words (in the generators of a two-generator subgroup of $PSL(2, \mathbb{C})$) have been used as an aid in proving discreteness result by Gehring- Martin, by Gabai-Meyerhoff-N.Thurston, by Keen and Series and Wright, by Parker, by Minsky and by many others. Different families of words are used in different settings to reach conclusions about discreteness properties of subgroups of $PSL(2, \mathbb{C})$.

In 2000 I began the investigation of connections between various of these families and formalized the concept of *informative words* [30]. The goal is to find connections between these families so that results obtained in one setting will give rise to new results in another setting. Some of these connections and useful applications have already been obtained [25, 30, 34, 36]. Others are in progress. The overall goal is to put all of these word families into a unified algebraic setting. This is the main focus of my current research.

5. Current Work

Currently I am working on several projects: (1) Keen and I want to use the complex length function to establish results about the ordering of the intersection points of the palindromes along the core geodesic. There are two important orders on rational numbers, the standard order and the order given by Farey level. This, along with complex length, should give an ordering along the L-line. (2) I am studying more general motions on the parameter space. For example, the composition of *good* words changes the commutator parameter. Moving around the parameter space using algorithmic words does not. The changes in the parameters for nsdc space do not correspond to quasiconformal deformations. (3) I want to develop the concept of the $PSL(2,\mathbb{R})$ as an analog (using the hyperbolic metric) of the Euclidean algorithm, a *so-called* non-Euclidean Euclidean algorithm. (4) I hope to explain the theoretical basis behind Yamamto's example.

I give some details of (3) and (4):

(3) The Non-Euclidean Euclidean Algorithm

The heuristic idea is the following: the G-M discreteness algorithm can be recast as a type of Euclidean Algorithm applied to the non-Euclidean distances that arise in the G-M algorithm. That is, in reducing the trace of the matrices, one is reducing the hyperbolic translation length and the calculations with the greatest integers in the $\frac{\log R}{\log K}$, where R and K are the multipliers of the hyperbolic generators and doing a Euclidean algorithm type calculation here. The full algorithm runs through elliptics and parabolics neither of which have multipliers but these can also be worked into this conceptual scheme. Proving that the original algorithm had polynomial complexity was difficult and required clever arguments because of several steps that increased the complexity double exponentially (see [20, 22] and my student Yicheng- Jiang's thesis). Computing the computational complexity of the recast non-Euclidean Euclidean algorithm where computation begins with $\log K$ yields polynomial complexity. The complexity of the algorithm is intimately connected with the numbers of self-intersections of curves on the surface when the group in question is discrete ([25]). Geodesics on surfaces, their lengths and intersection properties is a topic of current interest in the field.

(4) On Yamamoto's Example of a Classical Schottky Group.

In 1990 Yamamoto gave an explicit construction of a single non-classical Schottky group on two generators, G_ϵ. This example has not been understood well enough to generalize. It is easy to verify that it is a Schottky group. The computations to prove that it is a non-classical group for $\epsilon = 10^{-20}$ are delicate. I believe that Yamamoto is calculating the Hausdorff dimension of the limit set and proving that his example is non-classical by proving that the Hausdorff dimension of the limit set is too big. That is, I think that the proof can be given using the work of Beardon on the Hausdorff dimensions of limit sets of Schottky groups and Doyle's bound. Once this is done, I believe that Yamamoto's example can be generalized.

6. Abstracts of Research Papers

References

[1] * Compact Riemann Surfaces with Conformal Involutions, *Proc. Amer. Math. Soc.* **37** (1973), 105-107.

This paper uses adapted bases to characterize Riemann surfaces admitting a conformal involution in terms of their period matrices. The characterization was used in my thesis [2] and extended in [1] (see also [4]).

[2] * On the Moduli of Compact Riemann Surfaces with a Finite Number of Punctures in *Discontinuous Groups and Riemann Surfaces*, Annals of Math. Studies **79** (1974), 181-205.

In this paper (which includes and extends part of my thesis), I prove that the moduli space for compact surfaces with a finite number of punctures has a quasiprojective structure. This means that it can be embedded in a complex projective space as the difference of two algebraic varieties. This had been proved by W. J. Bailey in the compact case.

[3] On Conjugacy Classes in the Teichmüller Modular Group, *Mich. Math. J.* **23** (1976), 53-63.

This paper derives a formula for the number of conjugacy of elements of prime order p in the mapping class group, (the Teichmüller Modular group) of a compact Riemann surface of genus $g \geq 2$ with $k \geq 0$ punctures. The argument is combinatorial and extends some work of W.J. Harvey and C. Maclachlan.

[4] * A Matrix Representation for Automorphisms of Riemann Surfaces, *Linear Algebra and its Applications* **17** (1977), 139-147.

This paper extends [1] from involutions to prime order automorphisms on a compact surface. It gives an explicit description of the action of the homeomorphism on the fundamental group and on the first homology group. It extends results of Serre (1960) and Accola (1967).

[5] An Example about Normalizers in Mapping Class Groups, *Proc. Amer. Math. Soc.* **69** (1978), 138-147.

This paper answers in the negative a question raised by Birman by producing elements of the mapping-class group that are not in the normalizer of any element of finite order. The construction relies on results from [4] and techniques of Raymond and Tollefson.

[6] Intersection Matrices for Adapted Bases (with David Patterson), in *Riemann Surfaces and Related Topics*, Annals of Math. Studies, **97** (1981), 149-166.

We compute the intersection matrix for the adapted homology basis of an element of order greater than two and indicate the existence of an algorithm for replacing the matrix with respect to an adapted by a symplectic matrix representation of the action. I returned to this theme in [31, 33].

[7] A Remark about Components of Relative Teichmüller Spaces, *Canad. Math. Bull.* **24** (1981), 245-6.

The main result here is that the number of components of the relative (the mod n, $n > 2$) Teichmüller space for a surface of genus $g \geq 2$ is one. This paper answered part of a question raised by Earle. The result also yielded as a corollary the fact that the relative Riemann space is a complex algebraic variety.

[8] *On the Nielsen Type and the Classification of the Mapping Class Group, *Advances in Math.* **40** (1981), 68-96.

During the 1970s automorphisms of Riemann surfaces was a very active area of reserach as was the analytic approach to the Teichmüller theory of surfaces and the Teichmüller modular group. Thurston brought the topology of surfaces to the fore. There is a deep connection between Nieslen's work and Thurston's classification of surface diffeomorphisms. In particular, I established a connection between the pairs of integers, *the Nielsen types*, that Nielsen associated to a mapping-class and its family of lifts to the unit disc and Thurston's classiciation and showed that Thurston's classification could be obtained from Nielsen's and vice-versa.

Given a representative of a mapping class of a surface, Nielsen considered all of its lifts to the unit disc and associated a pair of integers, which I termed its *Nielsen type*, to each lift as well as to each lift of any power of the representative of the mapping class. The totality of the Nielsen types of all of these lifts classify a mapping-class. It is a finer classification than Thurston's. Roughly speaking, if the first integer is non zero for some lift, then the mapping class is reducible; if the first integer is zero for all lifts, then the mapping-class is pseudo-Anosov and the set of second integers in the pairs give the number of prongs of the singularities of the foliation as well as the number of singularities; and if the second integer is zero for all lifts, then the mapping class is of finite order.

Subsequent related papers are [9], [11], [10] and [12].

[9] Determining Thurston Classes Using Nielsen Types, *Trans. Amer. Math. Soc.* **272** (1982), 669-675.

This paper, a continuation of [8], proves that an example that Nielsen studied extensively, Nielsen's example #13, is a pseudo-Anosov mapping.

[10] Structures of Elliptic Irreducible Subgroups of the Mapping-class Group, *Proc. London Math. Soc.* **47** (1983), 27-42.

This paper initiates a study all possible product relations for elements of the mapping-class group under Thurston's classification. The notion of reducible and irreducible mapping classes was extended to reducible and irreducible groups of mapping classes and all irreducible elliptic subgroups of the mapping-class group are classified. The full set of examples sought was completed in [18].

[11] On the Existence of Cyclic Surface Kernels for Pairs of Fuchsian Groups (with Robert Gilman), *J. London Math. Soc.* **30** (1984), 451-464.

Nielsen-Thurston theory and group theoretic techniques are used to determine when two finite reducible elements of the mapping class group generate an infinite order subgroup. All cases where there exists cyclic surface kernels for certain signature pairs that correspond to the significant minimal cases are classified.

[12] A Characterization of Finite Subgroups of the Mapping-class Group, in *Proc. 1985 Alta Conference*, Annals of Math. Studies (1987), 433-442.

Let G be a group of diffeomorphisms acting on a surface S of genus $g \geq 2$. Let $L(G)$ be the group lifts of elements of G to the unit disc. We prove that every periodic subgroup of the mapping class group is finite. We use this result to characterize groups G whose image in the mapping class group is a finite group by the Nielsen types of the elements of $L(G)$. Our methods yield an explicit construction for the Fuchsian group that solves the Nielsen realization problem when there is a realization.

[13] *A Geometric Approach to the Hyperbolic Jorgensen Inequality, *Bull. Amer. Math. Soc.* **16** (1987), 91-92.

Jørgensen's inequality gives a necessary condition for a (non-elementary) group generated by two elements A and B of $SL(2,\mathbb{C})$ to be discrete, namely that $|Tr([A,B]) - 2| + |Tr^2(A) - 4| \geq 1$, where $[A,B]$ is the multiplicative commutator and Tr the trace.

In this paper and the subsequent paper, I used multipliers and cross-ratios of fixed points of pairs of hyperbolic Möbius tranformations to give an alternate geometric formulation of Jørgensen's inequality and a short new purely agebraic proof. Unlike the standard proof of Jørgensen's inequality, the new proof did not involve iteration of sequences of elements of the group. It did use geometric entities and inequalities derived in [14]. Parts of the G-M algorithm [17] were an outgrowth of this paper.

[14] Inequalities in Discrete Subgroups of $PSL(2,\mathbb{R})$, *Canad. J.* **XL** (1988), 115-130.

This papers contains various inequalities that imply discreteness. To give a flavor of the results here we let g and h be hyperbolic tranformations and let $\sqrt{K} > 1$ and $\sqrt{R} > 1$ be their respective multipliers and $0 < C < 1$ the (normalized) cross-ratio of their fixed points. Define $Q^2 = Q^2(K,R) = (\frac{\sqrt{K}+\sqrt{R}}{\sqrt{RK}+1)})^2$ and $S^2 = S^2(K,R) = (\frac{\sqrt{K}-\sqrt{R}}{\sqrt{RK}-1)})^2$.
Then $C \leq Q^2 \Rightarrow G = \langle g,h \rangle$ is discrete.

[15] On the Existence of Elliptic Elements in Discrete Groups, *Holomorphic Functions and Moduli, II* Springer-Verlag, N.Y. (1988) 23-27.

Inequalities involving S^2, Q^2 and C obtained in [14] are used to locate elliptic elements in two hyperbolic generator groups. This involves analysis of Q^2 and S^2 as functions of the multipiliers. In particular, the qualitative result is that elliptic elements occur *early* and *often*, that is, as short words in the initial generators and as many words in the initial generators.

[16] * A Geometric Approach to Jorgensen's Inequality, *Advances in Math.* **85(2)** (1991), 193-197.

The details of the result announced in [13] appear here. In particular, let $f(X) = \frac{X}{(X-1)^2}$. Then Jørgensen's Inequality translates to the statement $|\frac{f(C)}{f(K)f(R)}| + |\frac{1}{f(K)}| \geq 1$. The inequality appears as a consequence of applying the function f to various inequalities such as $C \leq Q^2$ obtained in [14] (and given above) and observations including the observation that f is a decreasing funcion of K and R for multipliers greater than one and that for positive X, $f(X) = \frac{1}{X}$.

[17] * An Algorithm for Two-generator Discrete Groups (with Bernard Maskit), *Mich. Math. J.* **38** (1991) 13-32.

This paper gives a geometric treatment of the $PSL(2,\mathbb{R})$ two-generators discreteness problem. The full solution to the problem had to be formulated in terms of an algorithm. In this paper Maskit and I treated six of the seven algorithmic cases, known collectively as "the intertwining cases".

For a given pair of generators, when the algorithm does not determine discreteness or non-discreteness directly from the Jørgensen-Poincare tests, it produces the next set of generators to consider. We prove that it outputs a definite answer to the discrete question after considering a finite number of pairs of generators. It stops because it is trace minimizing, an essential concept deveoped by Rosenberger and Purzitsky.

In $PSL(2,\mathbb{C})$ there is no known discreteness algorithm. However, results have been obtained by Sakuma and Ser Tan, among others, using the Jørgensen-Poincare dichotomy (originally due to Riley), which gives a *procedure*, something that does not necessarily terminate, as opposed to what computer scientsts would call an algorithm.

[18] Recent Developments in Nielsen Theory and Discrete Groups, in *Nielsen Theory and Dynamical Systems*, AMS CONM Mathematics **152** (1993), 159-176.

This paper surveyed results in Nielsen theory and Dynamical systems roughly ten years after my paper on the Nielsen-Thurston type ([8]). The survey included results of Gabai on the Nielsen realization problem and my paper [12] and also included some new results on product relations in mapping class groups. The quest for possible examples of products of every type begun in [10] is completed. By product relations is meant types of products using the Bers-Thurston classification. For example, can one find two pseudo-Anosovs whose product is finite order and reducible? The new results completed the proof that every possible product relation can occur in the mapping class group.

[19] * Two-generator Discrete Subgroups of $PSL(2,\mathbb{R})$, *Memoirs of the AMS # 117*, **561** (1995), 200 pages.

In [17] Maskit and I had previously dealt with the intertwining cases of two generator Fuchsian groups (i.e., we did not include an algorithmic and geometric treatment for the case of pairs of generators with intersecting axes). In this paper the geometry of intersecting axes case is considered and a geometric algorithm for that case is developed.

The paper also gives some algorithms for orders of elliptic elements whose entries lie in finite extensions of the rationals and initiates a discussion of algorithms, what an algorithm is and why an algorithm is needed for the discreteness problem. The discussion of algorithms is fully developed in [20].

[20] * Algorithms, Complexity and Discreteness Criteria in $PSL(2,\mathbb{C})$ *Journal D'Analyse Mathematique* **73** (1997), 91-114.

This paper contains treatments of the discreteness algorithm from [17, 19]: as a geometric algorithm (where the atomic steps are geometric in nature); a Blum-Shub-Smale machine which allows standard mathematical operations in $\mathbb{C}$ as atomic; and a Turing Machine algorithm, where the atomic steps of the algorithm can be implemented on a computer. For the Turing machine algorithm the entries in the two matrices are assumed to be algebraic numbers. The computational complexity of each form of the algorithm is computed. The complexity analysis shows that discreteness problem requires an algorithmic solution.

One important step in bounding the complexity of the algorithm is bounding the length of the words the algorithm must consider. This analysis played a role in my work with Keen [25] on intersection numbers of curves on a three holed sphere.

[21] A Discreteness Condition for Subgroups of $PSL(2,\mathbb{C})$, in *Proceedings of the Bers Colloquium*, AMS Contemp. Math. Series **211** (1997), 261-267.

This paper introduces a new class of two-generator subgroups of $PSL(2,\mathbb{C})$ groups, *non-separating disjoint circle groups* (nsdc groups, for short). An nsdc group is defined by the existence of certain triples of circles on the complex sphere, no one of which separates the other two, and which the generators of a specific degree two extension of the group fix. In particular, it is shown that if a group has the nsdc property, then it is a Kleinian group, that is, a discrete group. Further it is shown that an nsdc group is free and is a classical Schottky group (see also [28]).

My current student Karan Puri is generalizing nsdc groups to higher dimensional hyperbolic spaces. The generalization requires reformulations of some standard results of three-dimensional hyperbolic geometry and extensions of Wilker's approach. In particular, Puri gives factorization theorems in higher dimensions that are nonstandard and somewhat unexpected. Basmajian and Maskit are also obtaining factorization theorems, but under somewhat different assumptions.

[22] Complexity of a Turing Machine Discreteness Algorithm, in *The Tradition of Ahlfors and Bers, Proceedings of the AB98 Colloquium* AMS CONM **256** (2000), 165-171.

In this paper, an improved bound for the computational complexity of the Turing Maching $PSL(2,\mathbb{R})$ algorithm of [19, 20] is obtained. Subsequently my graduate student, Y. C. Jiang, used this to produce a polynomial bound for the complexity (Ann. Acad. Sci. Fenn. (2001)).

[23] Kleinian Groups with Real Parameters, (with F.W. Gehring and G.M. Martin), *Communications in Contemporary Math* (2001), 1-23.

This paper is an exploration of moduli for Kleinian groups. We find all Kleinian groups with real parameters, that is, all real points in the analytic space of two-generator Kleinian groups with one generator an elliptic of order two. Geometrically this is a slice of the space of all two-generator discrete Kleinian groups analogous to the Riley slice, but of a very different nature. The types of such Kleinian groups fall into three categories. The analysis of one of the three catagories is an application of my discreteness algorithm for intersecting axes [19].

[24] Alternate Discreteness Tests in *Birman Conf. Proceedings*, AMS/IP Studies in Advanced Mathematics **24** (2001), 41-47.

This paper derives some quick and simple discreteness tests and a complete alternate computational form of the discreteness algorithm that uses mutipliers and cross-ratios. For example, assume $G = \langle g, h \rangle$ is generated by a pair of hyperbolic elements with multipiliers $\sqrt{K} > 1$ and $\sqrt{R} > 1$ and let C $(0 < C < 1)$ be the be normalized cross ratio of the fixed points of the two transformations. If $KC < 1$ and $RC < 1$, then G is not discrete if $C < \frac{1}{2}$.

Some of the computations used here are the precursor of the Non-Euclidean-Euclidean Algroithm and the alternate computational complexity analysis (see section 5 on Current Work for more details.)

[25] * Word Sequence and Intersection Numbers (with Linda Keen) in *Proceedings of Iboamerican Conference, Guanajato, 2000* AMS CONM **311** (2002), 331-349.

There are various families of words used in computational and iterative discreteness problems, such as the *killer words* of Gabai-Myerhoff-Thurston, the *good words* of Gehring-Martin, Farey words used by Keen-Series and Minsky, and the algorithmic words of Gilman, Maskit and Jiang (see [30] for a discussion of all of these families). This paper establishes the connection between the Farey words and the algorithmic words, using the F-sequence of a word. The F-sequence is a type of continued fraction expansion. The connection between the F-sequence of a given word and self-intersections of the curve on the surface corresponding to that word is made explicit. The number of self-intersections of the curves is connected to the computational complexity of the algorithm found in [20]

[26] The Geometry of Two-Generator Groups: Hyperelliptic Handlebodies (with Linda Keen), *Geometriae Dedicata* **110** (2005), 159-190.

A hyperelliptic Riemann surface is a surface of genus g that has a conformal involution with precisely $2g + 2$ fixed points, its Weierstrass points. This paper generalizes the notion of a hyperelliptic Riemann surface of genus two to a hyperelliptic three-manifold. Such a three manifold is genus two solid handlebody whose boundary is a hyperelliptic Riemann surface. We investigate the geometry of the solid handlebody and the convex core, which is also a hyperelliptic Riemann surface. The handlebody has a unique order-two isometry fixing six unique geodesic line segments, the *Weierstrass lines* of the handlebody. The manifold is foliated by surfaces equidistant from the convex core, each fixed by the isometry of order two. The restriction of this involution to the equidistant surface fixes six *generalized Weierstrass points* on the surface.

[27] Boundaries for Two-Parabolic Schottky Groups, in *Spaces of Kleinian Groups* London Math Soc Lecture Notes **329** (2005), 283-299.

This paper surveys various results about the moduli and parameters of two-parabolic generator Kleinian groups including results of Lyndon-Ullman, Wright, Keen-Series, and Gilman-Waterman [29] on parameter spaces and the Riley slice; results of Marden and Jørgensen-Maskit-Marden on representation spaces; and results of Hidalgo-Maskit on noded Schottky space. The paper contains some new results that arise from the organization of the disparate results into a unified setting.

[28] Planar families of Discrete Groups (with Linda Keen) in *The geometry of Riemann Surfaces and Abelian Varieties*, AMS CONM **397** (2006), 79-88.

We study certain families of discrete groups built from a given group with the non-separating disjoint circle property, the nsdc property. A two-generator nsdc group determines a unique planar family of such groups. We find real moduli for these families. Our constructions are geometric and do not use the techniques of quasiconformal mappings or coverings.

[29] *Classical Two-Parabolic T-Schottky groups (with Peter Waterman), *J. D'Analyse Mathematique*, **XCVIII** (2006), 1-42.

Recall the definition of a *classical Schottky group* G. Let $C_i, C_i'(i = 1, ..., n)$ be $2n$ oriented Euclidean circles (or straight lines) that are disjoint that bound a region contained in the intersection of their exteriors. Let g_i be a Möbius transformation that maps the exterior of C_i to the interior of C_i'. Then the group generated by the g_i is a *classical Schottky* group of rank n. A group G of rank n is a classical Schottky group if it has such a set of generators and circles. If we allow arbitary tangencies between the Schottky circles, either at parabolic fixed points or non-parabolic points, between circles that are paired or not paired, we have a classical T-Schottky group. The T stands for "all tangencies allowed".

We find algebraic equations that describe the boundary of the parameter space for the classical T-Schottky space of a Kleinian group generated by two parabolic elements. The groups we consider correspond to doubly cusped manifolds with at least two double cusps or equivalently a noded Riemann surface of genus two with at least two distinct nodes. We also give a description of the boundary of the space of Kleinian groups that are non-separating disjoint circle groups. Our results yield new trace inequalities that imply discreteness, necessary and sufficient trace conditions for a two parabolic generator to be classical T-Schottky, and a sufficient trace condition that parameters of a T-Schottky group must satisfy. Additionally, our results allow us to explicitly construct the groups in a one complex parameter family of non-classical T-Schottky groups and to identify all of the non-classical groups in the so-called Riley slice of Schottky space. The existence of non-classical Schottky groups was first shown by Marden. The only known explicit construction of such a group is due to Yamamoto.

[30] * Informative Words and Discreteness, in *The Proc. of the Rosenbergerfest* AMS CONM **421** (2007), 147-155.

As mentioned before, there are various word families used in computational and iterative discreteness problems, such as the *killer words* of Gabai-Myerhoff-N.Thurston, the *good words* of Gehring Martin, Farey words used by Keen-Series and Minsky, and the algorithmic words of Gilman and Jiang.

In [25] Keen and I established the connection between the Farey words and the algorithmic words, using the F-sequence of a word. Gaven Martin and Tim Marshall have found that most of the killer words are good words and have a program for finding a full set of killer words all of which are good words.

This paper contains a survey of the families and the above connections and some new result. There is a mutliplication defined on certain pairs of Farey words, words that are so-called *Farey neighbors*. There is a semi-group operation defined on *good* words. In this paper a connection between the two is found. The connection and its ramifications are to be developed more fully.

In the meantime, the list of the families of words and their connections continues to grow with the addition of primitives and palindromes (see [34, 36]). The goal is to find a unified algebraic treatment for all of the informative words.

[31] Canonical Symplectic Representations for Prime Order Conjugacy Classes in the Mapping Class Group, *Journal of Algebra*, **318**, (2007) 430-455.

In this paper we find a unique normal form for the symplectic matrix representation of the conjugacy class of a prime order element of the mapping-class group. We fine tune the Schreier-Reidemeister rewriting process to extend the notion of adapted homology bases and we find an explicit *adapted homotopy basis* and a corresponding *adapted presentation* for the appropriate fundamental group.

We give a necessary and sufficient condition for a prime order symplectic matrix to be the image of a prime order element in the mapping-class group. We construct an algorithm to find the unique normal form given the conjugacy invariants of the mapping-class.

There is a trade off. Let h be any representative of a given mapping class, let $\mathcal{A}$ be an adapted homologyy basis with intersection matrix $I_{\mathcal{A}}$ and let $M_{\mathcal{A}}$ be the matrix of the action of h on $\mathcal{A}$. Let $\mathcal{C}$ be a canonical homology basis with the standard intersection matrix $I_{\mathcal{C}}$ and let $M_{\mathcal{C}}$ the matrix of the action of h on $\mathcal{C}$. The mapping-class can be described in terms of the pair $(I_{\mathcal{C}}, M_{\mathcal{C}})$ or the pair $(I_{\mathcal{A}}, M_{\mathcal{A}})$. In one case the first matrix is in a simple form with the second matrix of the pair being much more complicated and in the other case it is the second matrix that is in a simple form with the first matrix of the pair more complicated. All four matrices are described and we construct an algorithm to go back and forth from the one pair to the other.

[32] * The Structure of Two Parabolic Space: Parabolic Dust and Iteration, *Geometriae Dedicata* **131** (2008), 27-48.

A non-elementary Möbius group generated by two-parabolics is determined up to conjugation by one complex parameter and the parameter space, the parameter space for representations of groups generated by two parabolics into $PSL(2,\mathbb{C})$ modulo conjugacy, has been extensively studied. In this paper, we use the results of [29] to obtain an additional structure for the parameter space, which we term the *two-parabolic space*. This structure allows us to identify groups that contain additional conjugacy classes of primitive parabolics, which following Series-Mumford-Wright we call *parabolic dust groups*, non-free groups off the real axis, and groups that are both parabolic dust and non-free; some of these contain $\mathbb{Z} \times \mathbb{Z}$ subgroups. The structure theorem also attaches additional geometric structure to discrete and non-discrete groups lying in given regions of the parameter space including a new explicit geometric construction of some non-classical T-Schottky groups.

My student Andrew Silverio is working on the general method of moving around the diagram of subgroups related to the representation space suggested by the vertical and lateral moves discussed in the paper.

[33] Prime Order Automorphisms of Riemann Surfaces, in *Proceedings of the International Workshop on Teichmüller theory and Moduli Problems*, HRI Lecture Notes Series, Ramanujan Mathematical Society, India (2009) 21 pages, in press.

Recently there has been renewed interest in the mapping-class group of a compact surface of genus $g \geq 2$ and also in its finite order elements. A finite order element of the mapping-class group will have a representative that is a conformal automorphism on some Riemann surface of genus g.

Here we extend the proof of the existence of an *adapted homology basis* for a prime order automorphism to apply to fixed point free automorphisms and find some new consequences of the existence of an adapted basis. We construct an explicit example of such a basis and compute its intersection matrix. We connect results to the Thurston terminology.

[34] * Discreteness Criteria and the Hyperbolic Geometry of Palindromes (with Linda Keen), Journal of Conformal Geometry and Dynamics **13** (2009), 76-90.

We consider non-elementary representations of two generator free groups in $PSL(2,\mathbb{C})$, not necessarily discrete or free, $G = \langle A, B \rangle$. A word in A and B, $W(A,B)$, is a palindrome if it reads the same forwards and backwards. A word in a free group is *primitive* if it is part of a minimal generating set. Conjugacy classes of primitive elements of the free group on two generators can be identified with the positive rational numbers.

We study the geometry of palindromes. That is, we study palindromes and their action in $\mathbb{H}^3$ whether or not the two generator group, G, being represented is discrete. We show that there is a *core geodesic* $\mathbf{L}$ in the convex hull of the limit set of G and use it to prove three results: the first is that there are well defined maps from the non-negative rationals and from primitive elements or conjugacy classes of elements to $\mathbf{L}$; the second is that G is geometrically finite if and only if the axis of every non-parabolic palindromic word in G intersects $\mathbf{L}$ in a compact interval; the third is a description of the relation of the pleating locus of the convex hull boundary to the core geodesic and to palindromic elements. This explains in part why Jørgensen and Sandler were so interested in palindromes.

[35] Cutting Sequences and Palindromes, (with Linda L. Keen), in *Proceedings of a conference in honor of W. Harvey* London Math. Soc. 24 pages, in press.

In this paper we describe a geometric technique that ties together and gives new proofs of several more or less well-known theorems about primitive and palindromic words in a rank two free group.

[36] * Enumerating Palindromes in Rank Two Free Groups (with Linda Keen), submitted to Journal of Algebra, (undergoing suggested revisions) intially 11 pages; current version $\sim 15 - 19$ pages.

Let $F = \langle a, b \rangle$ be a rank two free group. We derive a new iteration scheme for primtives in F. The scheme gives either the unique palindrome in the conjugacy class or expresses the word in the conjugacy class as a unique product of two unique palindromes and also expresses all primitive pairs in such a form. We denote these primitive words by $E_{p/q}$ where p/q is rational number reduced to lowest terms. We prove that $E_{p/q}$ is a palindrome if pq is even and the unique product of two unique palindromes if pq is odd. We prove that the pairs $(E_{p/q}, E_{r/s})$ generate the group when $|ps - rq| = 1$. This latter statement improves the previously known result that held only for pq and rs both even. The derivation of the enumeration scheme itself automatically gives a new proof of the known result about primitives being conjugate to either a palindrome or a product of palindromes.

7. GRADUATE TEXTS

[CVI] *Complex Analysis in the spirit of Lipman Bers*, Graduate text (with I. Kra and R. Rodriguez), Springer-Verlag, GTM **245** (2007).

This is a graduate text for a first semester course in Complex Variables, our Math 623. The organization of the material is unique.

The first chapters of the text contain material that is standard to any complex variables course, that is, material on analytic functions that would be taught in any undergraduate complex variables course or during the first few weeks of any graduate complex variables course. However, the material is organized in a non-standard way where the emphasis is on the study of the many equivalent ways of understanding the concept of an analytic function. The many ways of formulating the concept are summarized in what is termed *the Fundamental Theorem* of functions of a complex variable. The organization of the analyticity conditions into a single unifying theorem demonstrates the emphasis on clarity and elegance is the hallmark of Lipman Bers' mathematical style. Here it provides a conceptual framework for results that are highly technical and often dry. The second part of the text deals with more advanced topics. The text prepares students for a standard qualifying exam in a graduate PhD program.

[CVII] *Topics in Complex Analysis and Hyperbolic Geometry*, Graduate text, companion volume to [CVI], (with I. Kra and R. Rodriguez), in preparation (targeted completion date in the distant future), under contract to be published by Springer-Verlag.

Complex Variables and Riemann Surfaces can be considered to be the cradle of much modern mathematics. This includes, but is not limited to, such fields as differential geometry, combinatorial and computational group theory, topology, dynamical systems, number theory, algebraic geometry, hyperbolic geometry, geometric function theory, several complex variables and complex geometry.

Volume II is intended to be used after Volume I in a first course to prepare graduate students for research in one of several possible directions including hyperbolic geometry, Kleinian groups, Teichmuller theory, several complex variables and parts of number theory or algebraic geometry. It could also be used as a final course for students planning to do research in another area of mathematics but who want to be conversant with the basic problems in these areas.

8. Other Publications

These abstracts do not contain references to other publications such as conference proceedings that I edited, articles on experimental mathematics courses and education reform, lecture notes, popular articles in the *Notices*. These are all listed on my complete CV and copies of many of them are in the file containing pdf copies of my publications.

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