Black-Scholes Pricing: Stochastic and Game-Theoretic

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- Review: Price and Probability.
- The Classical Black-Scholes Protocol.
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- The Classical Black-Scholes Formula.
- A Dividend-Paying Security.
- Vovk's Black-Scholes Protocol.
- Vovk's Black-Scholes Formula.

PRICE AND PROBABILITY

```
$\mathcal{K}_0 := \alpha$.
FOR $n = 1, \dots, N$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
```

Upper Price for a Variable y:

$\overline{\mathbb{E}}\, y :=$ smallest initial stake Skeptic
can parlay into y or more
at the end of the game

$= \inf\{\mathcal{L}(\Box) | \mathcal{L}$ is a martingale and
$\mathcal{L}(x_1, \dots, x_N) \geq y(x_1, \dots, x_N)\}$.

$\mathcal{L}(\Box)$ is the martingale's initial value.

Suppose Skeptic is willing to sell a variable to the public at any price at which he can replicate it with no risk of loss. Then $\overline{\mathbb{E}}\, y$ is his *minimum selling price* for y.

Upper Price for a Variable y:

$$\begin{aligned}\overline{\mathbb{E}}\, y := &\text{ smallest initial stake Skeptic} \\ &\text{ can parlay into } y \text{ or more} \\ &\text{ at the end of the game} \\ = &\text{ Skeptic's minimum selling price for } y.\end{aligned}$$

Proposition

$$\overline{\mathbb{E}}\, y_1 + \overline{\mathbb{E}}\, y_2 \geq \overline{\mathbb{E}}[y_1 + y_2]. \tag{1}$$

This follows from the fact that the sum of two martingales is a martingale (add the strategies).

Buying y for α is the same as selling $-y$ for $-\alpha$. So $-\overline{\mathbb{E}}\,-y$ is Skeptic's maximum buying price for y. We call this its lower price:

$$\underline{\mathbb{E}}\, y := -\overline{\mathbb{E}}\,-y.$$

By (1), $\overline{\mathbb{E}}\, y - \underline{\mathbb{E}}\, y \geq \overline{\mathbb{E}}\, 0$, which is 0, because Skeptic cannot make money for certain. So

$$\overline{\mathbb{E}}\, y \geq \underline{\mathbb{E}}\, y.$$

GENERALIZING THE GAME

- $\mathbf{M}$ is a linear space.
- $\mathbf{X}$ is a nonempty set.
- $\lambda : \mathbf{M} \times \mathbf{X} \to \mathbb{R}$ is linear in its first argument.

```
FOR $n = 1, \dots, N$:
  Skeptic announces $M_n \in \mathbf{M}$.
  Reality announces $x_n \in \mathbf{X}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + \lambda(M_n, x_n)$.
```

Now $\mathbf{X}^N$ is the sample space. As usual, we call a capital process for Skeptic a *martingale*. We call the protocol *coherent* if for every $M \in \mathbf{M}$ there exists $x \in \mathbf{X}$ with $\lambda(M, x) \leq 0$.

$$\begin{aligned} \overline{\mathbb{E}}\, y :=& \text{ smallest initial stake Skeptic can parlay into} \\ & \quad y \text{ or more at the end of the game} \\ =& \inf\{\mathcal{L}(\square) | \mathcal{L} \text{ is a martingale and} \\ & \quad \mathcal{L}(x_1, \dots, x_N) \geq y(x_1, \dots, x_N)\} \\ =& \text{ Skeptic's minimum selling price for } y. \end{aligned}$$

$$\underline{\mathbb{E}}\, y := -\,\overline{\mathbb{E}}\, {-y} = \text{Skeptic's maximum buying price for } y.$$

THE BLACK-SCHOLES PROTOCOL

The price of a security $\mathcal{S}$ is determined by a game just like those we have been studying. If we write $S(t)$ for the price at time t, then we can write the game's protocol as follows.

```
Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$
Players: Investor, Market
Protocol:
  $\mathcal{I}(0) := 0$.
  Market announces $S(0) > 0$.
  FOR $t = 0, dt, 2dt, \ldots, T - dt$:
    Investor announces $\delta(t) \in \mathbb{R}$.
    Market announces $dS(t) \in \mathbb{R}$.
    $S(t + dt) := S(t) + dS(t)$.
    $\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t) dS(t)$.
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A *European option* on a stock $\mathcal{S}$ with maturity T is a security that pays the amount $U(S(T))$ at time T, where U is a known function. If $\overline{\mathbb{E}}\, U(S(T)) = \underline{\mathbb{E}}\, U(S(T))$, then we say that *the option is priced*.

Geometric Brownian Motion

```
Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$
Players: Investor, Market
Protocol:
  $\mathcal{I}(0) := 0$.
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    Investor announces $\delta(t) \in \mathbb{R}$.
    Market announces $dS(t) \in \mathbb{R}$.
    $S(t + dt) := S(t) + dS(t)$.
    $\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t)dS(t)$.
```

Without additional assumptions, most options are not priced in this game.

The classical Black-Scholes theory adds the assumption that $S(t)$ follows a geometric Brownian motion. In other words, Market's moves must obey

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

where $dW(t)$ is Gaussian with mean zero and variance dt.

STOCHASTIC BLACK-SCHOLES

Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$

Players: Investor, Market

Protocol:

```
$\mathcal{I}(0) := 0$.
Market announces $S(0) > 0$.
FOR $t = 0, dt, 2dt, \dots, T - dt$:
    Investor announces $\delta(t) \in \mathbb{R}$.
    Market announces $dS(t) \in \mathbb{R}$.
    $S(t + dt) := S(t) + dS(t)$.
    $\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t) dS(t)$.
```

Constraint on Market: Market must choose $dS(t)$ randomly: $dS(t) = \mu S(t) dt + \sigma S(t) dW(t)$, where $W(t)$ is a standard Brownian motion.

With this constraint on Market, $U(S(T))$ is priced:

$$\overline{\mathbb{E}}\, U(S(T)) = \underline{\mathbb{E}}\, U(S(T)) = \int_{-\infty}^{\infty} U\left(S(0)e^{z}\right) \mathcal{N}_{-\sigma^2 T/2,\, \sigma^2 T}(dz).$$

Geometric Brownian Motion

Black-Scholes assumes that Market's moves must obey

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

where $dW(t)$ is Gaussian with mean zero and variance dt.

The Black-Scholes theory uses three consequences of this assumption:

1. **The $\sqrt{dt}$ effect.** The $dS(t)$ have order of magnitude $(dt)^{1/2}$.
2. **Standard deviation proportional to price.** The expected value of $(dS(t))^2$ just before Market makes the move $dS(t)$ is approximately $\sigma^2 S^2(t)dt$.
3. **By the law of large numbers:** When the $(dS(t))^2$ are added, they can be replaced by their expected values, $\sigma^2 S^2(t)dt$.

Point 3 is doubtful, because it is too asymptotic. In real market games, dt is perhaps a day, and N is at most in the thousands.

How the law of large numbers is used

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)$$

$$(dS(t))^2 \approx \sigma^2 S^2(t)(dW(t))^2$$

Because $dW(t)$ is Gaussian with mean zero and variance dt, $(dW(t))^2$ has mean dt and standard deviation Kdt:

$$(dW(t))^2 = dt + z,$$

where z is a fluctuation of order dt. Summing over all N increments, we obtain

$$\sum_{n=0}^{N-1} (dW(ndt))^2 = T + \sum_{n=0}^{N-1} z_n.$$

- $\sum_{n=0}^{N-1} z_n$ has a total variance $2Tdt$. So we may neglect it. We say that the z_n cancel each other out.

- More generally, if the $(dW(t))^2$ are added only after being multiplied by slowly varying coefficients, such as $\sigma^2(S(t))^2$, we can still expect the z_n to cancel out. So we simply replace each $(dW(t))^2$ in the sum with dt.

Here it is crucial that dt be sufficiently small. In financial applications, dt is not small enough!

Deriving the Black-Scholes Formula

Suppose, optimistically, that $U(S(T))$ is priced by some function $\overline{U}(s,t)$ of the current time t and the current price s of the stock. Our task is to find the function $\overline{U}$.

Considering only terms of order $(dt)^{1/2}$ or dt we obtain

$$d\overline{U}(S(t),t) \approx \frac{\partial \overline{U}}{\partial s} dS(t) + \frac{\partial \overline{U}}{\partial t} dt + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(dS(t))^2.$$

- The term in $dS(t)$ is of order $(dt)^{1/2}$.
- The term in dt is of order dt.
- The term in $(dS(t))^2$ is of order dt.

The terms of order dt must be included because their coefficients are always positive and hence their cumulative effect (there are T/dt of them) will be nonnegligible. Individually, the $dS(t)$ are much larger, but because they oscillate between positive and negative values while their coefficient varies slowly, their total effect may be comparable to that of the dt terms.

$$d\overline{U}(S(t),t) \approx \frac{\partial \overline{U}}{\partial s} dS(t) + \frac{\partial \overline{U}}{\partial t} dt + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(dS(t))^2.$$

$S(t)$'s following a geometric Brownian motion means that $(dS(t))^2 \approx \sigma^2 S^2(t)(dW(t))^2$. So the last term in the expansion becomes

$$\frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}\sigma^2 S^2(t)(dW(t))^2.$$

Because the coefficient of $(dW(t))^2$ varies slowly, we can replace it by dt, obtaining

$$d\overline{U}(S(t),t) \approx \frac{\partial \overline{U}}{\partial s} dS(t) + \left(\frac{\partial \overline{U}}{\partial t} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 \overline{U}}{\partial s^2}\right) dt.$$

So $\overline{U}(S(t),t)$ will be a martingale if

$$\frac{\partial \overline{U}}{\partial t} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 \overline{U}}{\partial s^2} = 0.$$

This is the Black-Scholes equation.

Black-Scholes Equation:

$$\frac{\partial \overline{U}}{\partial t} + \frac{1}{2}\sigma^2 S^2(t) \frac{\partial^2 \overline{U}}{\partial s^2} = 0.$$

For any smooth function $\overline{U}(s,t)$,

$$d\overline{U}(S(t),t) \approx \frac{\partial \overline{U}}{\partial s} dS(t) + \left(\frac{\partial \overline{U}}{\partial t} + \frac{1}{2}\sigma^2 S^2(t) \frac{\partial^2 \overline{U}}{\partial s^2} \right) dt.$$

If $\overline{U}$ satisfies the Black-Scholes equation, then Investor can obtain the capital process $\overline{U}(S(t),t)$ by setting his move $\delta(t)$ equal to $\frac{\partial \overline{U}}{\partial s}$. In other words, he holds $\frac{\partial \overline{U}}{\partial s}(S(t),t)$ shares of the security $\mathcal{S}$ from time t to time $t + dt$.

The solution of the Black-Scholes equation with final condition $\overline{U}(s,T) = U(s)$ is

$$\overline{U}(s,t) = \int_{-\infty}^{\infty} U\left(se^z\right) \mathcal{N}_{-\sigma^2(T-t)/2,\, \sigma^2(T-t)}(dz).$$

This is the Black-Scholes formula!

Reorganize the derivation to put the dubious use of the law of large numbers at the end:

$$\boxed{d\overline{U}(S(t),t) \approx \frac{\partial\overline{U}}{\partial s}dS(t) + \frac{\partial\overline{U}}{\partial t}dt + \frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}(dS(t))^2}$$

Consider an investor who holds
$\partial\overline{U}/\partial s$ shares of $\mathcal{S}$, and
$-(1/\sigma^2)\partial\overline{U}/\partial t$ shares of a security $\mathcal{D}$
from t to $t+dt$.

- The capital gain from t to $t+dt$ for a share of $\mathcal{S}$ is $dS(t)$. So the $\partial\overline{U}/\partial s$ shares produce the term in dt.
- **Suppose** the price per share of $\mathcal{D}$ at time t is $\sigma^2(T-t)$. Then the capital gain per share from t to $t+dt$ is $-\sigma^2 dt$. This produces the term in dt.
- **Suppose** each share of $\mathcal{D}$ pays the dividend

  $$-\frac{\sigma^2}{\partial\overline{U}/\partial t}\frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}(dS(t))^2$$

  for the period from t to $t+dt$. This produces the term in $(dS(t))^2$.

Under these curious assumptions,

—$d\overline{U}(S(t),t)$ is the change in the investor's capital,

—$\overline{U}(S(t),t)$ is a martingale, and therefore

—$\overline{U}(S(0),0)$ prices $\overline{U}(S(T),T)$ and hence prices $U(S(T))$ if $\overline{U}$ satisfies $\overline{U}(s,T)=U(s)$.

Why should a security $\mathcal{D}$ that pays

$$-\frac{\sigma^2}{\partial\overline{U}/\partial t}\frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}(dS(t))^2$$

as a dividend per share for the period from t to $t+dt$ have the price per share $\sigma^2(T-t)$?

The Black-Scholes equation,

$$\frac{\partial\overline{U}}{\partial t}+\frac{1}{2}\sigma^2S^2(t)\frac{\partial^2\overline{U}}{\partial s^2}=0,$$

helps. If $\overline{U}$ satisfies it, then

$$-\frac{\sigma^2}{\partial\overline{U}/\partial t}\frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}(dS(t))^2=\left(\frac{dS(t)}{S(t)}\right)^2.$$

The expected value of the sum of $(dS(t)/S(t))^2$ from t to T is $\sigma^2(T-t)$ under geometric Brownian motion. So if $\mathcal{D}$ is traded, and its buyers and sellers believe $S(t)$ follows geometric Brownian motion, then $\overline{U}(S(t),t)$ will be a martingale as required, and $U(S(T))$ will be priced by the Black-Scholes formula.

The final step is to make **dubious use of the law of large numbers** to conclude that $(dS(t)/S(t))^2$ is equal to σ^2dt, so that the dividend from holding $\mathcal{D}$ is exactly canceled out by its capital loss in each period, so that there is no point holding it.

Purely Game-Theoretic Black-Scholes

Investor trades in two securities: $\mathcal{S}$, which pays no dividends and $\mathcal{D}$, which pays the dividend $(dS(t)/S(t))^2$.

Parameters: $T > 0$ and $N \in \mathbb{N}$; $dt := T/N$
Players: Investor, Market
Protocol:

```
  Market announces $S(0) > 0$ and $D(0) > 0$.
  $\mathcal{I}(0) := 0$.
  FOR $t = 0, dt, 2dt, \ldots, T - dt$:
    Investor announces $\delta(t) \in \mathbb{R}$ and $\lambda(t) \in \mathbb{R}$.
    Market announces $dS(t) \in \mathbb{R}$ and $dD(t) \in \mathbb{R}$.
    $S(t + dt) := S(t) + dS(t)$.
    $D(t + dt) := D(t) + dD(t)$.
    $\mathcal{I}(t + dt) := \mathcal{I}(t) + \delta(t) dS(t)$
                    $+\lambda(t) \left( dD(t) + (dS(t)/S(t))^2 \right)$.
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Constraints on Market: (1) $D(t) > 0$ for $0 < t < T$ and $D(T) = 0$, (2) $S(t) \geq 0$ for all t, and (3) the wildness of Market's moves is constrained.

Once D pays its last dividend, at time T, it is worthless: $D(T) = 0$. So Market is constrained to make his $dD(t)$ add to $-D(0)$.

$$d\mathcal{I}(t) = \delta(t) dS(t) + \lambda(t) \left(dD(t) + (dS(t)/S(t))^2 \right)$$

$$d\overline{U}(S(t), D(t)) \approx \frac{\partial \overline{U}}{\partial s} dS(t) + \frac{\partial \overline{U}}{\partial D} dD(t) + \frac{1}{2} \frac{\partial^2 \overline{U}}{\partial s^2} (dS(t))^2$$

Game-theoretic Black-Scholes equation:
We need

$$\delta(t) = \frac{\partial \overline{U}}{\partial s}, \qquad \lambda(t) = \frac{\partial \overline{U}}{\partial D}, \qquad \frac{\lambda(t)}{S^2(t)} = \frac{1}{2} \frac{\partial^2 \overline{U}}{\partial s^2}.$$

The two equations involving $\lambda(t)$ require that the function $\overline{U}$ satisfy

$$-\frac{\partial \overline{U}}{\partial D} + \frac{1}{2} s^2 \frac{\partial^2 \overline{U}}{\partial s^2} = 0$$

for all s and all $D > 0$.

Game-theoretic Black-Scholes formula:
With initial condition $\overline{U}(s, 0) = U(s)$, the solution is

$$\overline{U}(s, D) = \int_{-\infty}^{\infty} U\left(s e^{z}\right) \mathcal{N}_{-D/2, D}(dz).$$

To summarize, the price at time t for the European option $\mathcal{U}$ in a market where both the underlying security $\mathcal{S}$ and a volatility security $\mathcal{D}$ with dividend $(dS(t)/S(t))^2$ are traded is

$$\mathcal{U}(t) = \int_{-\infty}^{\infty} U\left(S(t)e^{z}\right) \mathcal{N}_{-D(t)/2, D(t)}(dz).$$

To hedge this price, we hold a continuously changing portfolio, containing

$$\frac{\partial \overline{U}}{\partial s}(S(t), D(t)) \text{ shares of } \mathcal{S}$$

and

$$\frac{\partial \overline{U}}{\partial D}(S(t), D(t)) \text{ shares of } \mathcal{D}$$

at time t.

Our game-theoretic Black-Scholes theory can be contrasted with two versions of the stochastic theory:

- **Textbook Stochastic Black-Scholes:**
  Security $\mathcal{S}$ is priced by the market. Its price $S(t)$ is assumed to follow geometric Brownian motion; σ^2 can be estimated from past $dS(t)$. All options are priced by plugging the estimate of σ^2 into the Black-Scholes formula.

- **Stochastic Black-Scholes in Practice:**
  Security $\mathcal{S}$ is priced by the market. Puts/calls on $\mathcal{S}$ are also priced by the market. (These form a two-dimensional array: a range of strikes and a range of maturities.) Inconsistencies in the put/call prices of show that the assumption of geometric Brownian motion for $S(t)$ is faulty (volatility smile). So ad hoc adjustments are required to price other options.

- **Game-Theoretic Black-Scholes:**
  Security $\mathcal{S}$ is priced by the market. Dividend-paying security $\mathcal{D}$ is also priced by the market (this is only a one-dimensional array: a range of maturities). All other options on $\mathcal{S}$ are priced by plugging the market price of $\mathcal{D}$ into the Black-Scholes formula. No stochastic assumptions or ad hoc adjustments are required.

We are calling for a far-reaching change in how option exchanges are organized. The change will be hard to sell and complex to implement but should greatly increase efficiency.