CONTINUOUS COMPOUNDING

Invest $1 for one year at interest rate r.

- *Annual compounding*: you get $(1+r).
- *Semi-annual compounding*: you get $\$(1 + (r/2))^2$.
- *Continuous compounding*: you get $\$e^r$.

Invest $1 for T years at interest rate r.

- *Annual compounding*: you get $\$(1+r)^T$.
- *Semi-annual compounding*: you get $\$(1 + (r/2))^{2T}$.
- *Continuous compounding*: you get $\$e^{rT}$.

Invest $S for T years at interest rate r.

- *Annual compounding*: you get $\$S(1+r)^T$.
- *Semi-annual compounding*: you get $\$S(1 + (r/2))^{2T}$.
- *Continuous compounding*: you get $\$Se^{rT}$.

Mathematical notes:

- e = 2.718…
- $e^r = \lim_{n\to\infty} (1 + (r/n))^n$
- When r is close to zero, $e^r \approx 1+r$.
- $e^0 = 1$
- $e^r e^s = e^{r+s}$
- $e^r e^{-r} = e^0 = 1$
- $(e^r)^T = e^{rT}$

CONVERTING BETWEEN CONTINUOUS AND ANNUAL

$$Ae^{rT} = A(1+R)^T$$

$$e^{rT} = (1+R)^T$$

$$e^r = 1+R$$

$$r = \ln(1+R)$$

$$R = e^r - 1$$

CONVERTING BETWEEN CONTINUOUS AND m TIMES PER ANNUM

$$Ae^{rT} = A\left(1+\frac{R_m}{m}\right)^{mT}$$

$$e^r = \left(1+\frac{R_m}{m}\right)^m$$

$$r = m\ln\left(1+\frac{R_m}{m}\right)$$

$$R_m = m\left(e^{\frac{r}{m}} - 1\right)$$

FORWARD PRICE WITH CONTINUOUS COMPOUNDING

$$F_0 = S_0e^{rT}$$

If $F_0 > S_0e^{rT}$ or $F_0 < S_0e^{rT}$, then an arbitrageur can make money.

$F_0 > S_0e^{rT}$

Now:

Borrow \$$S_0$.
Spend the \$$S_0$ to buy an ounce of gold.
Go short in a forward contract
(agree to sell one ounce for \$$F_0$ after T years).

After T years:

Sell the gold for \$$F_0$.
Pay off the loan with \$$S_0e^{rT}$.
Profit: \$$(F_0 - S_0e^{rT})$

$F_0 < S_0e^{rT}$

Now:

Borrow an ounce of gold from Joe.
Sell the ounce of gold for \$$S_0$ and put the \$$S_0$ in the bank.
Go long in a forward contract
(agree to buy one ounce for \$$F_0$ after T years).

After T years:

Buy an ounce of gold for \$$F_0$ and return it to Joe.
Collect \$$S_0e^{rT}$ from the bank.
Profit: \$$(S_0e^{rT} - F_0)$

If you are holding gold for speculation, you can do this without finding someone else to borrow the gold from.

FORWARD PRICE OF ASSET WITH KNOWN INCOME

Say the asset is a bond with coupons that have present value I.

$$F_0 = (S_0 - I)e^{rT}$$

If $F_0 > (S_0 - I)e^{rT}$ or $F_0 < (S_0 - I)e^{rT}$, then an arbitrageur can make money.

$F_0 > (S_0 - I)e^{rT}$

Now:

Borrow $\$S_0$

Spend the $\$S_0$ to buy the bond.

Go short in a forward contract

(agree to sell the bond for $\$F_0$ after T years).

After T years:

Collect income of Ie^{rT} from the bond.

Sell the bond for $\$F_0$.

Pay off the loan with $\$S_0e^{rT}$.

Profit: $\$(F_0 + Ie^{rT} - S_0e^{rT})$

$F_0 < (S_0 - I)e^{rT}$

Now:

Borrow the bond from Joe (& pay him $I for the coupons).

Sell the bond for $\$S_0$ and put $\$(S_0 - I)$ in the bank.

Go long in a forward contract

(agree to buy the bond for $\$F_0$ after T years).

After T years:

Buy the bond for $\$F_0$ and return it to Joe.

Collect $\$(S_0 - I)e^{rT}$ from the bank.

Profit: $\$[(S_0 - I)e^{rT} - F_0]$

FORWARD PRICE OF ASSET WITH KNOWN DIVIDEND YIELD
Say a stock that pays continuous dividends at rate q.

$$F_0 = S_0e^{(r-q)T}$$

If $F_0 > S_0e^{(r-q)T}$ or $F_0 < S_0e^{(r-q)T}$, then an arbitrageur can make money.

$F_0 > S_0e^{(r-q)T}$

Now:

Borrow $\$S_0$.
Spend the $\$S_0$ to buy one share.
Go short in a forward contract on e^{qT} shares.
(agree to sell e^{qT} shares for $\$F_0$ per share after T years).

During the next T years:

Reinvest all dividends in the stock.

After T years:

Sell e^{qT} shares for $\$F_0e^{qT}$.
Pay off the loan with $\$S_0e^{rT}$.
Profit: $\$(F_0e^{qT} – S_0e^{rT})$

$F_0 < S_0e^{(r-q)T}$

Now:

Borrow a share from Joe.
Sell the share for $\$S_0$ and put the $\$S_0$ in the bank.
Go long in a forward contract on e^{qT} shares
(agree to buy e^{qT} shares for $\$F_0$ per share after T years).

During the next T years:

Continue to go short in the stock to pay Joe his dividends.

After T years:

Buy e^{qT} shares for $\$F_0e^{qT}$ and return them to Joe.
Collect $\$S_0e^{rT}$ from the bank.
Profit: $\$(S_0e^{rT} – F_0e^{qT})$

We must distinguish between the value of a forward contract and the forward price.

value = the price you should pay for the forward contract

delivery price = the price the contract requires you to pay at maturity

forward price = the delivery price of a forward contract negotiated today (fixed so that the value is zero)

Example: The delivery price of a forward that was negotiated yesterday is yesterday's forward price.

f = *value* of contract today
K = *delivery price* of contract
F = (today's) *forward price*

$$\mathbf{f = (F_0 - K)e^{-rT}}$$

VALUE OF OLD FORWARD WITH DELIVERY PRICE K

$$f = (F_0 – K)e^{-rT}$$

If $f > (F_0 – K)e^{-rT}$ **or** $f < (F_0 – K)e^{-rT}$**, then an arbitrageur can make money.**

$f > (F_0 – K)e^{-rT}$

Now:

Sell one old forward (delivery price K).

(agree to sell one unit for \$K after T years).

Put the proceeds (\$f) in the bank.

Go long in one new forward (delivery price F_0).

(agree to buy one unit for \$$F_0$ after T years).

After T years:

Collect fe^{rT} from the bank.

Buy one unit for \$$F_0$ and sell it for \$K.

Profit: \$$[fe^{rT} – F_0 + K]$

$f < (F_0 – K)e^{-rT}$

Now:

Borrow \$f.

Buy old forward (delivery price K).

(agree to buy one unit for \$K after T years).

Go short in one new forward (delivery price F_0).

(agree to sell one unit for \$$F_0$ after T years).

After T years:

Pay the bank \$$fe^{rT}$.

Buy one unit for \$K and sell it for \$$F_0$.

Profit: \$$[F_0 – K – fe^{rT}]$

SUMMARY (page 77)

Asset	**Value of long forward contract with delivery price K** $f = (F_0 - K)e^{-rT}$ $= F_0e^{-rT} - Ke^{-rT}$	**Forward price** F_0
Provides no income	$S_0 - Ke^{-rT}$	S_0e^{rT}
Provides known income with present value I	$S_0 - I - Ke^{-rT}$	$(S_0 - I)e^{rT}$
Provides known dividend yield q	$S_0e^{-qT} - Ke^{-rT}$	$S_0e^{(r-q)T}$

Investment vs Consumption Commodities

Forward Price F_0

Spot Price S_0

Asset	Investment Commodity	Consumption Commodity
Provides no income	$F_0 = S_0e^{rT}$	F_0 £ S_0e^{rT}
Provides known income with present value I	$F_0 = (S_0 - I)e^{rT}$	F_0 £ $(S_0 - I)e^{rT}$
Provides known dividend yield q	$F_0 = S_0e^{(r-q)T}$	F_0 £ $S_0e^{(r-q)T}$

Storage cost is treated as negative income.

Storage cost with present value U	$F_0 = (S_0 + U)e^{rT}$	F_0 £ $(S_0 + U)e^{rT}$
Storage cost at rate u	$F_0 = S_0e^{(r+u)T}$	F_0 £ $S_0e^{(r+u)T}$

You can have both income and storage:

Storage cost u and income q	$F_0 = S_0e^{(r+u-q)T}$	F_0 £ $S_0e^{(r+u-q)T}$

Cost of Carry = interest + storage – income = r + u – q = c:

Cost of carry c	$F_0 = S_0e^{cT}$	F_0 £ S_0e^{cT}

When **F_0 £ S_0e^{cT}**, the y such that **$F_0 = S_0e^{cT}$** is called the *convenience yield*.

Hierarchy of arguments:

- Strongest: arguments based on exact arbitrage
- Strong: arguments based on approximate arbitrage (if you don't take too large a position and there is not a lot of other money taking the same position)
- Doubtful: any argument based on the assumption that market prices are expected values with respect to a probability distribution
- Very doubtful: any argument based on CAPM

Recall that CAPM says that the expected return R_p on a portfolio p with beta β will obey

$$R_p - R_f \approx \beta(R_m - R_f).$$

Problems:

- The model is not well supported empirically.
- β may not be known (i.e., the historical data may not predict the future)
- Even if the model is right and β is known, the relation is only approximate, and the noise may be large.

Hedging using index futures:

In theory, you can hedge a portfolio of stocks with a known beta β by going short in

$$\beta P/A$$

future contracts, where P is the dollar value of the portfolio, and A is the dollar value of the asset underlying one futures contract.

This should convert the portfolio's return over the period of the hedge into the return on a risk free bond.

Problem 3.32

- P = \$50 million
- $\beta = 0.87$
- Hedge with S&P500 futures. Each contract is 250 times the index. The current value of the index is 1250. So the dollar value of the "asset underlying the contract" is \$1250x250 = \$312,500.
- The manager wants to hedge for 2 months, using futures with maturity of 3 months.
- $\beta P/A = 0.87 \times 50{,}000{,}000 / 312{,}500 \approx 139$. So short 139 contracts.
- We assume that the dividend yield on the S&P500 is q = 0.03 and the risk-free rate is r =0.06.

The theoretical futures price now is $\mathbf{S_0e^{(r-q)T}}$, and T = 0.25 (three months). The futures price two months from now depends on the value of the index then.

Problem 3.32

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- We assume that the dividend yield on the S&P500 is q = 0.03 and the risk-free rate is r =0.06.

The theoretical futures price now is

$$\mathbf{S_0e^{(r\text{-}q)T} = 1250e^{(0.06 - 0.03)0.25} = 1259.41}$$

The futures price two months from now depends on the value of the index then. If the index is at 1,000 two months from now, then the futures price will be $1{,}000\mathbf{e^{(r\text{-}q)T}}$ with T = 1 month = 0.08333:

$$\mathbf{S_0e^{(r\text{-}q)T} = 1000e^{(0.06 - 0.03)0.08333} = 1002.50.}$$

So the gain on the short position will be

$$(1259.41 - 1002.50) \times 250 \times 139 = \$8{,}927{,}530.$$

The index drop indicates a capital loss of –250/1250 = -20% by the market, offset by a 0.5% dividend (two months at 3% per annum), for a net return of -19.5%. This is less than the risk-free rate of 1% by 20.5%. CAPM says that the portfolio should be less than the risk-free rate of 1% by 0.87 x 20.5% = 17.835%. In other words, it should have a return of –16,835%, or a loss of 0.16825 x 50,000 = $8,417,500. So the manager records a net gain of

$$\$8{,}927{,}530 - \$8{,}417{,}500 = \$510{,}030.$$

The problem requires that you make a similar computation for other assumptions about the change in the value of the index.

Problem 3.29.

Continuously compounded rates:

• Risk-free	0.0925
• Storage cost for gold:	0.0050
• Bank's rate for cash loan:	0.1044
• Bank's rate for gold loan	0.0198

The interest on gold is paid in gold: if you borrow 1 ounce for one year, you must pay back 1.02 ounces.

Dicuss how the bank's interest rate for gold compares with its interest rate for cash.

Hints:

- The bank compensates itself for default risk and administrative costs by charging more than the risk-free rate for loaning cash. How much is the premium?
- How much is its premium for lending gold, taking into account the storage cost that it saves itself by loaning the gold?