

# Kleinian Groups with Real Parameters

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Dedicated to Albert Marden on the occasion of his 65<sup>th</sup> birthday

## Abstract

We find all real points of the analytic space of two generator Möbius groups with one generator elliptic of order two. Geometrically this is a certain slice through the space of two generator discrete groups, analogous to the Riley slice, though of a very different nature. We obtain applications concerning the general structure of the space of all two generator Kleinian groups and various universal constraints for Fuchsian groups.

## 1 Introduction

A two-generator Möbius group  $\langle f, g \rangle$  is determined up to conjugacy by three complex numbers, its parameters. There are many choices of parameters. A particularly useful choice are the trace parameters,  $(\gamma(f, g), \beta(f), \beta(g))$  for the commutator  $[f, g]$  and the generators  $f$  and  $g$  defined in (2.1) below. These parameters first appeared implicitly in work of Shimizu and Leutbecher and later, more explicitly, in work of Jørgensen [14]. This work was focused on inequalities for discrete groups in an effort to quantify various statements about them.

Here we take a slightly different viewpoint and consider the space of parameters for discrete groups (which we view as the space of discrete groups up to conjugacy) and discuss its structure. In particular we investigate the

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real points of this space and completely describe a certain two-dimensional slice through this space.

There is interesting related work. The one complex dimensional set of all parameters of discrete groups of the form  $(\gamma, 0, 0)$  representing four times punctured spheres is known as the Riley slice through Schottky space. We have come to know a great deal about this slice due to the work of several authors, among them Riley [19] and Keen and Series [16]. A recent deep result of Yair Minsky [21] implies the boundary of the unbounded component of this space for once punctured torus groups is a Jordan curve and Minsky's techniques generalize to the Riley slice.

There are also important connections with geometry and topology. For example, the Riley slice consists of all discrete groups generated by two parabolic elements and therefore contains such things as all hyperbolic 2-bridge knot and link complements.

Finally computer investigations show that the Riley slice has a very complicated fractal boundary. On the other hand our slice has boundary components which are lines and branches of hyperbolas. It is curious that these two slices through this 3-complex dimensional space are so different.

If  $G = \langle f, g \rangle$  is a discrete group with parameters  $(\gamma, \beta, \beta')$  where  $\gamma \neq \beta, 0$ , then there is a discrete group  $H = \langle f, h \rangle$  with parameters  $(\gamma, \beta, -4)$ , i.e., where  $h$  is of order 2. See Theorem 1.2. This reduces the problem of finding the parameters for a discrete group to the case where one parameter is fixed and real.

In this paper we analyze the space of discrete groups with parameters  $(\gamma, \beta, -4)$  where  $\gamma$  and  $\beta$  are real. This is the two dimensional slice we shall describe. It gives a necessary condition on the parameters  $(\gamma, \beta, \beta')$  for any discrete group when  $\gamma$  and  $\beta$  are real. Our main result is that the real  $\gamma\beta$ -plane is composed of three types of regions:

1. A region in which all  $(\gamma, \beta)$  correspond to discrete free groups; this region has six components three of which are bounded by branches of hyperbolas.
2. A region in which  $(\gamma, \beta)$  correspond to discrete groups in  $SL(2, \mathbb{R})$  and certain symmetric images; this region consists of three infinite strips.
3. A region whose three components are regions between the infinite strips and the hyperbolas.

There is a symmetry group of order six acting on the  $\gamma\beta$ -plane which permutes the three hyperbolas, the three infinite strips and the three bounded regions.

The paper is organized as follows. Section 2 contains definitions and preliminary results. The free part is analyzed in Section 3, the non-free Fuchsian part in Section 4 and the regions between the strips and the hyperbolas in Section 5. In Section 6 we gather together the results of the preceding three sections to obtain the main result, Theorem 6.1 which gives necessary conditions for a discrete group to have two real parameters. This yields an alternate derivation for the nondiscrete polygon, a polygon which gives nondiscrete groups with finitely many exceptions and was derived using iteration theory in [4]. This extends the region described by Jørgensen's inequality.

## 2 Preliminaries

Let  $\mathbf{M}$  denote the group of all Möbius transformations of the extended complex plane  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . We associate with a Möbius transformation

$$f = \frac{az + b}{cz + d} \in \mathbf{M}, \quad ad - bc = 1,$$

one of the two matrices that induce the action of  $f$  on the extended plane

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

and set  $\mathrm{tr}(f) = \mathrm{tr}(A)$  where  $\mathrm{tr}(A)$  denotes the trace of  $A$ . For a two generator group, once a matrix corresponding to each generator is chosen, the matrices associated to all other elements of the group are determined. Next for each  $f$  and  $g$  in  $\mathbf{M}$  we let  $[f, g]$  denote the commutator  $fgf^{-1}g^{-1}$ . We call the three complex numbers

$$\beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4, \quad \gamma(f, g) = \mathrm{tr}([f, g]) - 2 \quad (2.1)$$

the parameters of the two generator group  $\langle f, g \rangle$  and write

$$\mathrm{par}(\langle f, g \rangle) = (\gamma(f, g), \beta(f), \beta(g)). \quad (2.2)$$

These parameters are independent of the choice of representations for  $f$  and  $g$  and they determine  $\langle f, g \rangle$  up to conjugacy in the following sense. See, for example, Lemma 2.2 and Remark 2.6 in [5].

**Lemma 2.3** *If  $\tilde{f}, \tilde{g}, f, g \in \mathbf{M}$  and if*

$$\gamma(\tilde{f}, \tilde{g}) = \gamma(f, g) \neq 0, \quad \beta(\tilde{f}) = \beta(f), \quad \beta(\tilde{g}) = \beta(g),$$

*then there exists  $\phi \in \mathbf{M}$  which conjugates  $\tilde{f}$  to  $f$  and  $\tilde{g}$  to  $g$  or  $g^{-1}$ .*

We will often make use of the following identity.

**Lemma 2.4** *If  $f, g \in \mathbf{M}$  and if  $g$  is of order 2, then*

$$\beta(fg) = \gamma(f, g) - \beta(f) - 4.$$

**Proof.** By the Fricke identity, see for example §1.3 in [3],

$$\begin{aligned} \gamma(f, g) &= \text{tr}([f, g]) - 2 \\ &= \text{tr}^2(f) + \text{tr}^2(g) + \text{tr}^2(fg) - \text{tr}(f)\text{tr}(g)\text{tr}(fg) - 4 \\ &= \beta(f) + \beta(fg) + 4 \end{aligned}$$

since  $\text{tr}(g) = 0$ .  $\square$

A group of Möbius transformations  $G$  is *discrete* if the identity is isolated in  $G$ . It is natural to direct one's attention to the discrete *nonelementary* groups, those that do not contain an abelian subgroup of finite index. A discrete nonelementary group is called a *Kleinian group*.

We recall Lemma 2.29 from [8].

**Lemma 2.5** *If  $\langle f, h \rangle$  is a discrete subgroup of  $\mathbf{M}$  with  $\gamma(f, h) \neq 0$  and  $\gamma(f, h) \neq \beta(f)$ , then there exist elliptics  $h_1$  and  $h_2$  of order two such that  $\langle f, h_1 \rangle$  and  $\langle f, h_2 \rangle$  are discrete with*

$$\gamma(f, h_1) = \gamma(f, h) \quad \text{and} \quad \gamma(f, h_2) = \beta(f) - \gamma(f, h).$$

**Theorem 2.6** *If*

$$\text{par}(\langle f, g \rangle) = (\gamma, \beta, \beta'),$$

*then there exist  $h_1, h_2 \in \mathbf{M}$  of order 2 such that*

$$\text{par}(\langle f, h_1 \rangle) = (\gamma, \beta, -4), \quad \text{par}(\langle f, h_2 \rangle) = (\beta - \gamma, \beta, -4).$$

*Moreover*

1.  $\langle f, h_1 \rangle$  is discrete if and only if  $\langle f, h_2 \rangle$  is.
2.  $\langle f, h_1 \rangle$  is nonelementary if and only if  $\langle f, h_2 \rangle$  is.

If in addition  $\gamma \neq 0, \beta$ , then

3.  $\langle f, h_1 \rangle$  and  $\langle f, h_2 \rangle$  are discrete whenever  $\langle f, g \rangle$  is,
4.  $\langle f, h_1 \rangle$  and  $\langle f, h_2 \rangle$  are free on their generators whenever  $\langle f, g \rangle$  is.

**Proof.** The existence of  $h_1$  and  $h_2$  follows from Lemma 2.5 when  $\gamma \neq 0, \beta$ . If  $\beta \neq 0$  and  $\gamma = 0$  or  $\gamma = \beta$ , choose  $h_1$  and  $h_2$  so that one shares and the other interchanges the fixed points of  $f$ . If  $\gamma = \beta = 0$ , choose  $h_1$  and  $h_2$  so that both have a fixed point in common with  $f$ . Conclusion 1 then holds in each case. Conclusion 2 follows from enumerating the cases when  $(\gamma, \beta, -4)$  are the parameters of a nonelementary group.

Next

$$\text{par}(\langle f, h_j f h_j^{-1} \rangle) = (\gamma(\gamma - \beta), \beta, \beta) = \text{par}(\langle f, g f g^{-1} \rangle)$$

for  $j = 1, 2$ , by (2.2) of [8]. Hence if  $\gamma \neq 0, \beta$ , then  $\gamma(\gamma - \beta) \neq 0$  and Lemma 2.3 implies that  $\langle f, h_j \rangle$  has a subgroup, of at most index two, with generators which are conjugate by  $\phi_j \in \mathbf{M}$  to those of the subgroup of  $\langle f, g f g^{-1} \rangle$  of  $\langle f, g \rangle$ . Conclusions 3 and 4 then follow from this and the following easily verified fact.  $\square$

**Lemma 2.7** *If  $\langle f, g \rangle$  is free on its generators, then so is the subgroup  $\langle f, g f g^{-1} \rangle$ . The converse is also true when  $g$  is of order 2.*

The elementary discrete groups are classified, consisting of the spherical and euclidean triangle groups together with their subgroups. It is not difficult to find all the parameters for these groups and here we shall concentrate on the nonelementary discrete i.e., Kleinian, case. We therefore define  $\mathcal{D}_{2\text{-gen}}$  to be that subset of  $\mathbb{C}^3$  consisting of all parameters of two generator nonelementary discrete groups,

$$\mathcal{D}_{2\text{-gen}} = \{\text{par}(G) : G = \langle f, g \rangle \text{ is a discrete nonelementary group}\}.$$

When viewed in this way  $\mathcal{D}_{2\text{-gen}}$  is closed and components of  $\mathcal{D}_{2\text{-gen}}$  consist entirely of isomorphic groups. This follows from work of Jørgensen

[14]; see also [17] for further discussion. The set  $\mathcal{D}_{2-gen}$  is highly complicated with fractal like boundary. There are isolated points of  $\mathcal{D}_{2-gen}$  - triangle groups and Mostow rigid groups such as various hyperbolic two bridge knot complement groups and certain associated groups arising from Dehn fillings. There are also components of  $\mathcal{D}_{2-gen}$  with nonempty interior - Schottky and quasifuchsian groups.

Theorem 2.6 shows that there is a natural projection preserving discreteness given by

$$(\gamma, \beta, \beta') \rightarrow (\gamma, \beta, -4)$$

and hence it is natural to study the subset of  $\mathbb{C}^2$  defined by

$$\mathcal{D} = \{(\gamma, \beta) : (\gamma, \beta, -4) \in \mathcal{D}_{2-gen}\} \quad (2.8)$$

consisting of the parameters of all discrete nonelementary two generator groups with one generator of order two.

**Definition 2.9** *Let  $\mathcal{X}$  denote the union of the four sets of points:*

$$\mathcal{X}_1 = \{(x, x) : x = -1, -2, -3\},$$

$$\mathcal{X}_2 = \{(-1, x) : x = -2, -3, -(5 \pm \sqrt{5})/2\},$$

$$\mathcal{X}_3 = \{(1 + x, x) : x = -(5 \pm \sqrt{5})/2\},$$

$$\mathcal{X}_4 = \{(x, -3) : x = -1, -2, -(3 \pm \sqrt{5})/2\}.$$

The set  $\mathcal{X}$  does not lie in  $\mathcal{D}$  since the corresponding groups are elementary. However this set does lie in the image of the map  $\mathcal{D}_{2-gen} \rightarrow \mathcal{D}$  defined by  $(\gamma, \beta, \beta') \mapsto (\gamma, \beta)$ . In fact away from the preimage of the set  $\mathcal{X}$  this map also preserves the property of being nonelementary.

In [6] we describe a family  $\mathcal{P}$  of polynomial mappings with integer coefficients in two variables, closed under composition in the first variable, which act on  $\mathcal{D}$  by the rule

$$(\gamma, \beta) \mapsto (p(\gamma, \beta), \beta) \quad (2.10)$$

for  $p \in \mathcal{P}$ , again away from the preimage of the set  $\mathcal{X}$ . For example, two of the smallest degree polynomials in the family  $\mathcal{P}$  are

$$\gamma(\gamma - \beta), \quad \gamma(\gamma - \beta + 1)^2. \quad (2.11)$$

It is clear this polynomial family preserves the real points of  $\mathcal{D}$ . There are other interesting polynomial actions on  $\mathcal{D}$ . For instance the shifted Chebyshev polynomials  $P_n(z) = 2(T_n(1 + z/2) - 1)$  act multiplicatively via the rule

$$(\gamma, \beta) \mapsto \frac{P_n(\beta)}{\beta}(\gamma, \beta). \quad (2.12)$$

See, for example, (4.12). Here  $T_n$  is the usual Chebyshev polynomial of one real or complex variable, [2]. The way this action arises is described in [7]. Since the Chebyshev polynomials have real coefficients, this action too will preserve the real points of  $\mathcal{D}$ .

Information about  $\mathcal{D}$  implies geometric information about *all* Fuchsian and Kleinian groups, that is all hyperbolic 2 and 3 manifolds and orbifolds. For instance Jørgensen's inequality [14] is equivalent to the statement

$$\{|\gamma| + |\beta| < 1\} \cap \mathcal{D} = \emptyset.$$

Indeed other inequalities, implying such things as collaring theorems, have been obtained from finding open subsets of  $\mathbb{C}^2$  which do not lie in

$\mathcal{D}$  via a disk covering procedure. See, for example, [6, 8].

The following result exhibits the symmetries of  $\mathcal{D}$ . Our subsequent description of the real points of  $\mathcal{D}$  show this to be the maximal real linear symmetry group.

**Theorem 2.13** *The linear mappings*

$$\begin{aligned} \phi_1(\gamma, \beta) &= (\beta - \gamma, \beta), & \phi_2(\gamma, \beta) &= (\gamma, \gamma - \beta - 4), \\ \phi_3(\gamma, \beta) &= (-\beta - 4, \gamma - \beta - 4), & \phi_4(\gamma, \beta) &= (\beta - \gamma, -\gamma - 4), \\ \phi_5(\gamma, \beta) &= (-\beta - 4, -\gamma - 4) \end{aligned}$$

*together with the identity form a group isomorphic to the symmetric group  $S_3$  with the following property: If  $(\gamma, \beta) \in \mathcal{D}$ , then*

$$\phi_i(\gamma, \beta) \in \mathcal{D}$$

*for  $i = 1, 2, \dots, 5$ .*

**Proof.** Suppose that  $(\gamma, \beta) \in \mathcal{D}$  and that  $G = \langle f, g \rangle$  is a discrete group with

$$\text{par}(\langle f, g \rangle) = (\gamma, \beta, -4).$$

Then Theorem 2.6 with  $g = h_1$ , implies there exists  $h = h_2$  of order 2 such that  $\langle f, h \rangle$  is discrete and nonelementary with

$$\text{par}(\langle f, h \rangle) = (\beta - \gamma, \beta, -4).$$

Thus  $\phi_1(\gamma, \beta) \in \mathcal{D}$ .

Next

$$\beta(fg) = \gamma - \beta - 4$$

by Lemma 2.4,

$$\text{par}(\langle fg, g \rangle) = (\gamma, \gamma - \beta - 4, -4)$$

and  $\phi_2(\gamma, \beta) \in \mathcal{D}$ .

Finally we observe that

$$\phi_3 = \phi_1 \circ \phi_2, \quad \phi_4 = \phi_2 \circ \phi_1, \quad \phi_5 = \phi_1 \circ \phi_2 \circ \phi_1,$$

and the proof is complete by what was proved above.  $\square$

The main purpose of this paper is to completely describe the real points of  $\mathcal{D}$ . Hence we set

$$\mathcal{R} = \mathcal{D} \cap \mathbb{R}^2. \tag{2.14}$$

The set  $\mathcal{R}$  is closed and it is easy to see that the symmetries of Theorem 2.13 preserve the set  $\mathcal{R}$ . Since Fuchsian groups are nonelementary discrete subgroups of  $PSL(2, \mathbb{R})$ , every two generator Fuchsian group with one generator of order two corresponds to a point in  $\mathcal{R}$ . However the set of such points constitutes only a small part of  $\mathcal{R}$ .

### 3 The Free Part of $\mathcal{R}$

The first and easiest part of  $\mathcal{R}$  to describe corresponds to the groups  $\langle f, g \rangle$ , with  $g$  of order 2, which are free on their generators.

**Theorem 3.1** *The parameters  $(\gamma, \beta, -4)$  are those of a discrete group which is free on its generators whenever  $(\gamma, \beta)$  lies in one of the following regions:*

Region 1A:  $\beta \geq 0$  and  $\gamma \leq -4$ .

Region 1B:  $4 \leq \beta + 4 \leq \gamma$ .

Region 1C:  $\beta + 4 \leq \gamma \leq -4$ .

Region 1D:  $\beta + 4 \leq -4/\gamma < 0$ .

Region 1E:  $\beta + 4 \leq 4/(\gamma - \beta) < 0$ .

Region 1F:  $\beta - \gamma \geq 4/\gamma > 0$ .

These regions are illustrated in Figure 1.

**Proof.** We begin by establishing Theorem 3.1 for Region 1A. First consider an annulus with inner radius  $\frac{1}{\lambda}$  and outer radius  $\lambda$  and a circle in the closure of the annulus with center at  $a$  and radius  $r$ . We want to construct a transformation  $f$  whose fundamental domain is this annulus and a transformation  $g$  given by reflection in the circle. If we do this we know by the Klein combination theorem ([18], Theorem VII.A.13.) that  $f$  and  $g$  generate a discrete group that is free on its generators. We begin with  $\beta$  and  $\gamma$  in the region and show how to determine  $\lambda$ ,  $a$  and  $r$  so that  $\beta = \beta(f)$  and  $\gamma = \gamma(f, g)$ .

Suppose that  $\beta > 0$  and let  $f(z) = \lambda^2 z$  with  $\beta(f) = \beta$ . Then  $\beta = (\lambda - 1/\lambda)^2$  and  $D = \{z : 1/\lambda < |z| < \lambda\}$  is a fundamental domain for the action of  $f$  on  $\mathbb{C} \setminus \{0, \infty\}$ . Next let  $g$

$$g(z) = \frac{r^2}{z - a} + a, \quad a = \frac{\lambda + 1/\lambda}{2}.$$

Then

$$\gamma = \gamma(f, g) = \beta \left(1 - \frac{a^2}{r^2}\right) = (\lambda - 1/\lambda)^2 \left(1 - \frac{(\lambda + 1/\lambda)^2}{4r^2}\right),$$

and by hypothesis

$$(\lambda + 1/\lambda)^2 \left(1 - \frac{(\lambda - 1/\lambda)^2}{4r^2}\right) = \gamma + 4 \leq 0$$

whence

$$r \leq \frac{\lambda - 1/\lambda}{2}.$$

Hence the isometric circle of  $g$  lies in  $\overline{D}$  and  $\langle f, g \rangle$  is free on its generators by the Klein combination theorem.

Suppose next that  $\beta = 0$  and let  $f(z) = z + 2$  and  $g(z) = r^2/z$ . Then  $D = \{z : -1 < \operatorname{Re}(z) < 1\}$  is fundamental domain for  $f$  and

$$\gamma = \gamma(f, g) = \frac{-4}{r^2}, \quad \frac{-4}{r^2} + 4 = \gamma + 4 \leq 0.$$

Hence  $r \leq 1$ , the isometric circle of  $g$  again lies in  $\overline{D}$  and  $\langle f, g \rangle$  is free on its generators. This completes the proof of Theorem 3.1 for Region 1A and hence for Regions 1B and 1C, the images of Region 1A under  $\phi_1$  and  $\phi_2$ .

For Region 1D, suppose that  $\beta < -4$  and let  $f(z) = -\lambda^2 z$  with  $\beta(f) = \beta$ . Then  $\beta = -(\lambda + 1/\lambda)^2$  and

$$D = \{z : 1/\lambda^2 < |z| < \lambda^2\} \cap \{z : 0 < \operatorname{Re}(z)\}$$

is a fundamental domain for  $f$ . If we let

$$g(z) = \frac{r^2}{z - a} + a, \quad a = \frac{\lambda^2 + 1/\lambda^2}{2},$$

then

$$\gamma = \gamma(f, g) = \beta \left( \frac{a^2}{r^2} - 1 \right) = (\lambda - 1/\lambda)^2 \left( 1 - \frac{(\lambda^2 + 1/\lambda^2)^2}{4r^2} \right)$$

and by hypothesis

$$(\lambda^2 + 1/\lambda^2)^2 \left( 1 - \frac{(\lambda^2 - 1/\lambda^2)^2}{4r^2} \right) = (\beta + 4)\gamma + 4 \leq 0$$

whence

$$r \leq \frac{\lambda^2 - 1/\lambda^2}{2}.$$

Again  $\overline{D}$  contains the isometric circle of  $g$  and  $\langle f, g \rangle$  is free on its generators. Thus Theorem 2.1 holds for Region 1D and its images, Regions 1E and 1F, under  $\phi_1$  and  $\phi_2$ .  $\square$

## 4 The Infinite Strips of $\mathcal{R}$

In this section we find those points corresponding to discrete groups in the following three infinite strips:

Strip 2A:  $-4 \leq \beta \leq 0$ .

Strip 2B:  $-4 \leq \gamma \leq 0$ .

Strip 2C:  $\beta \leq \gamma \leq \beta + 4$ .

See Figure 1. The main result is a list of the values of  $(\gamma, \beta)$  that correspond to discrete groups in these strips - Theorems 4.17. Each infinite strip consists of a family of parallel lines. An alternative formulation of the main result describes the discrete groups along these lines as the union of a continuous part and a discrete part. On each parallel line, the continuous part consists of two semi-infinite intervals and the discrete part a list of isolated points in the complementary open interval. This formulation is Theorem 4.19.

Our method is to observe first that discrete groups corresponding to points in the right half of strip 2A have an invariant disk or half plane and next use a result of Knapp's [15] to analyze these groups. Information about the groups corresponding to points in the full strip 2A and in strips 2B and 2C then follows from applying the symmetries in Theorem 2.13.

**Lemma 4.1** *If  $\langle f, g \rangle$  is a discrete group with real parameters  $(\gamma, \beta, -4)$  and if*

$$\gamma \neq 0, \quad \gamma \geq \beta \geq -4,$$

*then either  $\langle f, g \rangle$  has an invariant disk or half plane or  $\langle f, g \rangle$  is finite.*

**Proof.** We see that

$$\mathrm{tr}^2(f) = \beta + 4 \geq 0, \quad \mathrm{tr}^2(g) = 0$$

and that

$$\mathrm{tr}^2(fg^{-1}) = \mathrm{tr}^2(fg) = \beta(fg) + 4 = \gamma - \beta \geq 0$$

by Lemma 2.4. Hence  $\mathrm{tr}(f)$ ,  $\mathrm{tr}(g)$ ,  $\mathrm{tr}(fg)$  and  $\mathrm{tr}(fg^{-1})$  are all real and the desired conclusion follows from Lemma 5.23 of [6].  $\square$

We now consider those infinite discrete groups  $\langle f, g \rangle$  with parameters  $(\gamma, \beta, -4)$  for which  $(\gamma, \beta)$  lies in the semi-infinite strip

$$\{-4 \leq \beta \leq 0, \gamma \geq \beta\}. \quad (4.2)$$

Lemma 4.1 implies that all these groups have an invariant disk or half plane and hence can be completely described by the results of Knapp.

We begin with the case where  $f$  is a primitive elliptic of order  $q$ , that is, a Möbius transformation which corresponds to a rotation of angle  $\pm 2\pi/q$ . In this case

$$\beta(f) = -4 \sin^2(\pi/q). \quad (4.3)$$

**Lemma 4.4** *Suppose that  $\langle f, g \rangle$  has parameters  $(\gamma, \beta, -4)$  with  $\gamma \geq \beta$  and that  $f$  is a primitive elliptic of order  $q \geq 3$ . Then  $\langle f, g \rangle$  is a discrete group if and only if one of the following conditions holds:*

1.  $\gamma - \beta = 4 \cos^2(\pi/r)$  for some integer  $r > 2q/(q - 2)$ .
2.  $\gamma - \beta \geq 4$ .
3.  $\gamma - \beta = (\beta + 2)^2$  and  $q$  is odd.

**Proof.** A matrix  $C = (c_{ij}) \in SL(2, \mathbb{R})$  is said to have *extreme negative trace* if

$$\text{tr}(C) = -2 \cos(\pi/n)$$

where  $n$  is an integer;  $C$  is said to be *normalized* if, in addition,  $c_{12} > 0$ . If  $C$  corresponds to a primitive elliptic, then  $C$  or its inverse is normalized in the above sense. Hence by replacing  $f$  and/or  $g$  by their inverses, we may assume that  $f$  and  $g$  are represented by normalized matrices  $A$  and  $B$  in  $SL(2, \mathbb{R})$ .

Suppose that  $f$  and  $g$  have disjoint fixed points. Then Theorem 2.3 of [15] implies that  $\langle f, g \rangle$  is discrete if and only if  $\text{tr}(A)$ ,  $\text{tr}(B)$  and  $\text{tr}(AB)$  satisfy one of seven conditions, four of which are ruled out by the hypotheses that  $\text{ord}(f) \geq 3$  and  $\text{ord}(g) = 2$ . The remaining three conditions as listed in Theorem 2.3 of [15] are

- I.  $|\text{tr}(AB)| < 2$  and  $AB$  has extreme negative trace,

$$\text{II. } |\text{tr}(AB)| \geq 2,$$

$$\text{IV. } \text{tr}(AB) = -2 \cos(2\pi/q) \text{ and } q \text{ is odd.}$$

Proposition 2.2 in [15] states that in Case I, if  $r = \text{ord}(fg)$ , then  $\langle f, g \rangle$  is the  $(2, q, r)$  triangle group so that  $r > 2q/(q-2)$  while the proof of Proposition 2.2 shows that in Case IV  $\text{tr}(AB) \neq \cos(2\pi/n)$ . Since

$$\gamma - \beta = \beta(fg) + 4 = \text{tr}^2(fg) \quad 4 \cos^2(2\pi/q) = (\beta + 2)^2, \quad (4.5)$$

conclusions 1, 2, 3 in Lemma 4.4 follow from Proposition 2.1 in [15] and conditions I, II, IV above.

Conversely conditions 1, 2, 3 in Lemma 4.4 each imply that  $\gamma \neq 0$  and hence that  $f$  and  $g$  have disjoint fixed points. Next these hypotheses together with the proof of Proposition 2.2 of [15], imply that  $\text{tr}(AB)$  satisfies conditions I, II, IV above and hence, by Theorem 2.3 of [15], that  $\langle f, g \rangle$  is discrete.  $\square$

Lemma 4.4 and the symmetry  $\phi_1$  of Theorem 2.13 then yield the following values of  $\gamma$  for the case when  $\gamma < \beta$ .

**Lemma 4.6** *Suppose that  $\langle f, g \rangle$  has parameters  $(\gamma, \beta, -4)$  with  $\gamma < \beta$  and that  $f$  is a primitive elliptic of order  $q \geq 3$ . Then  $\langle f, g \rangle$  is a discrete group if and only if one of the following conditions holds:*

1.  $\gamma = -4 \cos^2(\pi/r)$  for some integer  $r > 2q/(q-2)$ .
2.  $\gamma \leq -4$ .
3.  $\gamma = -(\beta + 2)^2$  and  $q$  is odd.

Thus the sets of possible  $\gamma$  values arising in Lemmas 4.4 and 4.6 consist of a *continuous* part

$$C(\beta) = \{x : x \geq \beta + 4\} \cup \{x : x \leq -4\} \quad (4.7)$$

and *discrete* parts

$$D(\beta) = \{-4 \cos^2(\pi/r), \beta + 4 \cos^2(\pi/r) : r > 2q/(q-2)\} \quad (4.8)$$

if  $q$  is even and

$$D(\beta) = \{-4 \cos^2(\pi/r), \beta + 4 \cos^2(\pi/r) : r > 2q/(q-2)\} \cup \{-(\beta+2)^2, (\beta+1)(\beta+4)\} \quad (4.9)$$

if  $q$  is odd. Thus we can summarize the results of Lemmas 4.4 and 4.6 as follows.

**Theorem 4.10** *If  $\beta = -4 \sin^2(\pi/q)$  where  $q \geq 3$ , then  $(\gamma, \beta) \in \mathcal{R}$  if and only if*

$$\gamma \in \Gamma(\beta) = C(\beta) \cup D(\beta). \quad (4.11)$$

If  $f$  is not a primitive elliptic, then  $\beta(f) = -4 \sin^2(p\pi/q)$  where  $(p, q) = 1$  and there exists an integer  $m \in (1, q)$  so that  $f^m$  is primitive. Then  $\langle f, g \rangle = \langle f^m, g \rangle$ ,

$$\gamma(f, g) = \frac{\beta(f)}{\beta(f^m)} \gamma(f^m, g) = \frac{\sin^2(p\pi/q)}{\sin^2(\pi/q)} \gamma(f^m, g) \quad (4.12)$$

by [7] and  $\gamma(f^m, g)$  is one of the values listed in Theorem 4.10. Hence in this case we define

$$\Gamma(\beta) = \frac{\sin^2(p\pi/q)}{\sin^2(\pi/q)} \Gamma(-4 \sin^2 \pi/q). \quad (4.13)$$

There remains the case  $\beta = 0$  when  $f$  is parabolic. If  $\gamma \geq 0$ , then  $\langle f, g \rangle$  is discrete if and only if

$$\gamma \geq 4 \quad \text{or} \quad \gamma = 4 \cos^2(\pi/r)$$

for some integer  $r$ ,  $r \geq 3$  by Proposition 4.1 of [15]. If  $\gamma \leq 0$ , then the symmetry  $\phi_1$  in Theorem 2.13 shows that  $\langle f, g \rangle$  is discrete if and only if

$$\gamma \leq -4 \quad \text{or} \quad \gamma = -4 \cos^2(\pi/r)$$

where  $r$  is as above. Thus we define

$$\Gamma(0) = \{x : |x| \geq 4\} \cup \{\pm 4 \cos^2(\pi/r) : r \geq 3\} \quad (4.14)$$

The following result is then an immediate consequence of Theorem 4.10.

**Theorem 4.15** *Let  $-4 \leq \beta \leq 0$ . Then  $(\gamma, \beta) \in \mathcal{R}$  if and only if*

$$\beta = -4 \sin^2(p\pi/q), \quad (p, q) = 1 \quad \text{and} \quad \gamma \in \Gamma(\beta). \quad (4.16)$$

Notice here that  $\sin^2(p\pi/q)/\sin^2(\pi/q) \geq 1$ , so the smallest values of  $\gamma$  mostly occur when  $f$  is primitive. This, together with the observation that  $G = \langle f, g \rangle$  is Fuchsian only if  $(\gamma(f, g), \beta(f)) \in \mathcal{R}$  implies the following universal constraints on the commutator parameter in Fuchsian and Kleinian groups with an elliptic generator.

**Theorem 4.17** *Suppose that  $G = \langle f, g \rangle$  is a Kleinian group with  $f$  elliptic of order  $q$  and  $\gamma = \gamma(f, g) \in \mathbb{R}$ . Then  $\gamma \neq 0$ .*

1. *If  $\gamma > 0$ , then either  $\gamma \geq 2$  or  $\gamma$  is one of the values listed below:*

$$\begin{aligned} \gamma &= 4(\cos^2(2\pi/7) - \sin^2(\pi/7)), \\ \gamma &= 4(\cos^2(2\pi/9) - \sin^2(\pi/9)), \\ \gamma &= 4 \cos(2\pi/5), \\ \gamma &= 2 \cos(2\pi/5) + 1, \\ \gamma &= 2(\cos(2\pi/5) + \cos(2\pi/7)), \\ \gamma &= 4 \cos^2(\pi/7)(4 \cos^2(\pi/7) - 3), \\ \gamma &= 4 \cos^2(\pi/9)(4 \cos^2(\pi/9) - 3), \\ \gamma &= (4 \cos^2(\pi/7) - 1)^2(4 \cos^2(\pi/7) - 3), \\ \gamma &= 4 \cos^2(\pi/5)(4 \cos^2(\pi/5) - 2), \\ \gamma &= 4 \cos^2(\pi/q) - 3, \quad q \geq 7, \\ \gamma &= 4 \cos^2(\pi/q) - 2, \quad q \geq 5. \end{aligned}$$

2. *If  $\gamma < 0$ , then either  $\gamma \leq -4$  or  $\gamma = -4 \cos^2(2\pi/r)$  for  $r \geq 3$ ,  $r \neq 4$ .*

*All of the cases listed in 1. and 2. can occur.*

**Proof.** By Theorem 2.6 there exists  $h \in \mathbf{M}$  such that  $\langle f, h \rangle$  is discrete with  $\text{par}(\langle f, h \rangle) = (\gamma(f, g), \beta(f), -4)$ . Next  $\langle f, h \rangle$  is Kleinian if  $(\gamma, \beta) \notin \mathcal{X}$ , the exceptional set described in Definition 2.9. If  $(\gamma, \beta) \in \mathcal{X}$ , there is nothing to establish since all the corresponding values of  $\gamma$  occur in the above list in 2. Thus we may assume without loss of generality that  $\beta(g) = -4$ .

If  $p$  is congruent to  $\pm 1 \pmod q$ , then  $f$  and  $f^{-1}$  are primitive and the result follows from Lemmas 4.4 and 4.6 by identifying those values of the set  $D(-4 \sin^2(\pi/q))$  in (4.8) or (4.9), depending on the parity of  $q$ , which exceed 2 or are less than  $-4$ .

If  $p$  is not congruent to  $\pm 1 \pmod{q}$ , then

$$\frac{\sin^2(p\pi/q)}{\sin^2(\pi/q)} \geq \frac{\sin^2(2\pi/q)}{\sin^2(\pi/q)} = 4 \cos^2(\pi/q) = c(q) \geq 1 \quad (4.18)$$

since  $q \geq 3$ . Thus the only new values we might get which lie in the interval  $[-4, 2]$  are those identified in the primitive case which when multiplied by  $c(q)$  again lie in  $[-4, 2]$ . These values are readily identified after a little computation and appear on our list.

Finally the fact that each value occurs for a discrete group is a simple consequence of the only if part of Theorem 4.10.  $\square$

We can use the symmetries  $\phi_3$  and  $\phi_1$  of Theorem 2.13 to find the parameters of discrete groups in the other strips 2B and 2C. Combining these with Theorem 4.15 we have the following result.

**Theorem 4.19** *Suppose that  $\gamma$  and  $\beta$  are real.*

1. *If  $-4 \leq \beta \leq 0$ , then  $(\gamma, \beta) \in \mathcal{R}$  if and only if  $\beta = -4 \sin^2(p\pi/q)$ ,  $(p, q) = 1$  and  $\gamma \in \Gamma(\beta)$ .*
2. *If  $-4 \leq \gamma \leq 0$ , then  $(\gamma, \beta) \in \mathcal{R}$  if and only if  $\gamma = -4 \cos^2(p\pi/q)$ ,  $(p, q) = 1$  and  $-(\beta + 4) \in \Gamma(-\gamma - 4)$ .*
3. *If  $\beta \leq \gamma \leq \beta + 4$ , then  $(\gamma, \beta) \in \mathcal{R}$  if and only if  $\gamma = \beta + 4 \cos^2(p\pi/q)$ ,  $(p, q) = 1$  and  $\gamma \in \Gamma(\gamma - \beta - 4)$ .*

The following forms of the Fricke identity

$$\begin{aligned} \gamma(f, g) &= \beta(f) + \beta(fg) + 4, \\ \gamma(f, g) &= \beta(f) + \beta(fg^{-1}) + 4, \\ \beta(fg) &= \beta(fg^{-1}) = \gamma(f, g) - \beta(f) - 4 \end{aligned} \quad (4.20)$$

allow us to think of the three strips 2A, 2B, 2C respectively as the regions where  $f$ ,  $[f, g]$ ,  $fh$  are elliptic. Furthermore the region below the line  $\gamma = \beta$  corresponds to  $\beta(fh) \geq -4$ . Thus  $\langle f, h \rangle$  can only be Fuchsian above the line  $\beta \geq -4$  and below the line  $\gamma = \beta$ .

From this observation together with the symmetry group of Theorem 2.13 we can obtain a version of Theorem 4.17 giving various universal constraints for Kleinian groups with an elliptic commutator and  $\beta(f) \in \mathbb{R}$  as well as other related results. We leave it to the reader to formulate these results.

## 5 The Remaining Regions

We are left only with the three regions between the hyperbolas of Theorem 3.1 and the strips given above in §4. We consider only one such region (Figure 2) since the others will then follow from the symmetries in Theorem 2.13. The region we consider is

$$\Omega = \{(\gamma, \beta) : 0 < \gamma < \beta, \beta \leq \gamma + 4/\gamma\}. \quad (5.1)$$

The determination of the discreteness conditions in  $\Omega$  is based on the fact that a group  $G$  with parameters  $(\gamma, \beta, -4)$  in  $\Omega$  has a subgroup  $F$  of index two with parameters  $(\gamma(\gamma - \beta), \beta, \beta)$ . Since a group and a subgroup of finite index are simultaneously discrete or non-discrete and simultaneously elementary or non-elementary, it will suffice to analyse discreteness properties and the parameters for  $F$ . We first show that  $F$  has an invariant disc and is generated by a pair of hyperbolics with intersecting axes and an elliptic commutator. Thus the discreteness of  $F$  can be determined by using the triangle discreteness algorithm of [12]. That is, we will apply Theorems 3.1.1 and Theorem 3.1.2 of [12] to obtain the necessary and sufficient discreteness conditions on the parameters for  $F$  given below in Theorem 5.4. These conditions are then translated into conditions on the parameters for  $G$  to give our main theorem, Theorem 5.5.

We begin with two lemmas that establish the relationship between  $G$  and  $F$ . We omit the proof of the following easy lemma.

**Lemma 5.2** *If  $(x, y) \in \Omega$ , then  $-4 < x(x - y) < 0$ .*

**Lemma 5.3** *Suppose that  $G = \langle f, g \rangle$  with  $\text{par}(G) = (\gamma, \beta, -4)$  and  $(\gamma, \beta) \in \Omega$ . Then  $F = \langle f, gfg^{-1} \rangle$  has an invariant disk or half plane and  $\text{par}(F) = (\gamma(\gamma - \beta), \beta, \beta)$ . Further  $F$  is generated by a pair of hyperbolics with intersecting axes and an elliptic commutator.*

**Proof:** Let  $h = gfg^{-1}$ . Then  $\text{tr}(f) = \text{tr}(h) \in \mathbb{R}$ . and

$$\text{tr}(fh^{-1}) = \text{tr}(fgf^{-1}g^{-1}) = \gamma + 2, \quad \text{tr}(fh) = \text{tr}(fgfg^{-1}) = \beta - \gamma + 2,$$

both of which are real. The first assertion now follows from Lemma 4.1 since  $\beta > 0$  implies the group  $F$  has a hyperbolic element and is therefore not purely elliptic. The trace identity

$$\gamma(f, gfg^{-1}) = \gamma(f, g)(\gamma(f, g) - \beta(f))$$

can be deduced from the Fricke identity. See (2.2) of [8]. Since  $0 < \gamma < \beta$ ,  $f$  is hyperbolic and

$$-4 < \gamma(\gamma - \beta) = \gamma(f, gfg^{-1}) < \beta(f), \quad \gamma(f, gfg^{-1}) \neq 0$$

or  $|\text{tr}[f, h]| < 2$ , so that  $[f, h]$  is elliptic. It is well known that a pair of hyperbolics with an invariant disk have intersecting axes (intersecting interior to or on the boundary of that disk) if and only if the trace of their commutator is less than or equal to 2. See p. 119 Cor. 3.4 [11].  $\square$

Our goal is to prove:

**Theorem 5.4** (*Discreteness theorem for  $F$ .*) *Let  $G = \langle f, g \rangle$ ,  $\text{par}(G) = (\gamma, \beta, -4)$  and  $(\gamma, \beta) \in \Omega$  and  $F = \langle f, gfg^{-1} \rangle$ . Then  $F$  is discrete if and only if there is an integer  $q$  such that one of the following holds.*

1.  $\gamma(\gamma - \beta) = -4 \cos^2(\pi/q)$ ,  $\beta \geq 4 \cos(2\pi/q)$ ,  $q \geq 3$ ,
2.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $\beta = (1 + 2 \cos(2\pi/q))^2 - 4$ ,  $q \geq 7$ ,  $(q, 6) = 1$ ,
3.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $\beta = (1 + 2 \cos(2\pi/q))^2 - 4$ ,  $q \geq 4$ ,  $(q, 3) = 1$ ,
4.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/q)$ ,  $\beta = 4 \cos(2\pi/q)$ ,  $q \geq 5$ ,  $(q, 2) = 1$ ,
5.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/7)$ ,  $\beta = 8 \cos(2\pi/7) + 4 \cos^2(2\pi/7)$ .

As an immediate corollary we have the following discreteness theorem for  $G$ .

**Theorem 5.5** (*Discreteness theorem for  $G$* ) *Let  $G = \langle f, g \rangle$ ,  $\text{par}(G) = (\gamma, \beta, -4)$  and  $(\gamma, \beta) \in \Omega$ . Then  $G$  is Kleinian if and only if there is an integer  $q$  such that one of the following holds.*

1.  $\gamma(\gamma - \beta) = -4 \cos^2(\pi/q)$ ,  $\beta \geq 4 \cos(2\pi/q)$ ,  $q \geq 3$ ,
2.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $\beta = (1 + 2 \cos(2\pi/q))^2 - 4$ ,  $q \geq 7$ ,  $(q, 6) = 1$ ,
3.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $\beta = (1 + 2 \cos(2\pi/q))^2 - 4$ ,  $q \geq 4$ ,  $(q, 3) = 1$ ,
4.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/q)$ ,  $\beta = 4 \cos(2\pi/q)$ ,  $q \geq 5$ ,  $(q, 2) = 1$ ,
5.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/7)$ ,  $\beta = 8 \cos(2\pi/7) + 4 \cos^2(2\pi/7)$ .

By Lemma 5.3,  $F$  is generated by a pair of hyperbolics  $f$  and  $gfg^{-1} = h$  with intersecting axes. The determination of the discreteness of such a group requires two steps. The first step is to apply the triangle algorithm (section 2.5 of [12]). This algorithm associates to the pair of generators an initial triangle, denoted  $T_{f,h}$ , and replaces the triangle by successive triangles and

corresponding (Nielsen equivalent) generators for the group until one arrives at an acute triangle  $Act_{f,h}$ , known as the *acute triangle* determined by the initial pair of generators. The discreteness theorem (Theorem 3.1.1 of [12]) gives a necessary and sufficient discreteness test that applies to  $Act_{f,h}$ , the acute triangle.

The hyperbolics  $f$  and  $gfg^{-1} = h$  determine a hyperbolic triangle  $T = T_{f,h}$  as follows: Let  $p_2$  be the point of intersection of the their axes. Pick  $p_1$  along the axis of  $f$  and  $p_3$  along the axis of  $h$  so that  $f = E_{p_1}E_{p_2}$  and  $h = E_{p_3}E_{p_2}$ , where  $E_a$  denotes the half-turn whose fixed point is  $a$ . We let  $T$  be the triangle with vertices  $p_1, p_2, p_3$  and note that the lengths of the sides of  $T$  are half the translation lengths of the transformations  $f$ ,  $h = gfg^{-1}$ , and  $fh^{-1} = fgg^{-1}g^{-1}$ . Since  $f$  and  $h$  are conjugate,  $T$  is an isosceles triangle. Let  $T'$  be the triangle with vertices  $p'_1, p_2, p_3$  where  $p'_1 = E_{p_2}(p_1)$ .  $T'$  has sides half the translation lengths of  $f$ ,  $h$  and  $fh$ . We let  $\theta$  be the angle of  $T$  at the vertex  $p_2$ . An acute hyperbolic triangle is a hyperbolic triangle all of whose angles are less than or equal to  $\pi/2$ .

**Lemma 5.6** *If  $trf = trh$ , then the triangle  $T$  is an acute triangle if and only if*

$$trfh^{-1} \leq trfh.$$

*If  $T$  is not an acute triangle, then  $T'$  is. Either  $T$  or  $T'$  coincides with  $Act_{f,h}$ , the acute triangle.*

**Proof:** Since  $trf = trh$ , both  $T$  and  $T'$  are isosceles triangles. We apply Theorem 2.1.3 of [12] to the group whose two generators are  $f$  and  $h$  to see that  $trfh^{-1} \leq trfh$  if and only if  $\theta \leq \pi/2$ . If  $\delta$  is the *equal angle*, the angle opposite the equal sides, in any hyperbolic isosceles triangle, then  $\delta < \pi/2$ , since the triangle must have positive area. Thus  $T$  is an acute triangle precisely when  $\theta \leq \pi/2$ . When  $\theta > \pi/2$ ,  $T'$  must be an acute triangle.

Next we assume that  $trfh^{-1} \leq trfh$  and construct  $Act_{f,h}$  the acute triangle according to the triangle algorithm, 2.5 [12]. We write  $S_f$  to signify that side  $S$  is a side whose length is half the translation length of  $f$ . The algorithm begins with the triangle with sides  $A$ ,  $B$  and  $C$  where  $A \leq B \leq C$ .

If  $trf \leq trfh^{-1}$ , this is  $T$ , the acute triangle with  $A_f$ ,  $B_h$  and  $C_{fh^{-1}}$ , and so we stop at step one.

If  $trfh^{-1} < trf$ , then we must replace  $f$  and/or  $h$  by its inverse so that the first triangle is either the triangle whose *ordered* sides are  $A_{fh^{-1}}$ ,  $B_{h^{-1}}$  and

$C_f$  or the triangle whose ordered side are  $A, B, D$  where  $D = D_{fh^{-2}}$ . Since the triangle with sides  $A, B, C$  is acute, either the algorithm began with this triangle or it began with the triangle  $A, B, D$  and stops after one step with  $T'$  when  $A, B, D$  is replaced by  $A, B, C$ .  $\square$ .

We recall for the reader the discreteness test, Theorems 3.1.1 of [12].

**Theorem 5.7 (Discreteness Condition [12]):** *Let  $F$  be a group generated by a pair of hyperbolics,  $f$  and  $h$  whose axes intersect and whose commutator is elliptic. Then  $F$  is discrete if and only if either*

1.  $[f, h]$  is the square of a primitive rotation or
2.  $\text{tr}([f, h]) = -2 \cos(2k\pi)/n$  where  $k = 2$  or  $3$ ,  $n$  is an integer with  $k \leq n/2$ , and
  - a. when  $k = 3$ ,  $\text{Act}_{f,h}$  is an equilateral triangle,
  - b. when  $k = 2$ , either  $\text{Act}_{f,h}$  is a right isosceles triangle or the isosceles triangle with sides  $M, N$  and  $N$  where  $\cosh D_0 = 1/(2 \sin \pi/7)$ ,  $\cosh M = \cosh^2 D_0 - \cos(2\pi/7) \sinh^2 D_0$ , and  $\cosh N = \cosh^2 D_0 - \cos(6\pi/7) \sinh^2 D_0$ .

Let  $D_0(2, 3)$  denote the side opposite the  $\pi/n$  angle in a  $(2, 3, n)$  hyperbolic triangle and  $D_0(2, 4)$  the side opposite the  $\pi/n$  angle in a  $(2, 4, n)$  hyperbolic triangle. We also have

**Theorem 5.8 ([12]):** *When  $F$  satisfies the hypotheses of Theorem 5.4, then*

1. in the case  $k = 3$ , if  $E$  denotes the side of the equilateral triangle, we have

$$\cosh E = \cosh^2 D_0(2, 3) - \cos(2\pi/3) \sinh^2 D_0(2, 3)$$

and

2. in the case that  $k = 2$ , if  $L$  denotes the legs of the right isosceles triangle,

$$\cosh L = \cosh D_0(2, 4).$$

**Proof:** We use theorem 3.1.2 of [12] and observe that the lengths are lengths of the sides of the standard acute triangles constructed in sections 8.1 and 8.2 of [12]). We use the variant of the hyperbolic law of sines (equation (6.7) p. 74 of [12]) to compute that  $\cosh D_0(2, 3) = 2 \cos(\pi/n)/\sqrt{3}$  and  $\cosh D_0(2, 4) = 2 \cos(\pi/n)/\sqrt{2}$  and the fact that  $\beta(f) = \text{tr}^2(f) - 4 = 4 \sinh^2(\tau(f)/2)$ , with  $\tau(f)$  the translation length of  $f$  being twice the appropriate side length of the acute triangle.  $\square$

**Proof** (of Theorem 5.4, the discreteness theorem for  $F$  :) We conclude that for the group  $F$

- if  $\text{tr}fh > \text{tr}fh^{-1} > \text{tr}f$ ,  $F$  is not discrete;
- if  $\text{tr}fh = \text{tr}fh^{-1} > \text{tr}f$ , then  $Act_{f,h}$  is a right isosceles triangle and  $F$  is discrete;
- if  $\text{tr}fh^{-1} = \text{tr}f$ , then  $Act_{f,h}$  is an equilateral triangle and  $F$  is discrete;
- if  $\text{tr}fh^{-1} < \text{tr}f$ , then  $F$  is not discrete unless we have the values for  $M$  and  $N$  given above.

This proves the forward implication of Theorem 5.4

To prove the converse, we assume that  $F$  is given and that  $\beta$  and  $\gamma$  are given. Let  $\gamma' = \gamma(\gamma - \beta)$  always denote numerical values given in Cases 1-5 of the theorem. Further assume  $\beta$  and  $\gamma'$  both satisfy numerical conditions given in the same case of the theorem. Note that when

$$\beta(f) = \beta(h) = \beta, \quad \gamma(\gamma - \beta) = \text{tr}[f, h] - 2,$$

the Fricke identity yields

$$\beta(fh) = \gamma^2 + 4\gamma, \quad \beta(fh^{-1}) = (\beta - \gamma)^2 + 4(\beta - \gamma)$$

or vice-versa.

If we first set

$$\begin{aligned} \gamma' &= \gamma(f, h) = -4 \cos^2(3\pi/q), \\ \beta &= \beta(f) = \beta(h) = (1 + 2 \cos(2\pi/q))^2 - 4, \\ \gamma_1 &= -2 + (\beta^2 + 4)^{1/2}, \end{aligned}$$

then the triple-angle formulas imply that  $\gamma_1$  and  $\gamma_1 - \beta$  are the two roots of  $\gamma(\gamma - \beta) = \gamma'$ . Hence either  $\beta(fh) = \beta$  or  $\beta(fh^{-1}) = \beta$ , so the triangle is equilateral and  $F$  is discrete.

If we next set

$$\gamma' = -4 \cos^2(2\pi/q), \quad \beta = 4 \cos(2\pi/q), \quad \gamma_1 = (\beta - 4)/2,$$

we may again verify that  $\text{tr}fh = \text{tr}fh^{-1} = \text{tr}^2f$  so that we have a right isosceles triangle and  $F$  is discrete.

Finally a messy computation shows that the values of  $\beta$  and  $\gamma'$  in Case 5 with  $\gamma_1 = (\beta^2 + 4)^{1/2} - 3$  give the special isosceles triangle. This establishes sufficiency and we are done provided that the restrictions on  $\beta$  and  $q$  are correct.

To complete the proof, observe that if  $[f, h]$  is the square of a primitive rotation then (the proof of Lemma 13.1.1 [12] shows that)  $\text{tr}[f, h] = -2 \cos \theta$  where  $\theta = \pm 2\pi/n$ ,  $n \geq 3$ . Since  $\cos 2\alpha = 2 \cos^2 \alpha - 1$ ,  $\gamma' = -4 \cos^2 \pi/n$  and there is no restriction on  $\beta$  in Case 1 other than  $\gamma(\gamma - \beta) = -4 \cos^2(\pi/q)$ . The restrictions on  $q$  come from the fact that when  $k = 3$  and  $(q, 3) \neq 1$  or  $(q, 6) \neq 1$  or  $k = 2$  with  $(q, 4) \neq 1$ ,  $[f, h]$  is the square of a primitive rotation and this reduces to Case 1.

This completes the necessary and sufficient conditions for  $F$  to be discrete.

□.

**Proof** of Theorem 5.5: To complete the proof observe that since  $f$  is hyperbolic and  $\gamma' < 0$ ,  $F$  is nonelementary. □

The curves corresponding to the parameters of discrete groups in Case 1 are the hyperbolas

$$\beta = \gamma + \frac{4 \cos^2(\pi/p)}{\gamma} \tag{5.9}$$

illustrated in Figure 3. Since the minimum value on each of the above hyperbolas is  $4 \cos(\pi/p)$ ,  $\beta \geq 4 \cos(\pi/p)$ .

Now using our symmetries we can determine the points in the other regions that we have not dealt with yet. Let

$$\Omega_1 = \{(\gamma, \beta) : \gamma > 0, -4 \geq \beta > -4 - 4/\gamma\} \tag{5.10}$$

and

$$\Omega_2 = \{(\gamma, \beta) : \beta < -4, \beta + 4/(\beta + 4) \leq \gamma < \beta\} \quad (5.11)$$

**Corollary 5.12** *Let  $G = \langle f, g \rangle$ ,  $\text{par}(G) = (\gamma, \beta, -4)$  and  $(\gamma, \beta) \in \Omega_1 \cup \Omega_2$ . Then  $G$  is Kleinian if and only if there is an integer  $q$  such that*

1.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $\gamma - \beta = (1 + 2 \cos(2\pi/q))^2$ ,  $q \geq 7, (q, 6) = 1$ ,
2.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $\gamma - \beta = (1 + 2 \cos(2\pi/q))^2$ ,  $q \geq 4, (q, 3) = 1$ ,
3.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/q)$ ,  $\gamma - \beta = 4 \cos(2\pi/q)$ ,  $q \geq 5, (q, 2) = 1$ ,
4.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/7)$ ,  $\gamma - \beta = 8 \cos(2\pi/7) + 4 \cos^2(2\pi/7)$ .

**Proof:** Apply symmetries  $\phi_1$  and  $\phi_3$ .

Finally (as a simple consequence of Theorem 5.5) we collect here those values which occur in the region  $\Omega$  when  $\beta \leq 2$ . We shall use this in the next section.

**Corollary 5.13** *If  $(\gamma, \beta) \in \Omega$  and if  $0 \leq \beta < 2$ , then one of the following occurs:*

1.  $\gamma \in \{2 \cos(\pi/7) - 1, 2 \cos(2\pi/7) - 1\}$ ,  $\beta = 2(\cos(2\pi/7) + \cos(\pi/7) - 1)$ ,
2.  $\gamma \in \{\sqrt{2}, \sqrt{2} - 1\}$ ,  $\beta = 2\sqrt{2} - 1$ ,
3.  $\gamma \in \{(\sqrt{5} + 1)/2, (\sqrt{5} - 1)/2, \}$   $\beta = \sqrt{5} - 1$ .

## 6 Final Remarks

Given the projection  $(\gamma, \beta, \beta') \rightarrow (\gamma, \beta)$  of  $\mathcal{D} \setminus \mathcal{X} \rightarrow \mathcal{D}$  mentioned in Section 2 we can formulate the following general necessary condition which a Kleinian group must satisfy when two of its parameters are real.

**Theorem 6.1** *Suppose that  $G = \langle f, g \rangle$  is a Kleinian group with  $\beta = \beta(f) \in \mathbb{R}$ ,  $\beta \neq -4$ , and  $\gamma = \gamma(f, g) \in \mathbb{R}$ . Then one of the following holds:*

1.  $(\gamma, \beta) \in \mathcal{X}$ .

2.  $\beta \geq 0$  and  $\gamma \leq -4$ .
3.  $4 \leq \beta + 4 \leq \gamma$ .
4.  $\beta + 4 \leq \gamma \leq -4$ .
5.  $\beta \leq -4 - 4/\gamma \leq 0$ .
6.  $\gamma \leq \beta + 4/(\beta + 4) \leq 0$ .
7.  $\beta \geq \gamma + 4/\gamma \geq 0$ .
8.  $\beta = -4 \sin^2(p\pi/q)$ ,  $(p, q) = 1$ ,  $q \geq 3$  and  $\gamma \in \Gamma(\beta)$ .
9.  $\gamma = -4 \cos^2(p\pi/q)$ ,  $(p, q) = 1$ ,  $q \geq 3$  and  $-(\beta + 4) \in \Gamma(-\gamma - 4)$ .
10.  $\gamma = \beta + 4 \cos^2(p\pi/q)$ ,  $(p, q) = 1$ ,  $q \geq 3$  and  $\gamma \in \Gamma(\gamma - \beta - 4)$ .
11.  $\gamma(\gamma - \beta) = -4 \cos^2(\pi/q)$ ,  $q \geq 3$  and  $\beta \geq 4 \cos(2\pi/q)$ .
12.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $q \geq 7$ ,  $(q, 6) = 1$  and  $\beta = (1 + 2 \cos(2\pi/q))^2 - 4$ .
13.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $q \geq 4$ ,  $(q, 3) = 1$  and  $\beta = (1 + 2 \cos(2\pi/q))^2 - 4$ .
14.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/7)$  and  $\beta = 8 \cos(2\pi/7) + 4 \cos^2(2\pi/7)$ .
15.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $q \geq 7$ ,  $(q, 6) = 1$  and  $\gamma - \beta = (1 + 2 \cos(2\pi/q))^2$ .
16.  $\gamma(\gamma - \beta) = -4 \cos^2(3\pi/q)$ ,  $q \geq 4$ ,  $(q, 3) = 1$  and  $\gamma - \beta = (1 + 2 \cos(2\pi/q))^2$ .
17.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/q)$ ,  $q \geq 5$ ,  $(q, 2) = 1$  and  $\gamma - \beta = 4 \cos(2\pi/q)$ .
18.  $\gamma(\gamma - \beta) = -4 \cos^2(2\pi/7)$  and  $\gamma - \beta = 8 \cos(2\pi/7) + 4 \cos^2(2\pi/7)$ .

Next, collecting the universal constraints we obtained in Theorems 4.17, Corollary 5.13 and their symmetries, we can identify the six sided polygon in  $\mathbb{R}^2$  shown in Figure 4 as containing just finitely many points corresponding to discrete nonelementary groups in its interior. This polygon was also obtained using iteration theory in [9].

The labeled points inside the polygon consist of the points of the exceptional set  $\mathcal{X}$  together with some of the parameters for the  $(2, 3, 7)$ ,  $(2, 4, 5)$  and  $(3, 3, 4)$ -triangle groups. These latter points are

1.  $\Delta(2, 3, 7)$ :

$$\beta = 2(\cos(2\pi/7) + \cos(\pi/7) - 1),$$

$$\gamma \in \{2 \cos(\pi/7) - 1, 2 \cos(\pi/7) - 2, 2 \cos(2\pi/7), 2 \cos(2\pi/7) - 1\},$$

2.  $\Delta(2, 4, 5)$ :

$$\beta = \sqrt{5} - 1, \quad \gamma \in \left\{ \frac{\sqrt{5}-3}{2}, \frac{\sqrt{5}-1}{3}, \frac{\sqrt{5}+1}{3} \right\},$$

3.  $\Delta(3, 3, 4)$ :

$$\beta = 2\sqrt{2} - 1, \quad \gamma \in \{\sqrt{2} - 1, \sqrt{2}\}.$$

Another feature to notice about this polygon is that it does not contain the parameters  $(\gamma, \beta, \beta')$  for any Kleinian group in its interior unless the point  $(\gamma, \beta)$  is labeled. It is possible that  $(\gamma, \beta, \beta')$  are the parameters of a Kleinian group, but that  $(\gamma, \beta, -4)$  are the parameters of one of the finite triangle groups,  $A_4$ ,  $A_5$ , or  $S_4$ . Furthermore, we can now consider the images of the polygon of Figure 4 under the symmetries. This polygon is shown in Figure 5. We have not marked the finite number of points inside which do correspond to Kleinian groups. They are easily computed from above.

Finally, we remark that the constraints we found on the commutator parameter in Theorems 4.17 and so forth are stronger than necessary to obtain the polygon of Figure 4. They actually give a somewhat larger polygon with infinitely many exceptions. However apart from the  $(2, 3, p)$  triangle groups and the groups free on generators of order two and three (and their Lie symmetries) there are only a finite number of exceptions.

## References

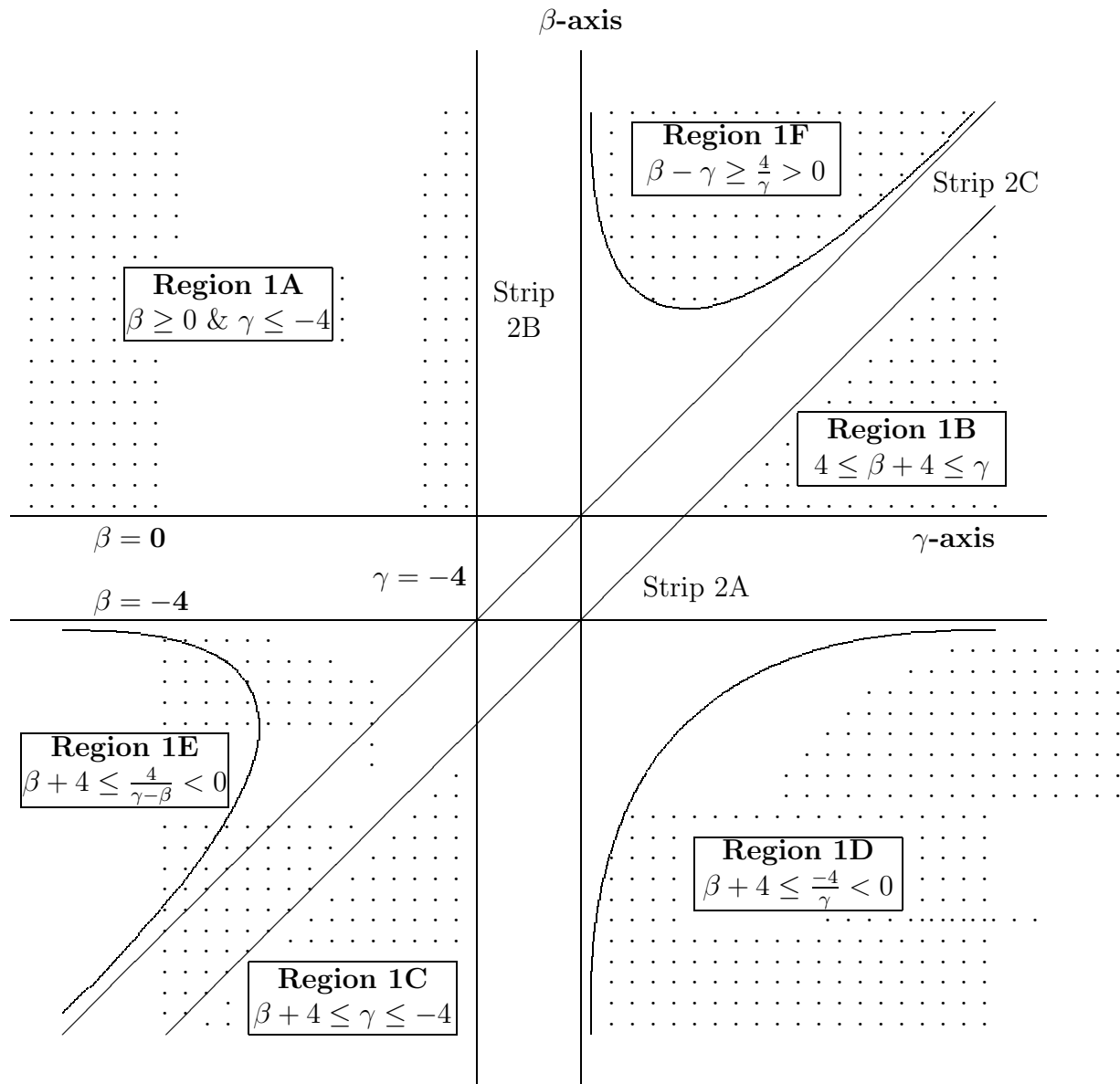
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groups.100

Figure 1: The discrete free groups

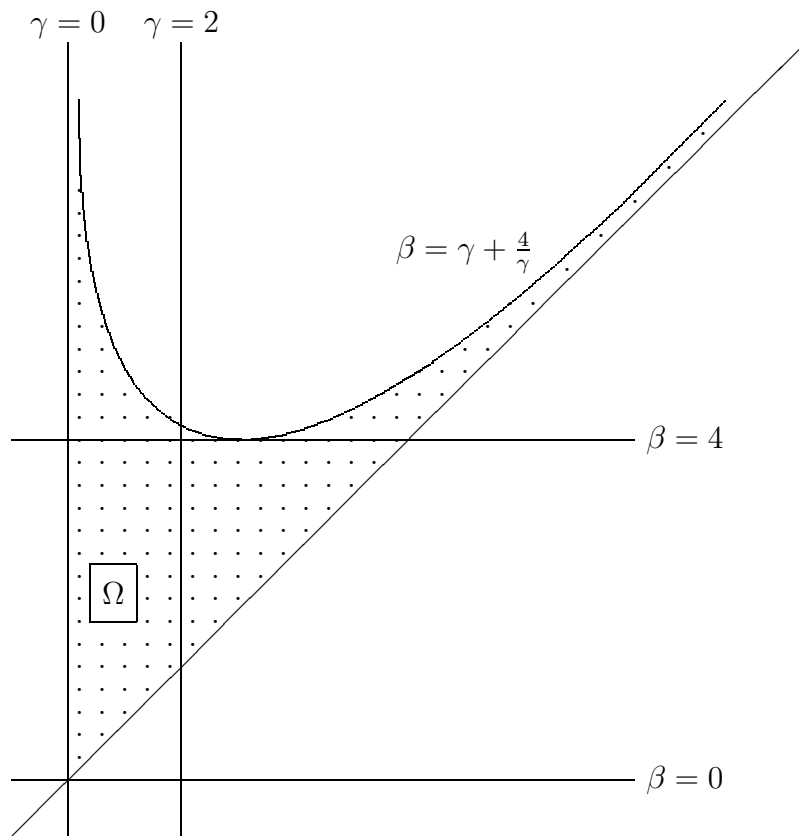


Figure 2: The region  $\Omega = \{(\gamma, \beta) : 0 < \gamma < \beta, \beta \leq \gamma + 4/\gamma\}$ .

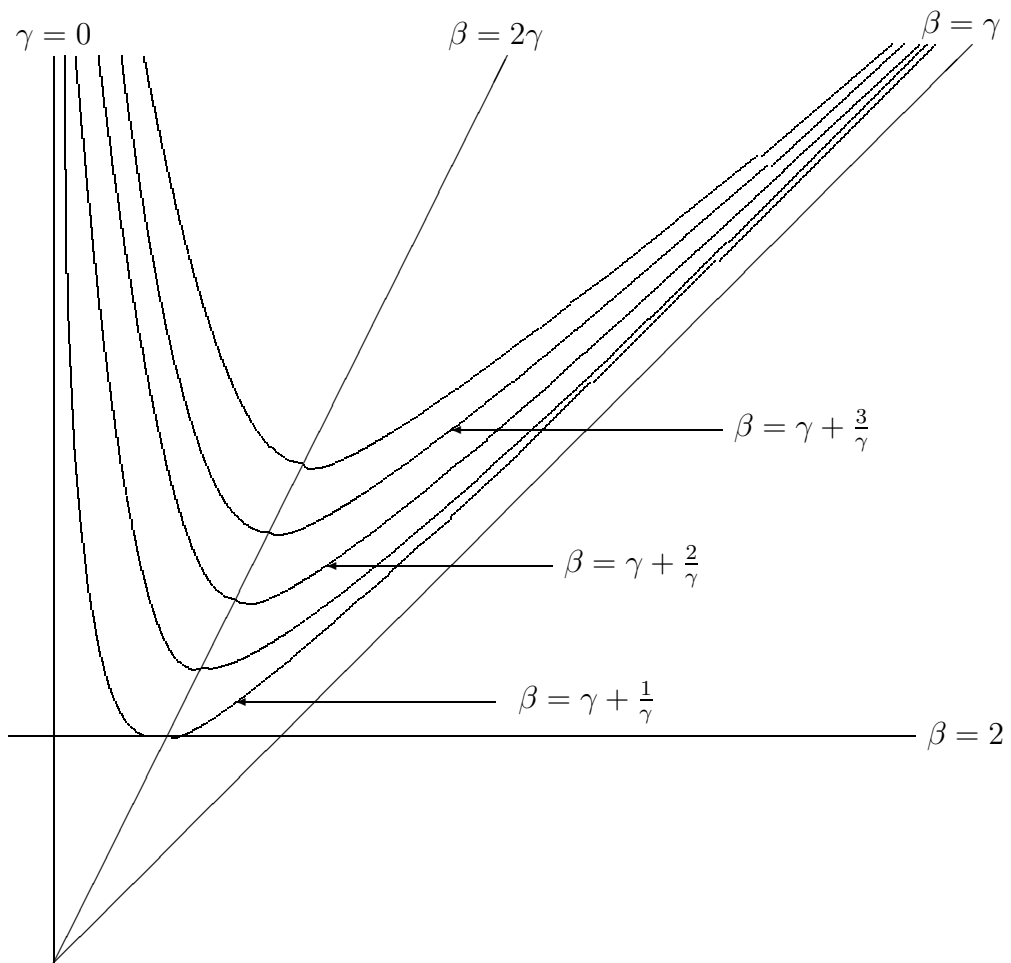


Figure 3: One parameter families of discrete groups

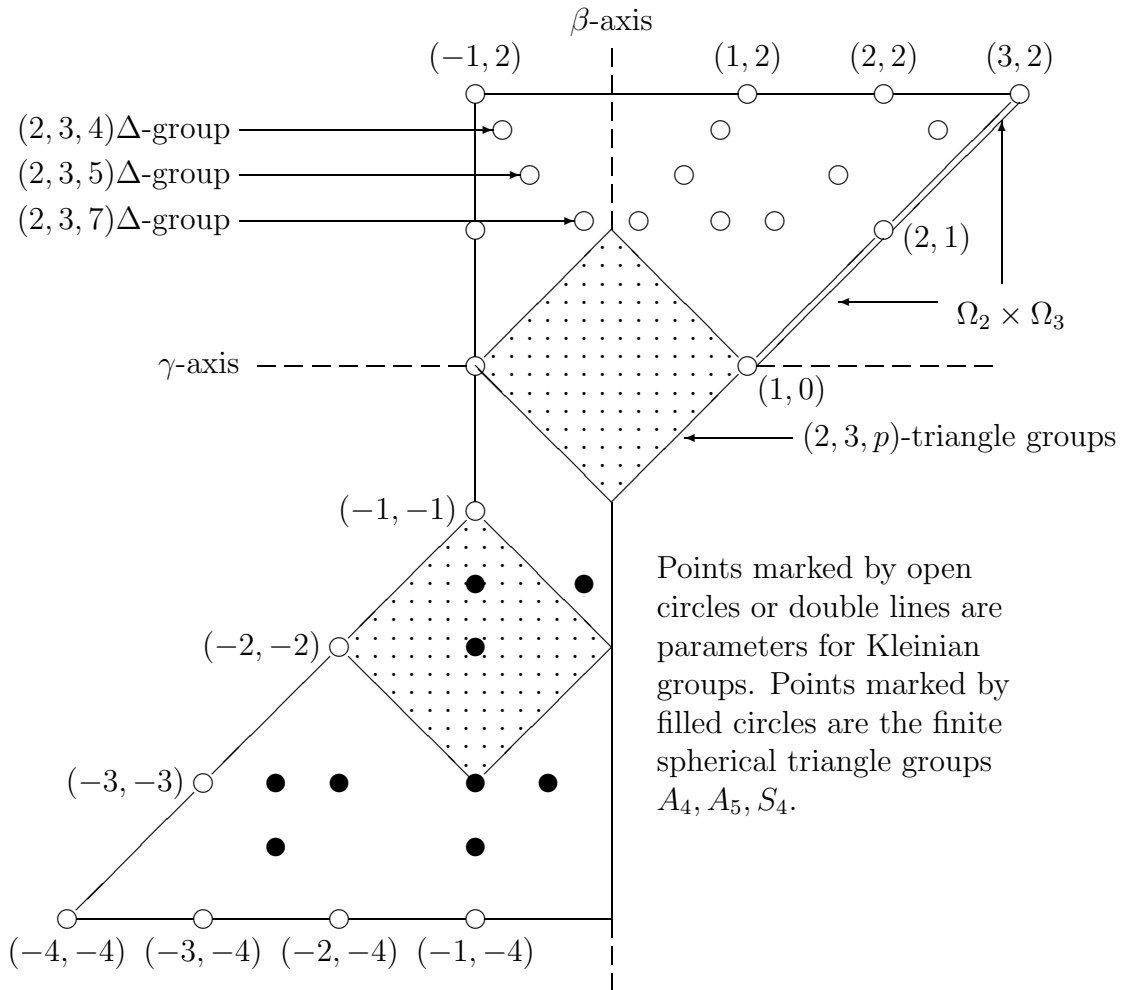


Figure 4: Polygon pictures. The upper square is a graphical representation of Jørgensen's inequality while the lower square is the variant found in [4].

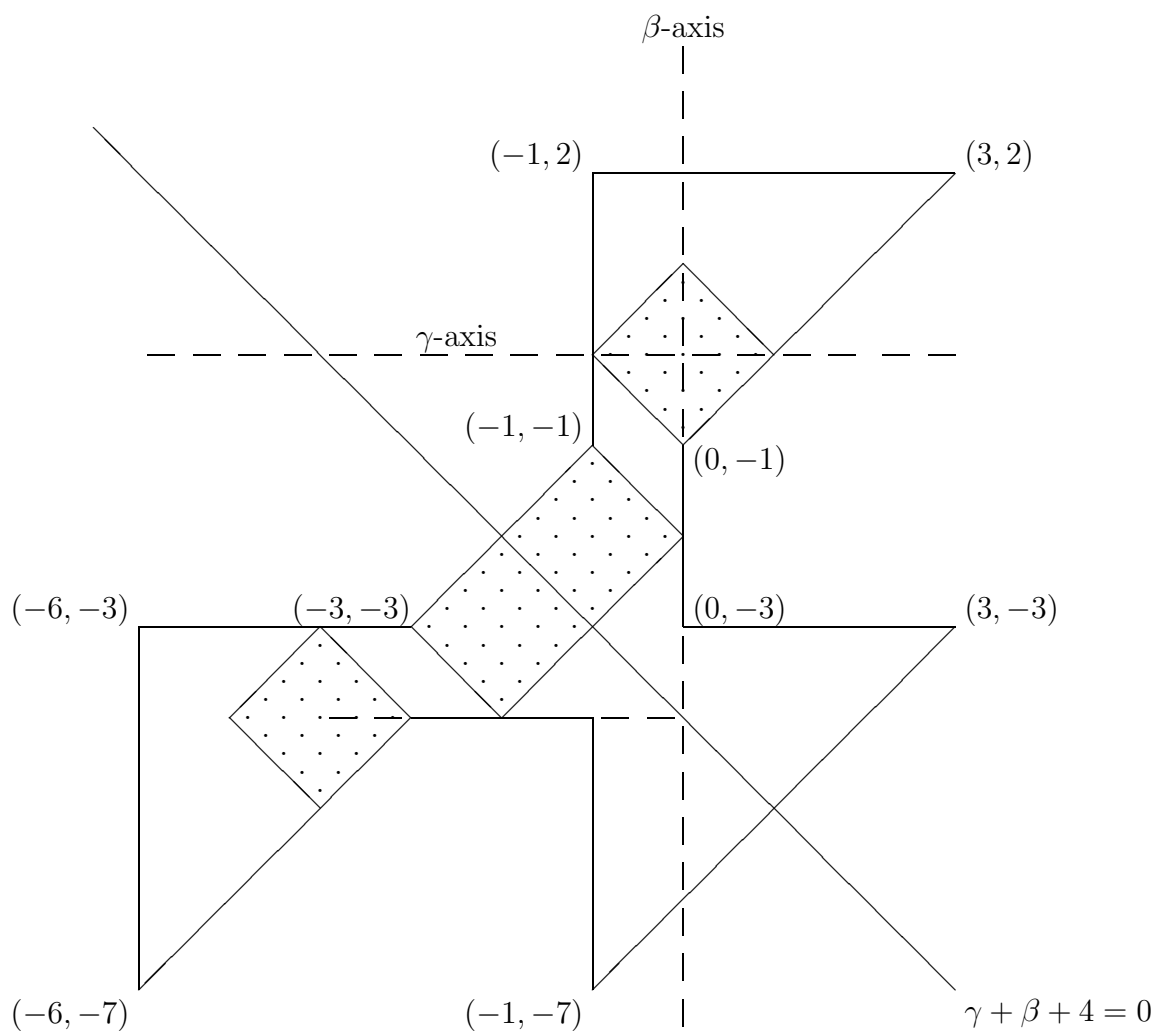


Figure 5: The extended polygon. The line  $\gamma + \beta + 4 = 0$  is the reflection line for the symmetry  $\phi_5$  and Theorem 2.12 shows that the entire picture is symmetric in this line. Hence the lower square is another version of Jørgensen's inequality (also discovered by Delin Tan [20]) while the rectangle which is bisected by the above line is the variant of Jørgensen's inequality from [4] plus its symmetric image.