Probability, Price, and the Central Limit Theorem

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- Review: The infinite-horizon fair-coin game for the strong law of large numbers.
- The finite-horizon fair-coin game for the weak law of large numbers.
- Bernoulli's Theorem.
- Price and Probability.
- De Moivre's Theorem.
- The One-Sided Central Limit Theorem.

REVIEW: THE (INFINITE-HORIZON) FAIR-COIN GAME FOR THE STRONG LAW OF LARGE NUMBERS

Players: Skeptic, Reality

Protocol:

$\mathcal{K}_0 = 1$.
FOR $n = 1, 2, \ldots$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Winner: Skeptic wins if (1) $\mathcal{K}_n$ is never negative and (2) either $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i = 0$ or $\lim_{n\to\infty} \mathcal{K}_n = \infty$. Otherwise Reality wins.

Skeptic has a winning strategy in this game. So we say...

- ...Skeptic *can force* $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i = 0$.
- ...$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i = 0$ happens *almost surely*.
- ...$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i = 0$ has *probability one*.

GENERALIZATION: INFINITE-HORIZON FAIR-COIN GAME WITH GOAL E

Players: Skeptic, Reality

Protocol:

$\mathcal{K}_0 = 1$.
FOR $n = 1, 2, \ldots$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Winner: Skeptic wins if (1) $\mathcal{K}_n \geq 0$ for all n and (2) either E happens or $\lim_{n\to\infty} \mathcal{K}_n = \infty$. Otherwise Reality wins.

If Skeptic has a winning strategy in this game, then we say. . .

- . . . Skeptic *can force* E.
- . . . E happens *almost surely*.
- . . . E has *probability one*.

THE (FINITE-HORIZON) FAIR-COIN GAME FOR THE WEAK LAW OF LARGE NUMBERS

Parameters: Natural number N, $\epsilon > 0$, $\alpha > 0$

Players: Skeptic, Reality

Protocol:

$\mathcal{K}_0 = \alpha$.
FOR $n = 1, 2, \ldots, N$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Winner: Skeptic wins if (1) $\mathcal{K}_n$ is never negative and (2) either $\mathcal{K}_N \geq 1$ or $|\mathcal{S}_N/N| < \epsilon$, where the process $\mathcal{S}$ is defined by $\mathcal{S}_0 := 0$ and $\mathcal{S}_n = \sum_{i=1}^{n} x_i$ for $n = 1, \ldots, N$.

Finitary in three respects:
- Finitely many rounds N.
- Skeptic tries to multiply his capital by α^{-1}, not by ∞.
- Skeptic wants $\mathcal{S}_N/N$ close to zero, but not infinitely close.

Bernoulli's Theorem Skeptic has a winning strategy if $N \geq 1/(\alpha\epsilon^2)$.

Lemma 1 *Set*

$$\mathcal{L}_n := \frac{\mathcal{S}_n^2 + N - n}{N} \text{ for } n = 0, 1, \ldots, N. \qquad (1)$$

This is a nonnegative martingale, and $\mathcal{L}_0 = 1$.

Proof Because $\mathcal{S}_n^2 - \mathcal{S}_{n-1}^2 = 2\mathcal{S}_{n-1}x_n + x_n^2$, the increment of $\mathcal{S}_n^2 - n$ is

$$(\mathcal{S}_n^2 - n) - (\mathcal{S}_{n-1}^2 - (n-1)) = 2\mathcal{S}_{n-1}x_n + (x_n^2 - 1).$$

Since $x_n^2 = 1$, $\mathcal{S}_n^2 - n$ is a martingale; it is obtained by starting with capital 0 and then buying $2\mathcal{S}_{n-1}$ tickets on the nth round. The process $\mathcal{L}_n$ is therefore also a martingale. By (1), $\mathcal{L}_0 = 1$ and $\mathcal{L}_n \geq 0$ for $n = 1, \ldots, N$.

Bernoulli's Theorem Skeptic has a winning strategy in the finite-horizon fair-coin game if $N \geq 1/(\alpha\epsilon^2)$.

Proof Suppose Skeptic starts with α and plays $\alpha\mathcal{P}$, where $\mathcal{P}$ is a strategy that produces the martingale $\mathcal{L}_n$ when he starts with 1. His capital at the end of the game is then $\alpha\mathcal{S}_N^2/N$, and if this is 1 or more, then he wins. Otherwise $\alpha\mathcal{S}_N^2/N < 1$. Multiplying this by $1/(\alpha\epsilon^2) \leq N$, we obtain $|\mathcal{S}_N/N| < \epsilon$; Skeptic again wins.

Bernoulli's Theorem with Probability

Protocol:

$\mathcal{K}_0 := \alpha$.
FOR $n = 1, \ldots, N$:
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in \{-1, 1\}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Upper and Lower Probability:

Consider an event E ($E \subseteq \Omega$, where $\Omega = \{-1,1\}^N$). Its *upper probability* is

$$\overline{\mathbb{P}}\, E := \inf\{\alpha \mid \text{Skeptic can parlay } \alpha \text{ into at least 1 if } E \text{ happens and at least 0 otherwise}\}.$$

Its *lower probability* is

$$\underline{\mathbb{P}}\, E := 1 - \overline{\mathbb{P}}\, E^{\mathsf{c}},$$

where E^{c} is the complement of E; $E^{\mathsf{c}} := \Omega \setminus E$.

Now consider the event $E = \{|\mathcal{S}/N| \geq \epsilon\}$.

According to Bernoullli's theorem Skeptic has the desired strategy if $N \geq 1/(\alpha\epsilon^2)$, or $\alpha \geq 1/(N\epsilon^2)$. So

$$\overline{\mathbb{P}}\left\{\left|\frac{\mathcal{S}_N}{N}\right| \geq \epsilon\right\} \leq \frac{1}{N\epsilon^2}.$$

Equivalently,

$$\underline{\mathbb{P}}\left\{\left|\frac{\mathcal{S}_N}{N}\right| < \epsilon\right\} \geq 1 - \frac{1}{N\epsilon^2}.$$

PRICE AND PROBABILITY

$\mathcal{K}_0 := \alpha$.
FOR $n = 1, \ldots, N$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Upper Price for a Variable y:

$\overline{\mathbb{E}}\, y :=$ smallest initial stake Skeptic can parlay into y or more at the end of the game

$$= \inf\{\mathcal{L}(\Box) | \mathcal{L} \text{ is a martingale and } \mathcal{L}(x_1, \ldots, x_N) \geq y(x_1, \ldots, x_N)\}.$$

$\mathcal{L}(\Box)$ is the martingale's initial value.

Suppose Skeptic is willing to sell a variable to the public at any price at which he can replicate it with no risk of loss. Then $\overline{\mathbb{E}}\, y$ is his *minimum selling price* for y.

Upper Price for a Variable y:

$$\overline{\mathbb{E}}\, y := \text{smallest initial stake Skeptic can parlay into } y \text{ or more at the end of the game}$$
$$= \text{Skeptic's minimum selling price for } y.$$

Proposition

$$\overline{\mathbb{E}}\, y_1 + \overline{\mathbb{E}}\, y_2 \geq \overline{\mathbb{E}}[y_1 + y_2]. \tag{2}$$

This follows from the fact that the sum of two martingales is a martingale (add the strategies).

Buying y for α is the same as selling $-y$ for $-\alpha$. So $-\overline{\mathbb{E}}\,-y$ is Skeptic's maximum buying price for y. We call this its lower price:

$$\underline{\mathbb{E}}\, y := -\overline{\mathbb{E}}\,-y.$$

By (2), $\overline{\mathbb{E}}\, y - \underline{\mathbb{E}}\, y \geq \overline{\mathbb{E}}\, 0$, which is 0, because Skeptic cannot make money for certain. So

$$\overline{\mathbb{E}}\, y \geq \underline{\mathbb{E}}\, y.$$

Probability from Price

$\overline{\mathbb{E}}\, y =$ Skeptic's minimum selling price for y.
$\underline{\mathbb{E}}\, y =$ Skeptic's maximum buying price for y.

We recover the concepts of upper and lower probability when we set

$$\overline{\mathbb{P}}\, E := \overline{\mathbb{E}}\, \mathcal{I}_E \qquad \text{and} \qquad \underline{\mathbb{P}}\, E := \underline{\mathbb{E}}\, \mathcal{I}_E,$$

where $\mathcal{I}_E$ is the indicator variable for E.

$\overline{\mathbb{P}}\, E := \overline{\mathbb{E}}\, \mathcal{I}_E =$ smallest initial stake Skeptic can parlay into at least 1 if E happens and at least 0 otherwise

$$\underline{\mathbb{P}}\, E = \underline{\mathbb{E}}\, \mathcal{I}_E = -\overline{\mathbb{E}} - \mathcal{I}_E = -\overline{\mathbb{E}}[-1 + \mathcal{I}_E]$$

$$= 1 - \overline{\mathbb{E}}\, \mathcal{I}_E = 1 - \overline{\mathbb{P}}\, E.$$

THE CENTRAL LIMIT THEOREM

We consider only coin-tossing (DeMoivre's theorem). For simplicity, we now score Heads as $1/\sqrt{N}$ and Tails as $-1/\sqrt{N}$.

FOR $n = 1, \ldots, N$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Set $\mathcal{S}_n := \sum_{i=1}^{n} x_i$.

Consider a smooth function U.

De Moivre's Theorem For N sufficiently large, both $\overline{\mathbb{E}}\, U(\mathcal{S}_N)$ and $\underline{\mathbb{E}}\, U(\mathcal{S}_N)$ are arbitrarily close to $\int_{-\infty}^{\infty} U(z)\, \mathcal{N}_{0,1}(dz)$.

How do we prove De Moivre's theorem?

$$\mathcal{S}_n := \sum_{i=1}^{n} x_i.$$

We want to know the price at time 0 of the payoff $U(\mathcal{S}_N)$ at time N. Let us also consider its price at time n. Intuitively, this should depend on $\mathcal{S}_n$, the value of the sum so far. Assume, optimistically, that the price at time n is given by a function of two variables, $\overline{U}(s, D)$: the price at time n is $\overline{U}(\mathcal{S}_n, \frac{N-n}{N})$.

Successive prices are

$$\overline{U}(0,1), \overline{U}(\mathcal{S}_1, \tfrac{N-1}{N}), \ldots$$
$$\ldots, \overline{U}(\mathcal{S}_{N-1}, \tfrac{1}{N}), \overline{U}(\mathcal{S}_N, 0),$$

These must be the successive values of a martingale.

- $\overline{U}(\mathcal{S}_N, 0)$ must equal $U(\mathcal{S}_N)$.

- $\overline{U}(0,1)$ is the price that interests us.

We want to choose $\overline{U}(s, D)$ so that

$$\overline{U}(0,1), \overline{U}(\mathcal{S}_1, \tfrac{N-1}{N}), \ldots$$
$$\ldots, \overline{U}(\mathcal{S}_{N-1}, \tfrac{1}{N}), \overline{U}(\mathcal{S}_N, 0)$$

is a martingale with $\overline{U}(\mathcal{S}_N, 0) = U(\mathcal{S}_N)$.

Consider the increments in s, D, and $\overline{U}$:

- $\Delta s_n = x_n = \pm \frac{1}{\sqrt{N}}$.
- $\Delta D_n = -\frac{1}{N}$.
- $\Delta \overline{U}_n = \overline{U}(\mathcal{S}_n, \frac{N-n}{N}) - \overline{U}(\mathcal{S}_{n-1}, \frac{N-n+1}{N})$.

Study $\Delta\overline{U}$ with a Taylor's expansion:

$$\Delta\overline{U} \approx \frac{\partial \overline{U}}{\partial s}\Delta s + \frac{\partial \overline{U}}{\partial D}\Delta D + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(\Delta s)^2$$

$$= \frac{\partial \overline{U}}{\partial s} x - \left(\frac{\partial \overline{U}}{\partial D} - \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}\right)\frac{1}{N}.$$

$$\Delta \overline{U} \approx \frac{\partial \overline{U}}{\partial s} x - \left(\frac{\partial \overline{U}}{\partial D} - \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}\right)\frac{1}{N}.$$

We need the second term to go away, which requires

$$\frac{\partial \overline{U}}{\partial D} = \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}$$

Then we obtain the desired martingale by buying $\frac{\partial \overline{U}}{\partial s}$ x-tickets on the nth round. In other words, we set

$$M_n := \frac{\partial \overline{U}}{\partial s}(\mathcal{S}_{n-1}, \tfrac{N-n+1}{N}).$$

The partial differential equation

$$\frac{\partial \overline{U}}{\partial D} = \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}$$

is the *heat equation*. Laplace showed that its solution is a Gaussian integral.

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With the initial condition $\overline{U}(s,0) = U(s)$, the solution is

$$\begin{aligned}\overline{U}(s,D) &= \int_{-\infty}^{\infty} U(z)\,\mathcal{N}_{s,D}(dz)\\ &= \int_{-\infty}^{\infty} U(s+z)\,\mathcal{N}_{0,D}(dz).\end{aligned}$$

So the initial price, $\overline{U}(0,1)$, is

$$\int_{-\infty}^{\infty} U(z)\,\mathcal{N}_{0,1}(dz).$$

The One-Sided Central Limit Theorem

Allow Reality to choose x anywhere in an interval instead of limiting her to two values.

> FOR $n = 1, \ldots, N$:
> Skeptic announces $M_n \in \mathbb{R}$.
> Reality announces $x_n \in [-1/\sqrt{N}, 1/\sqrt{N}]$.
> $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Now $\Delta s \leq \Delta D$ instead of $\Delta s = \Delta D$. The upper price will be given by a function $\overline{U}(s, D)$ that satisfies

- $\lim_{D \to 0} \overline{U}(s, D) = U(s)$,
- $\overline{U}(s, D) \geq U(s)$, and
- $\frac{\partial \overline{U}}{\partial D} = \frac{1}{2} \frac{\partial^2 \overline{U}}{\partial s^2}$

This is diffusion with heat sources that keep the temperature at s from falling below $U(s)$.

Numerical results are easily obtained but differ from those for De Moivre's theorem.

De Moivre In the De Moivre case, where Reality must choose between two values on each round, the upper and lower prices are approximately equal. We obtain a nonzero belief that the final sum will be a certain distance away from zero:

$\overline{\underline{\mathbb{P}}}\{\lvert\mathcal{S}_N\rvert \geq 0.01\} = 0.992$
$\overline{\underline{\mathbb{P}}}\{\lvert\mathcal{S}_N\rvert \leq 3\} = 0.997$

One-Sided In the one-sided case, where Reality can choose from an interval, the upper and lower prices are very different. We do *not* obtain a nonzero belief that the final sum will be a certain distance away from zero:

Upper probabilities	**Lower probabilities**
$\overline{\mathbb{P}}\{\lvert\mathcal{S}_N\rvert \geq 0.01\} = 1.000$	$\underline{\mathbb{P}}\{\lvert\mathcal{S}_N\rvert \geq 0.01\} = 0$
$\overline{\mathbb{P}}\{\lvert\mathcal{S}_N\rvert \leq 3\} = 1$	$\underline{\mathbb{P}}\{\lvert\mathcal{S}_N\rvert \leq 3\} = 0.995$