

Agent-Based Simulation of Endogenous Coalitions: Some Small-Scale Examples

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Agent-based Computational Economics (ACE) has taken important steps in modeling environments in which agents may be heterogenous and interact directly with one another, so that structures, if observed, emerge from the agents' interactions rather than being assumed by the theorist. (Holland and Miller, Tesfatsion) Game theory has contributed to this study mainly through a limited range of noncooperative game models, particularly models of evolutionarily stable strategies and/or repeated social dilemmas. In these models, coalitions (groups of agents who commit themselves to a joint strategy) play no role. However, in Eric Maskin's words (2004), we live our lives in coalitions. This paper is a preliminary exploration of the potential of agent-based computer simulation as a tool for modeling coalition formation, thus, Agent-Based Simulation of Endogenous Coalitions (ABSEC). At this stage, the study will focus on small-scale examples, which can be fairly readily understood by direct examination, as a means of refining the simulation procedure with a view to the development of larger-scale, more "realistic" models.

1. The Problem of Endogenous Coalitions

There are several concepts of the solution of the game in cooperative game theory. Nevertheless, much of the literature relies on the same group of powerful simplifying assumptions. Among these are “transferable utility,” the representation of the cooperative game in coalition function form and the assumption that the cooperative game must be superadditive. “Transferable utility” means that the benefits gained by choosing a joint strategy can be costlessly redistributed among the members of the coalition so that all are better off. “Transferable utility” will be assumed in this paper. With transferable utility, we may assume that rational agents will choose the joint strategy that maximizes the total payout, and that total payout characterizes the *value* of the coalition. Therefore, it is common in cooperative game theory to ignore all the details and to focus on the total values the various coalitions can obtain, and assign to each coalition a number expressing that value. This assignment is called a “characteristic function” in mathematical set theory and is sometimes called the “coalition function” in cooperative game theory. Representation of the game in coalition function form is a second simplifying assumption in most cooperative game theory. A third simplifying assumption that is usually made is that the game is “superadditive,” that is, a coalition formed by the merger of two or more coalitions will realize a value at least as great as the sum of the values of the coalitions merged. It should be noted that superadditivity is more than an assumption. As Aumann and Dreze (1974 p. 233) note, there are arguments for superadditivity that are quite persuasive. But, as they also note, superadditivity is quite problematic in many economic applications. As they say, “‘acting together’ and sharing the proceeds may change the nature of the game.” (p. 233)

Together, these simplifying assumptions lead to a view of the world that differs from the world we seem to experience in ways that can be very important pragmatically.

To be specific,

- 1) Expressing the game in coalition function form means that it cannot be applied to any case in which there are *externalities*. Indeed, to use Skitovsky's language, pecuniary as well as technological externalities are excluded.
- 2) Superadditivity means that if agents are rational, the grand coalition will always form and will efficiently determine the strategies of every agent. This means it is simply not applicable to any case in which excessive centralization may cause problems. It also means that there can never be stable inefficiencies as problems to be solved. How many cooperative game theorists does it take to change a light bulb? None: if the light bulb needed changing, a coalition would already have formed to change it!

The world we observe, the world relevant to public policy, seems to be one in which many coalitions form and often act independently and indeed competitively with one another. In cooperative game theory such an array of distinct coalitions is called a "coalition structure." (Aumann and Dreze) Neither of these simplifying assumptions strictly excludes stable coalition structures, though if a coalition structure other than the grand coalition were to be stable in a superadditive game, some strong explanation would be needed. However, we would like a theory that would give us some insight as to just what coalition structures would be likely to form, and why, and game theory based on the

second and third simplifying assumptions is not helpful with that. This is the problem of endogenous formation of coalitions. (Carraro)

There are approaches to cooperative game theory that allow both for externalities and coalition structures. As early as 1963 Thrall and Lucas proposed a more complex way of assigning values to coalitions, the *partition function*, that allows for externalities in a broad sense. However, as Aumann and Dreze (1974 p. 233 note) remark, representation of the game in partition function form creates technical (mathematical) difficulties. In intractable mathematical problems, workable solutions may often be found through computer simulation. Accordingly, in this paper, games in partition function form will be studied by means of agent-based simulation of endogenous coalitions.

2. Games in Partition Function Form

Let N be an index set of agents in a game, $a_i \in N$, $i=1, \dots, n$, and let Π_N be the set of all partitions of N and $\mathcal{P} \in \Pi_N$. $\mathcal{P} = \{C_1, C_2, \dots, C_r, \emptyset\}$. $|C_i|$ will denote the number of members in C_i and $|\mathcal{P}|$ will denote the number of nonempty coalitions in \mathcal{P} . A pair $\{\mathcal{P}, C_i\}$ with $C_i \in \mathcal{P}$ is called an embedded coalition and ECL denotes the set of all embedded coalitions. The set ECL is defined for any N . A coalition value function $v(\mathcal{P}, C_i)$ assigns a real number value to coalition C_i in the context of the partition \mathcal{P} . $\Gamma = \{N, v(\mathcal{P}, C_i)\}$ comprises a game in partition function form.

If \mathcal{P} is a partition and $Q = \{B_1, \dots, B_s, \emptyset\}$ is a partition and $\forall i=1, \dots, s, \exists k \in \{1, \dots, r\} \ni C_k \neq \emptyset, B_i \subset C_k \in \mathcal{P}$, then Q is said to be a *refinement* of \mathcal{P} . Remark:

Formally, each partition is a refinement of itself. If $\mathcal{P} \neq \mathcal{Q}$, then \mathcal{Q} is said to be a proper refinement.

The *fine* partition is $\mathcal{P}_F = \{\{a_1\}, \{a_2\}, \dots, \{a_N\}\}$. Trivial lemma: $\forall \mathcal{P} \in \Pi_N, \mathcal{P} \neq \mathcal{P}_F, \mathcal{P}_F$ is a refinement of \mathcal{P} . Let $C_{\mathcal{P}(a_i)}$ denote $C \in \mathcal{P} \ni a_i \in C$. For any $\mathcal{P} \in \Pi_N, \mathcal{P}'(j) = \{C_1, C_2, \dots, C_{\mathcal{P}(a_j)} \setminus a_j, \{a_j\}, C_r, \emptyset\}$. We will refer to $\mathcal{P}'(j)$ as the first refinement of \mathcal{P} with respect to j .

For any $\mathcal{P} \in \Pi_N$ and $S \notin \mathcal{P}, \mathcal{P}'_S = \{B_1, B_2, \dots, B_m, \emptyset\} \ni$

1. $\exists i \in \{1, \dots, m\} \ni B_i = S$
2. $\forall j \neq i, j \in \{1, \dots, m\}, \exists h \in \{1, \dots, r\} \ni B_j = C_h \setminus (C_h \cap S)$
3. $\forall h \in \{1, \dots, r\}$, either $C_h \setminus (C_h \cap S) = \emptyset$ or $\forall j \neq i, j \in \{1, \dots, m\} \ni B_j = C_h \setminus (C_h \cap S)$

\mathcal{P}'_S will be called the residual partition of \mathcal{P} with respect to S .

For $\mathcal{P} \in \Pi_N, S \in \mathcal{P}$, a refinement \mathcal{Q} is said to be *granular* with respect to S, \mathcal{P} , iff

$\forall B \in \mathcal{Q}$, either $B = S$ or $\exists C \in \mathcal{P}, C \neq S, \ni B \subset C$. Trivial lemma: $\forall C \in \mathcal{P}, C \neq S,$

$\exists \mathcal{B} = \{B_j\} \subset \mathcal{Q} \ni C = \bigcup_{B_j \in \mathcal{B}} B_j$. Remark: A granular refinement of \mathcal{P} with respect to S is

one that retains S , leaving the but allows any reorganization of $N \setminus S$. Note that if \mathcal{P} is the grand coalition ($\{\{N\}\}, \{N\}$), it has no proper granular refinement.

For $\mathcal{P} \in \Pi_N, S \in \mathcal{P}$, a refinement \mathcal{Q} is said to be *particulate* with respect to S, \mathcal{P} , iff

$\forall B \in \mathcal{Q}$, either $B \subset S$ or $\exists C \in \mathcal{P}, C \neq S, \ni B = C$. Trivial lemma: for all $C \in \mathcal{P}, C \neq S, \exists B \in \mathcal{Q}$

$\ni B = C$.

Remark: A particulate refinement of \mathcal{P} with respect to S is one that subdivides only S , leaving the other coalitions in \mathcal{P} unchanged. Note that every partition is particulate with respect to the grand coalition ($\{\{N\}\}, \{N\}$).

We now consider a two contrasting examples of the value function. Consider game 1, the “Make Room Game,” a congestion game in which congestion occurs unless all three players concur in restraint of their own congestive activities. We find by inspection that the value of each coalition is unique, so that this game could equally be expressed in coalition function form, without any loss of information.

Game 1. “Make Room”

Coalitions coalition values

¹ {A}{B}{C}	3	3	3
² {A,B}{C}	6	3	
³ {A,C}{B}	6	3	
⁴ {B,C}{A}	6	3	
⁵ {A, B, C}	10		

Game 2. NIMBY

Coalitions coalition values

¹ {A}{B}{C}	3	3	3
² {A,B}{C}	8	6	
³ {A,C}{B}	8	6	
⁴ {B,C}{A}	8	6	
⁵ {A, B, C}	16		

Second, consider Game 2, NIMBY, a game of public good production in which the cost of a public good is less than the total benefit it conveys to a two-person coalition. The benefit is even greater when the cost is shared among all three agents in the grand coalition. Any coalition structure including a two-person or three-person coalition will result in the production of the public good, if the two-person coalition does not consider the benefit to the third party. In the case of a partition with a two-person and a three-person coalition, this creates a positive externality to the third person, so that the value of a singleton coalition depends on whether or not the other two parties coalesce. This game cannot be represented in coalition function form without some assumption to determine which value is assigned to the singleton coalitions.

NIMBY also provides an example of dominance cycles. The grand coalition will be dominated by one of the two-versus-one coalitions, since any agent can get a payoff of 6 by seceding from the grand coalition and taking advantage of the positive externality from the other two. However, each two-versus-one partition is dominated by the grand coalition by a payoff allocation that gives more than 8 (total) to the members of the two-person coalition and more than 6 to the singleton. Such cycles are sometimes considered as shortsighted, in that a farsighted agent would not abandon his present affiliation for another coalition and partition that are themselves unstable in a way to his disadvantage.

If the game is not superadditive then the grand coalition may not be efficient. We can define an efficient partition following Aumann and Dreze. First, if $\Gamma = \{N, v(\mathcal{P}, C_i)\}$ is a game in partition function form, let $\Gamma^* = \{N, v^*(\mathcal{P}, C_i)\}$ be the *superadditive cover* of Γ . The superadditive cover is defined as follows: Let \mathcal{P} be a partition and S the set of all

refinements of \mathcal{P} . (Trivially $\mathcal{P} \in \mathcal{S}$). Then $v^*(\mathcal{P}, C_i) = \text{MAX}_{Q \in \mathcal{S}} \sum_{\substack{B \in Q \\ B \subset C_i}} v(Q, B)$. That is, the value

of an embedded coalition in the superadditive cover is the maximum over all refinements of the partition of the sum of the values of the subsets of that coalition in the original game. Note that the superadditive cover is itself superadditive. A partition Q is efficient if

$$\sum_{B \in Q} v(B) = v^*(N).$$

That is, an efficient partition generates the maximum total value,

summed over all its coalitions, that the game admits of.

Games 1 and 2 are superadditive and in both cases only the grand coalition is efficient. By contrast consider game 3, a four-person game that we shall call the Marshall Game. With four persons there are 15 partitions. This game is not superadditive and is meant to capture, at a very small scale, the idea from Marshallian economics that some kinds of coalitions (business firms) may have an optimal scale. In this case the optimal scale envisioned is 2. The superadditive cover of this game is shown as Game 4. Notice that the only efficient partition in game 3 is 10, $\{1\ 2\}\{3\ 4\}$.

Game 3. Marshall Game

1	{ 1 } { 2 } { 3 } { 4 }	5, 4, 3, 2
2	{ 1 2 } { 3 } { 4 }	12, 2, 1
3	{ 1 3 } { 2 } { 4 }	12, 3, 1
4	{ 1 } { 2 3 } { 4 }	4, 10, 1
5	{ 1 2 3 } { 4 }	15, 3
6	{ 1 4 } { 2 } { 3 }	12, 3, 2
7	{ 1 } { 2 4 } { 3 }	4, 10, 2
8	{ 1 } { 2 } { 3 4 }	4, 3, 10
9	{ 1 2 4 } { 3 }	15, 3
10*	{ 1 2 } { 3 4 }	14, 12
11	{ 1 3 4 } { 2 }	12, 4
12	{ 1 3 } { 2 4 }	10, 10
13	{ 1 4 } { 2 3 }	10, 10
14	{ 1 } { 2 3 4 }	4, 12
15	{ 1 2 3 4 }	16

Game 4. The Superadditive Cover of the Marshall Game

1	{ 1 } { 2 } { 3 } { 4 }	5, 4, 3, 2
2	{ 1 2 } { 3 } { 4 }	12, 3, 2
3	{ 1 3 } { 2 } { 4 }	12, 4, 2
4	{ 1 } { 2 3 } { 4 }	5, 10, 2
5	{ 1 2 3 } { 4 }	15, 3
6	{ 1 4 } { 2 } { 3 }	12, 4, 3
7	{ 1 } { 2 4 } { 3 }	5, 10, 3
8	{ 1 } { 2 } { 3 4 }	5, 4, 10
9	{ 1 2 4 } { 3 }	15, 3
10	{ 1 2 } { 3 4 }	14, 12
11	{ 1 3 4 } { 2 }	14, 4
12	{ 1 3 } { 2 4 }	12, 10
13	{ 1 4 } { 2 3 }	10, 10
14	{ 1 } { 2 3 4 }	4, 13
15	{ 1 2 3 4 }	26

For programming purposes a partition is represented as a zero-based array with upper bound N , $\mathbf{P}(i)$. The number of coalitions in the partition is $\mathbf{P}(0)$, while $\mathbf{P}(i)$ is the number of the coalition to which agent i belongs. Without further restriction this notation would not be unique, as, for example, 1,1,2,3,1,3 and 2,2,3,1,2,1 would designate identical three-coalition partitions for a six-person game. Accordingly, as a convention the coalitions are enumerated in the order of the lowest-numbered member of each coalition. Thus only the first of these arrays is acceptable as a representation of a partition in this study, as the coalition that contains agent 1 must be coalition 1 and the singleton coalition that contains agent 3 must receive a lower number than the coalition comprising agents 4 and 6.

It is not difficult to write an algorithm to generate all possible partitions for a set of cardinality N . There seems to be no simple formula to compute the number of partitions for a set of cardinality N , but computation makes it clear that this number increases very rapidly with N . Table 1 shows the computed number of partitions for $N=1, \dots, 10$.

Table 1. Partitions of a Set of N

N	number of partitions
1	1
2	2
3	5
4	15
5	52
6	203
7	877
8	4140
9	21147
10	115975

This paper will explore some simulations of small-scale games, games of three and four persons. These small games can be understood relatively easily by inspection, while this will not be the case even for moderately large games. The purpose is to validate the simulation methods. However, this will be worthwhile to the extent that the simulations can be scaled to games of more realistic sizes, and this remains a difficult issue.

3. The Simulations

For the simulations, the first step is to generate the set of all partitions for a game of the appropriate size. For these small-scale examples the values of imbedded coalitions are entered manually. Each simulation is then initialized with the fine partition for the particular game. For the naïve dynamics, at each step thereafter, an embedded coalition is proposed that is not an element of the current partition. This is the candidate coalition. The proposed partition will consist of the candidate coalition and the residuals of all other coalitions from the current partition. The residual of a coalition is the coalition of all members who are not in the candidate coalition. (Further dissolution of the residual coalitions would be a further step in the simulation and may or may not occur.) If the members of the candidate coalition can all be better off than they are in the current partition then the proposed partition replaces the current partition; and the division of the net payoff is determined at random; otherwise the current partition is retained for the next step. Figure 1 shows the sequence of coalitions by coalition number for a representative simulation of the NIMBY game for this dynamics.

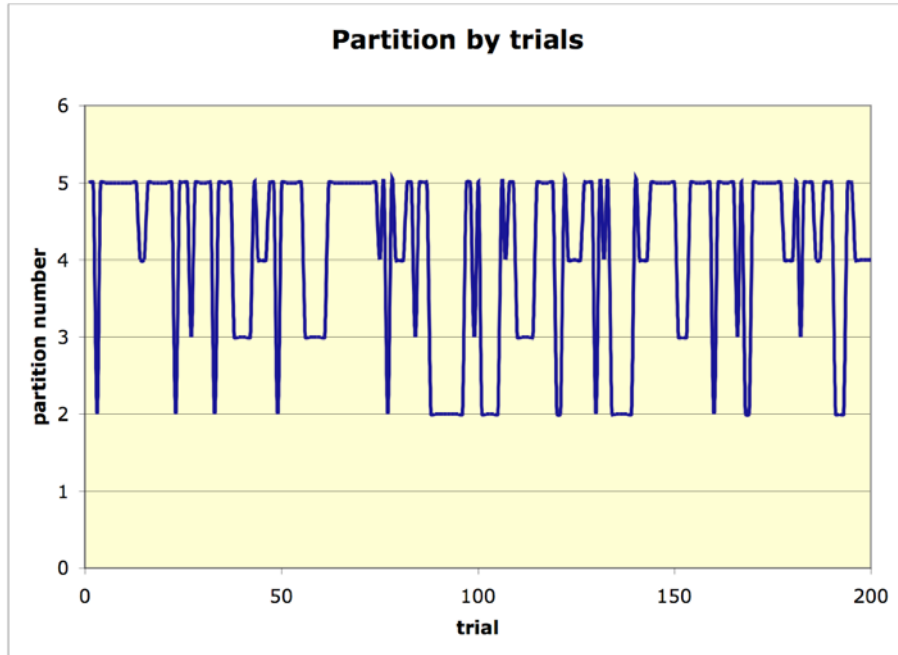


Figure 1. A Representative Sequence of Partitions

The naïve dynamics is based on a simple coalition dominance criterion. As noted in the previous section, this may lead to dominance cycles and consequently may be considered as lacking in foresight. Like rationality in general, foresight may be interpreted in an absolute or a relative and bounded way, and for agent-based computer simulation, this is a crucial point, since the simulation program must convey to the agent the information on the basis of which a foresighted decision may be made. Thus, in practice, bounded rationality (and therefore bounded foresight) is often unavoidable in agent-based simulations. In this study, a second sequence of simulations was conducted for each example in a way that should allow some (bounded) foresight. In the foresight simulations, a record is kept for each agent of the partitions, coalitions, and payoffs the agent has experienced. The first fifty iterations are conducted without reference to these records, as an exploratory learning sequence. An *aspiration* index is computed as the

agent's average payoff when that agent has been a member of the coalition with initiative. For example, suppose a three-person game begins with agents 1 and 2 forming a coalition that remains in place for the first three plays; and then, at the fourth play, 2 and 3 form a coalition that remains for the fifth play. At the sixth play, agent 1's aspiration is the average of his payoffs for the first three plays, agent 3's aspiration is the average of his payoffs for plays 4 and 5, while agent 2's aspirations are the average of his payoffs in all five rounds to date, as he has always been in the coalition that has taken the initiative. After the learning period, the agent will veto any new coalition with a payoff less than his aspiration. As in the naïve game, unanimity within the candidate coalition is required for the transition to be made.

a. The Make Room Game

Game 1, the Make Room Game, is a particularly simple game. As noted above it is superadditive and can be represented in coalition function form. Further, it is symmetrical, the grand coalition is efficient and dominates all other partitions, and the game is free of dominance cycles. It should be no surprise, then, that in the simulations the group move quickly to the grand coalition and remain there. This is equally true of farsighted and naïve simulations and this distinction does not make a difference for the Make Room Game. Table 2 shows the number of times a particular partition is observed, by number, for nine naïve simulations with nine distinct random seeds. The results for farsighted simulations were identical. Because the simulations stabilized in the first few iterations, the aspirations never played any role in transitions from one partition to another. This accounts for the identical results in the two series of simulations.

Table 2. Partitions Observed in Game 1

partition	random seeds		
	137030	250523	355516
1	2	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	198	200	200
	453382	546750	656110
1	1	0	2
2	0	0	0
3	0	0	0
4	0	0	0
5	199	200	198
	776298	849321	975664
1	0	0	1
2	0	0	0
3	0	0	0
4	0	0	0
5	200	200	199

b. The NIMBY Game and a Modification

The NIMBY game is also symmetrical and superadditive, and the grand coalition is efficient; but it is more complex in that it includes positive externalities and dominance cycles. In Maskin’s view, positive externalities may lead to the formation of coalition structures other than the grand coalition. As we saw in Figure 1, there is some evidence of cycles in the naïve simulations of the NIMBY game. Figure 2 shows the sequence of payoffs to the three agents in the same simulation (with random number seed 564750) which further indicates the cyclical nature of the simulation.

By contrast, with the farsighted simulation dynamics, the cyclical tendency was not observed after the learning period. Instead, each of the trials stabilized at one or another of the two-versus-one coalition structures. In this case, it might appear that the

positive externality was an obstacle to the formation of the efficient grand coalition. Table 3 summarizes the partitions seen in each of the eighteen simulations of this game. The average number of times a partition was observed is shown for the naïve and farsighted simulations, followed by the standard deviation. We notice that the (efficient) grand coalition occurred much more frequently in the naïve simulations. A result was that the payoffs to the agents were, on the whole, slightly better in the naïve simulations, as shown in Table 4. The standard deviations for the number of observations of partitions 2, 3, 4 in the farsighted simulations are quite large, reflecting the fact that in a particular simulation one of these partitions either was observed quite frequently (if the simulation stabilized on it) or very rarely (if not). The larger standard deviations in Table 4 reflect the fact that the distribution of payoffs in stabilized simulations tended to be somewhat unequal, while the payoffs in the naïve simulations continued to be roughly randomly distributed.

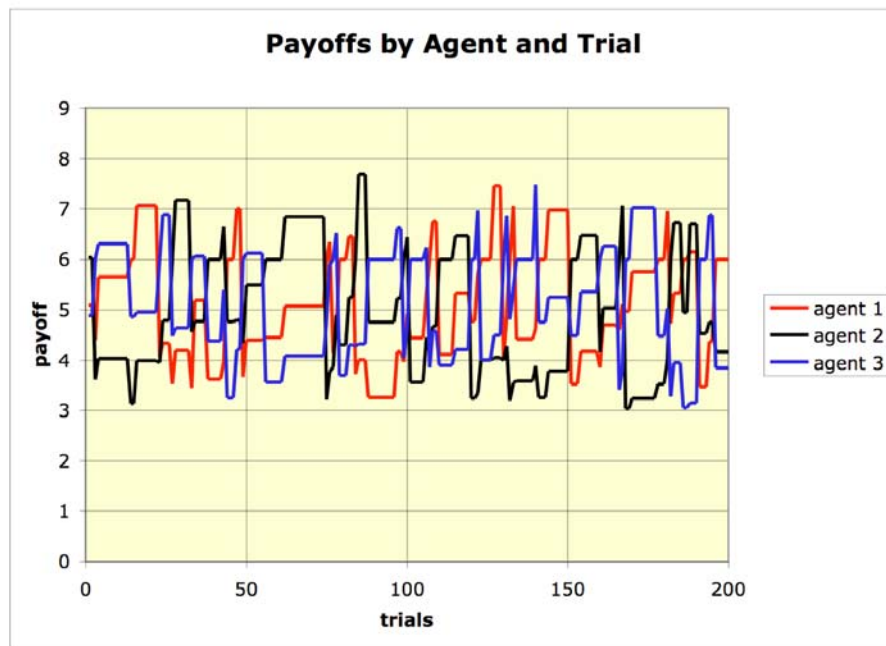


Figure 2. Payoffs in a Naïve Simulation of Game 2

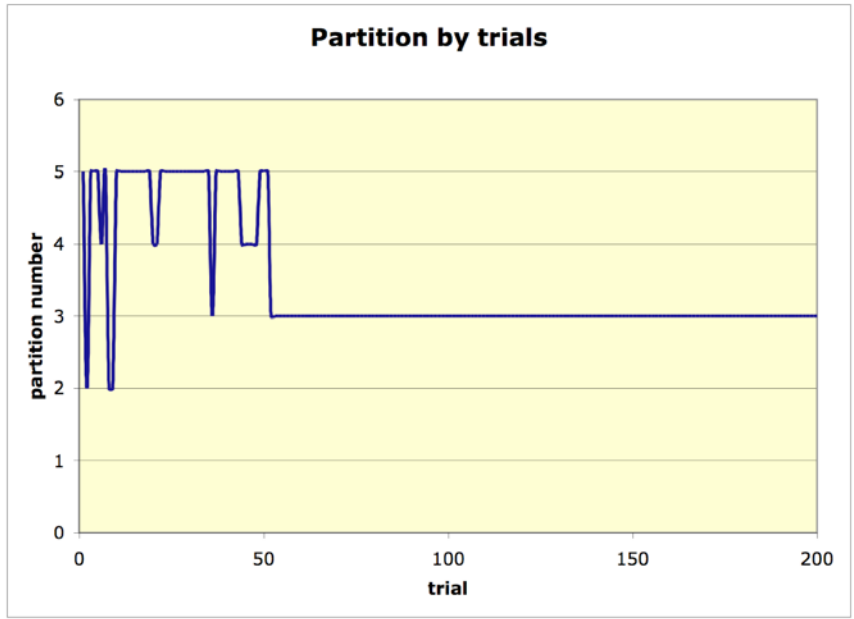


Figure 3. Partitions in an Otherwise Similar Farsighted Simulation

Table 3. Partitions Observed in Game 2

partition	naïve simulations		farsighted simulations	
	average	S. D	average	S. D.
1	0.33	0.50	0.22	0.44
2	29.89	5.84	40.33	64.85
3	32.00	7.97	74.22	76.50
4	29.33	9.01	56.67	74.03
5	108.44	11.01	28.56	6.52

The instability of the naïve simulations of the NIMBY game is attributable to the dominance cycles, not to the positive externality per se. This conclusion follows from simulations of the following slightly modified NIMBY game, Game 5. The modification is an increase in the value of the grand coalition so that it is dominant and undominated; dominance cycles are eliminated. Nevertheless, the random element in the determination of payoff shares results in some instability in some simulations. A majority of the

simulations of both types stabilize on the efficient grand coalition, while two of the nine naïve simulations do not stabilize, and four of the nine farsighted simulations ended with one of the two-against one partitions. It may be that these would stabilize on the grand coalition (with sufficiently equal payoff schedules) in more extended simulations. Results are summarized in tables 5 and 6.

Table 4. Payoffs in Game 2

naïve simulations				
agent	1	2	3	overall
average payoff	5.09	5.01	4.98	5.03
S. D.	0.14	0.18	0.07	0.14
farsighted simulations				
agent	1	2	3	overall
average payoff	4.76	5.10	4.42	4.76
S. D.	0.84	0.68	0.88	0.82

Game 5. A Modified NIMBY Game

1	{ 1 } { 2 } { 3 }	3, 3, 3
2	{ 1 2 } { 3 }	8, 6
3	{ 1 3 } { 2 }	8, 6
4	{ 1 } { 2 3 }	6, 8
5	{ 1 2 3 }	20

Table 5. Partitions Observed in Game 5

	naïve average	S. D.	farsighted average	S. D.
1	0.56	0.88	45.78	56.72
2	6.67	7.66	86.33	67.61
3	4.11	6.29	4.56	6.52
4	9.78	9.47	5.22	8.42
5	178.89	21.84	58.11	21.81

Table 6. Payoffs in Game 5

agent	naïve				farsighted			
	1	2	3	overall	1	2	3	overall
average	6.26	6.41	6.69	6.45	5.18	5.13	5.43	5.25
SD	0.40	0.48	0.55	0.47	1.82	1.63	2.23	1.84

c. An Exploiter Game

Now consider game 6, below. This game can be thought of as expressing exploitation in either of two senses. One the one hand, in the fine partition agent 3 may be seen as exploiting agents 1 and 2 because of their failure to work together, so that when 1 and 2 form a coalition they are able to eliminate the exploitation. On the other hand, it may be that in forming a coalition, 1 and 2 “gang up” on 3 to exploit him. In any case this game is highly unsymmetrical, but is superadditive and the grand coalition is efficient.

Game 6. The Exploiter Game

1	{ 1 } { 2 } { 3 }	1, 1, 11
2	{ 1 2 } { 3 }	14, 2
3	{ 1 3 } { 2 }	12, 1
4	{ 1 } { 2 3 }	1, 12
5	{ 1 2 3 }	18

In naïve simulations of Game 6, the efficient partition 5 was observed in a large majority of the cases, and could remain in place for multiple iterations, but there seems to have been no tendency for the simulations to stabilize. A typical instance is shown as Figure 4. By contrast, the farsighted simulations always stabilized either on the fine coalition or coalition 2, but the number of iterations needed to attain stability varied

widely. Several simulations remained at the efficient grand coalition for extended periods before moving to a stable partition. An example in which the coalition 2 is the stable outcome is shown in Figure 5. (Figures 4 and 5 were both simulations with the random seed 355516.) The partitions observed are summarized in Table 7. Here again, the payoffs to the agents are slightly better overall in the naïve simulations.

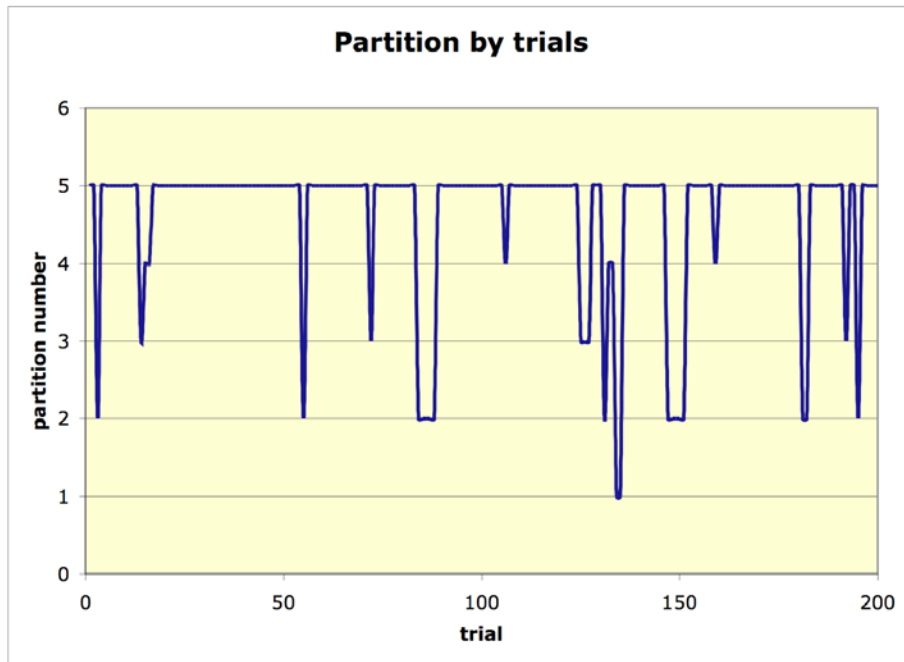


Figure 4. Partitions in a Typical Naïve Sequence in Game 6.



Figure 5. Partitions in a Typical Farsighted Sequence in Game 6.

Table 7. Partitions Observed in Game 6

	naïve		farsighted	
	average	S. D.	average	S. D.
1	5.44	4.07	0.44	0.73
2	12.22	5.47	17.67	38.70
3	8.22	4.63	8.11	14.37
4	5.78	2.39	24.67	40.42
5	168.33	10.16	149.11	49.79

Table 8. Payoffs in Game 6

agent	naïve				farsighted			
	1	2	3	overall	1	2	3	overall
average	5.63	5.65	6.12	5.80	5.18	5.13	5.43	5.25
SD	0.91	1.12	1.33	1.05	1.82	1.63	2.23	1.84

d. The Marshalllean Game

Now, consider again Game 3. As previously noted, this four-person game is not superadditive but has an efficient coalition structure corresponding to an optimal size and array of coalitions, namely, partition 10. In four of nine naïve simulations, the process stabilizes at the efficient partition. Figure 6 shows a representative case in which the simulation does stabilize. However, in other cases there is no evidence that the simulation reaches a stable state within the 200 iterations of the simulation. Figure 7 shows such an instance. When the naïve simulations do stabilize they do so at the efficient partition. The overall experience is summarized in Table 7. In the farsighted partitions stabilization takes place in seven of nine simulations and in a majority of cases stabilizes on partition 10. Figure 9 is a representative instance. In two cases the farsighted simulation stabilizes on, respectively, inefficient partitions 3 and 6. The overall experience is summarized in Tables 9 and 10.

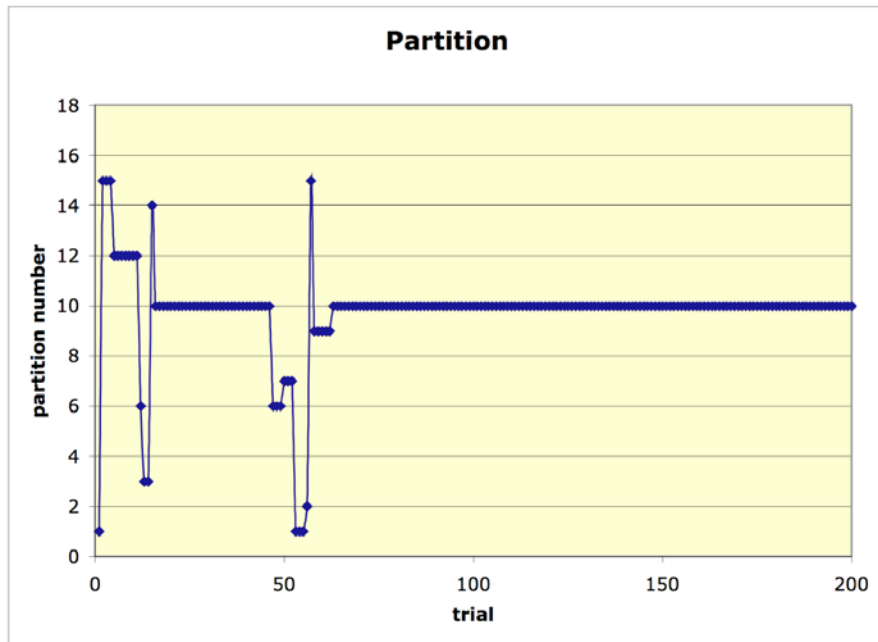


Figure 6. A Stabilized Naïve Simulation of Game 3.

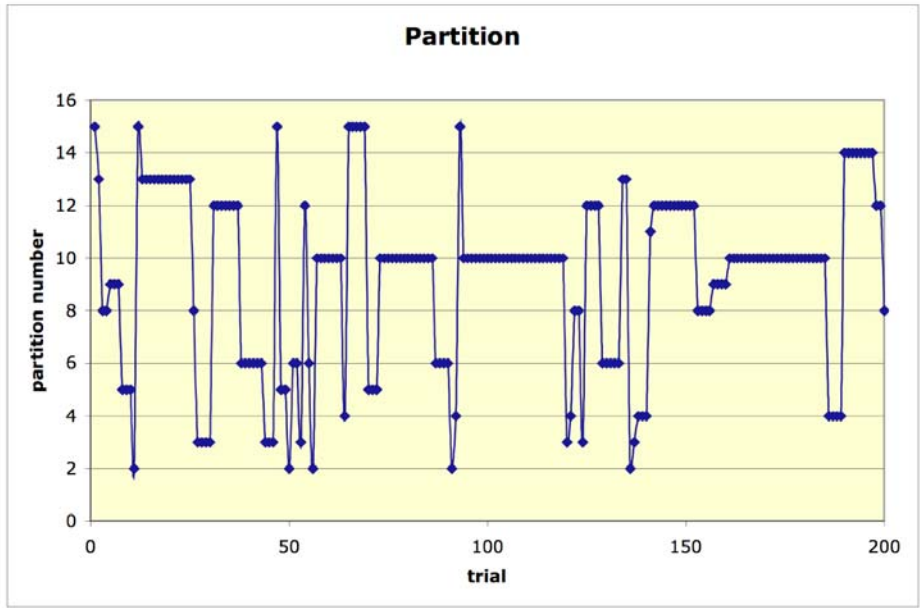


Figure 7. A Simulation that Does Not Stabilize in Game 3

Table 9. Partitions Observed in the Simulations of Game 3

partition	naïve		farsighted	
	average	S D	average	S D
1	1.78	1.64	0.44	1.01
2	4.56	3.47	4.22	3.56
3	10.78	10.08	23.00	39.55
4	4.67	4.09	3.33	7.09
5	5.11	4.86	3.44	5.15
6	15.33	13.39	29.78	48.96
7	7.56	8.53	7.78	10.56
8	11.56	8.00	6.22	6.10
9	4.78	5.70	1.78	3.11
10	83.78	56.94	109.11	65.92
11	1.56	1.33	0.22	0.44
12	17.44	8.06	2.22	3.07
13	15.56	11.88	3.44	6.86
14	4.67	5.22	0.89	1.36
15	10.89	8.57	4.11	2.52

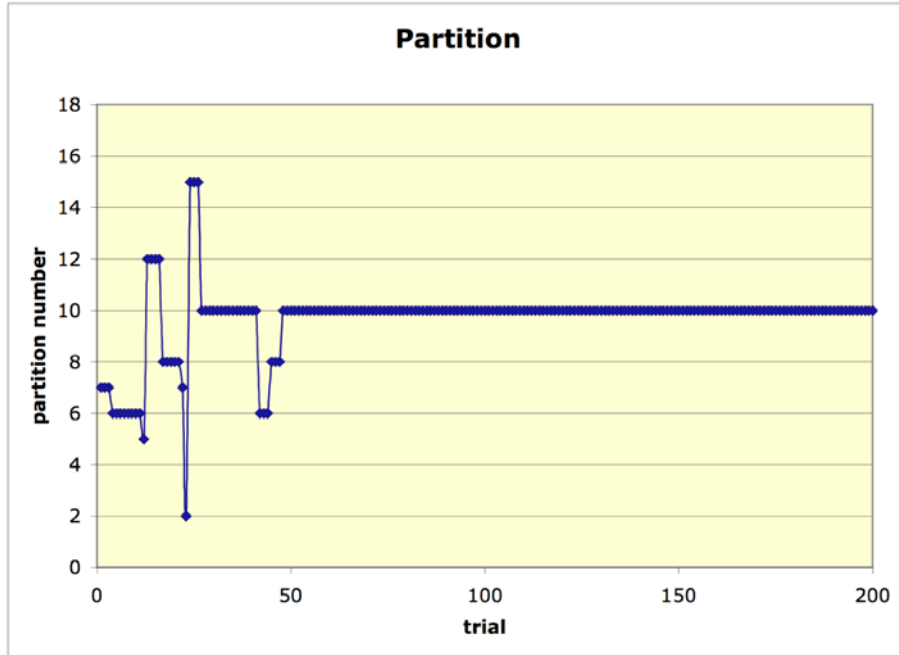


Figure 8. A Representative Farsighted Simulation of Game 3

Table 10. Payoffs in the Simulations of Game 3

naïve					
agent	1	2	3	4	overall
Average	6.56	5.36	4.80	4.28	5.25
SD	0.64	0.71	0.83	0.75	1.11
Farsighted					
agent	1	2	3	4	overall
Average	6.92	5.26	5.03	4.54	5.44
SD	0.88	0.97	1.06	1.27	1.35

e. Positive Externalities

Game 7 is a mildly unsymmetrical four-person game with many positive externalities. The game is, however, superadditive and the grand coalition is the only efficient partition. For all naïve simulations of this game, the process moved to the efficient grand coalition and remained there, the positive externalities creating no

obstacle. In the farsighted simulations the process also stabilized, and usually on the efficient grand coalition; but there were two exceptions that stabilized on partitions 11 and 13. Tables 11 and 12 summarize the partitions observed and payoffs respectively.

Game 7. Positive Externality Game

1	$\{1\}\{2\}\{3\}\{4\}$	2, 2, 2, 3
2	$\{12\}\{3\}\{4\}$	10, 3, 3
3	$\{13\}\{2\}\{4\}$	10, 3, 3
4	$\{1\}\{23\}\{4\}$	3, 10, 4
5	$\{123\}\{4\}$	20, 5
6	$\{14\}\{2\}\{3\}$	12, 4, 4
7	$\{1\}\{24\}\{3\}$	3, 10, 3
8	$\{1\}\{2\}\{34\}$	3, 3, 10
9	$\{124\}\{3\}$	20, 5
10	$\{12\}\{34\}$	12, 12
11	$\{134\}\{2\}$	22, 6
12	$\{13\}\{24\}$	12, 12
13	$\{14\}\{23\}$	14, 14
14	$\{1\}\{234\}$	5, 20
15	$\{1234\}$	30

Table 11. Partitions Observed in Simulations of Game 7

partition	naïve		farsighted	
	average	S D	average	S D
1	0.11	0.33	0.25	0.46
2	0.33	0.71	0.13	0.35
3	0.00	0.00	0.00	0.00
4	0.00	0.00	0.13	0.35
5	0.11	0.33	0.88	1.46
6	0.22	0.44	0.13	0.35
7	0.00	0.00	0.25	0.46
8	0.56	1.33	0.25	0.46
9	2.22	3.96	0.50	1.41
10	0.67	1.32	3.63	4.03
11	3.67	5.17	18.88	46.58
12	1.89	2.32	0.63	0.92
13	3.67	3.54	14.00	32.04
14	1.44	2.92	2.25	3.62
15	185.11	14.49	158.13	51.84

Table 12. Payoffs Observed in Simulations of Game 7

naïve agent	1	2	3	4	overall
Average	7.36	7.31	7.32	7.68	7.42
SD	1.03	0.65	0.76	1.08	0.87

Farsighted agent	1	2	3	4	overall
Average	6.76	7.19	7.31	8.14	7.35
SD	0.71	0.75	0.68	0.76	0.86

f. Countervailing Power

Game 8 is a four-person game suggested by Galbraith’s hypothesis of countervailing power. This game is not superadditive, and the only efficient partition is the fine partition. Generally, formation of coalitions decreases the average payoff, but with the following exceptions: if either 1 and 2 or 3 and 4 form two-person coalitions, the members of the coalition gain a substantial advantage while imposing a large negative externality on the other, unorganized pair; while if both {1,2} and {3,4} coalesce, the

outcome approximates the efficient fine partition but with some loss of its efficiency. This implies a complicated dominance cycle as the fine partition is dominated (via $\{1,2\}$) by partition 2 and (via $\{3,4\}$) by partition 8. Both of these, in turn, are dominated by partition 10, which in turn is dominated by the fine partition, although the fine partition is not accessible at one step from partition 10. However, depending on payoffs, which have a random component, partitions such as 3 and 4 may dominate 10 and are in turn dominated by 1, the fine partition.

In this game, the naïve simulations never stabilize. The farsighted simulations always stabilize, but never on the efficient fine partition. In several cases the farsighted simulations stabilize on partition 10, the countervailing power partition, but both partitions 2 and 8, in which one pair benefits at the other's expense, are also observed. Partition 8 was most frequently observed, but this is probably a sampling fluctuation. Table 13 summarizes the partitions observed and Table 14 summarizes the payoffs observed in both series of simulations. Payoffs are highly unequal in the farsighted simulations because they are highly unequal in partitions 2 and 8, which are relatively frequently observed.

Game 8. A Countervailing Power Game

1*	{ 1 } { 2 } { 3 } { 4 }	5, 5, 5, 5
2	{ 1 2 } { 3 } { 4 }	15, 1, 1
3	{ 1 3 } { 2 } { 4 }	9, 5, 5
4	{ 1 } { 2 3 } { 4 }	5, 9, 5
5	{ 1 2 3 } { 4 }	12, 5
6	{ 1 4 } { 2 } { 3 }	9, 5, 5
7	{ 1 } { 2 4 } { 3 }	5, 9, 5
8	{ 1 } { 2 } { 3 4 }	1, 1, 15
9	{ 1 2 4 } { 3 }	12, 5
10**	{ 1 2 } { 3 4 }	9, 9
11	{ 1 3 4 } { 2 }	12, 5
12	{ 1 3 } { 2 4 }	8, 8
13	{ 1 4 } { 2 3 }	8, 8
14	{ 1 } { 2 3 4 }	5, 12
15	{ 1 2 3 4 }	16

Table 13 Partitions Observed in Game 8

partition	naïve		farsighted	
	average	S D	average	S D
1	65.44	22.63	19.88	12.49
2	24.00	11.27	23.00	50.65
3	10.78	8.90	4.25	2.92
4	11.33	5.96	3.75	3.58
5	3.89	2.42	1.25	2.38
6	13.89	5.30	2.13	3.27
7	13.33	9.89	8.75	8.51
8	29.56	8.83	73.00	72.17
9	4.56	4.19	0.00	0.00
10	17.44	20.34	63.50	66.84
11	1.78	2.28	0.38	1.06
12	2.00	3.91	0.00	0.00
13	1.00	1.41	0.00	0.00
14	1.00	1.22	0.13	0.35
15	0.00	0.00	0.00	0.00

Table 14. Payoffs Observed in Game 8

naïve agent					
	1	2	3	4	overall
Average	4.55	4.47	4.85	4.68	4.64
SD	0.41	0.34	0.43	0.48	0.43
Farsighted agent					
	1	2	3	4	overall
Average	3.36	3.56	5.51	5.32	4.44
SD	1.83	1.78	2.02	1.88	2.04

g. A Game with a Dummy

In some cooperative game theory, the treatment of “dummy players” is a key assumption of the model. (Shapley, e.g.) If player k is a dummy player, then for any partition \mathcal{P} and coalition $C \neq \{k\}$ such that $k \in C$, $v(\mathcal{P}, C) - v(\mathcal{P}'(k), C \setminus \{k\}) = v(\{k\})$. (Recall, $\mathcal{P}'(k)$, the first refinement of \mathcal{P} with respect to k , is formed by dropping k from the coalition C). In other words, the dummy player never adds more to a coalition than his own value as a singleton coalition. Some definitions state also that $v(\{k\}) = 0$. For cooperative game solutions such as the Shapley value, the prediction is that a dummy player’s payout in a cooperative game will be exactly the dummy’s singleton value. Game 9 is a four-person superadditive game that can be represented in coalition function form, in which player 1 is a dummy player with singleton value 3.

Game 9. A Game with a Dummy Player

1	{ 1 } { 2 } { 3 } { 4 }	3, 9, 8, 7
2	{ 1 2 } { 3 } { 4 }	12, 8, 7
3	{ 1 3 } { 2 } { 4 }	11, 8, 7
4	{ 1 } { 2 3 } { 4 }	3, 20, 7
5	{ 1 2 3 } { 4 }	23, 7
6	{ 1 4 } { 2 } { 3 }	10, 9, 8
7	{ 1 } { 2 4 } { 3 }	3, 20, 8
8	{ 1 } { 2 } { 3 4 }	3, 9, 20
9	{ 1 2 4 } { 3 }	23, 8
10	{ 1 2 } { 3 4 }	12, 20
11	{ 1 3 4 } { 2 }	23, 9
12	{ 1 3 } { 2 4 }	11, 20
13	{ 1 4 } { 2 3 }	10, 20
14*	{ 1 } { 2 3 4 }	3, 40
15*	{ 1 2 3 4 }	43

Although Game 9 is superadditive, both partitions 15 (the grand coalition) and 14 were efficient. Indeed, a game with a dummy player cannot have a unique efficient partition: given an efficient partition the shift of the dummy player to any other coalition (including a singleton coalition) will leave the total value unchanged and so result in another efficient partition. For this game, all naïve simulations moved quickly to one of the efficient partitions, and stabilized at partition 14. The farsighted simulations behaved similarly except that in two cases, they remained at partition 15. The average payoffs were as shown in Table 15. We see that both families of simulations approximate the hypothetical payoff of 3 for the dummy agent, with some deviation, but also that the naïve simulations are actually a little more predictable than the farsighted ones in this case. This reflects the fact that in simulations in which the grand coalition occurs, the transition from that partition (in which the dummy will have some payoff over 3) to

partition 14 (in which the dummy is a singleton and receives just 3) either does not occur or takes longer in the farsighted simulations.

Table 15. Average Payoffs in Simulations of Game 9

naïve agent					
	1	2	3	4	overall
Average	3.36	14.96	12.98	11.63	10.73
SD	0.43	2.28	2.15	1.97	4.82
Farsighted agent					
	1	2	3	4	overall
Average	3.96	14.80	12.38	11.77	10.73
SD	0.88	1.60	1.72	2.27	4.43

While the dummy player hypothesis directs our attention to the payoffs, examination of the partitions observed in these simulations is useful. The rapid and stable convergence of these simulations to efficient partitions is partly a consequence of the fact that the efficient partitions are strongly dominant by a large margin. By contrast consider game 10, which replicates game 9 except for the payoffs to the last two partitions, which will be 3, 30 for partition 14 and 33 for partition 15. While these partitions can be stable with payments of exactly 10 to agents 2, 3, 4, the element of random variation in the distribution of payoffs will induce dominance cycles. Suppose, for example, we have {2, 3, 4} with payoffs 12, 9, 9. This is dominated by {3, 4} with 10, 10; which in turn is dominated by {2, 3} with 9, 11; which in turn is dominated by {2, 3, 4} with 10, 12, 8.

A representative example is shown in Figure 10, a naïve simulation of game 10 with random number seed 849321. For comparison, a naïve simulation of game 9 with the same random number seed is shown in Figure 11. Figure 12 shows the corresponding process of payments to the four participants in the simulation of game 10, with a visual impression of a cyclical process. The farsighted simulations of game 10 are less evidently

cyclical, stabilizing in 6 of 9 cases, but never at an efficient partition. Instead partitions 4 and 8 were both observed twice and 7 and 12 once. Table 16 compares the overall frequency of partitions observed in games 9 and 10.

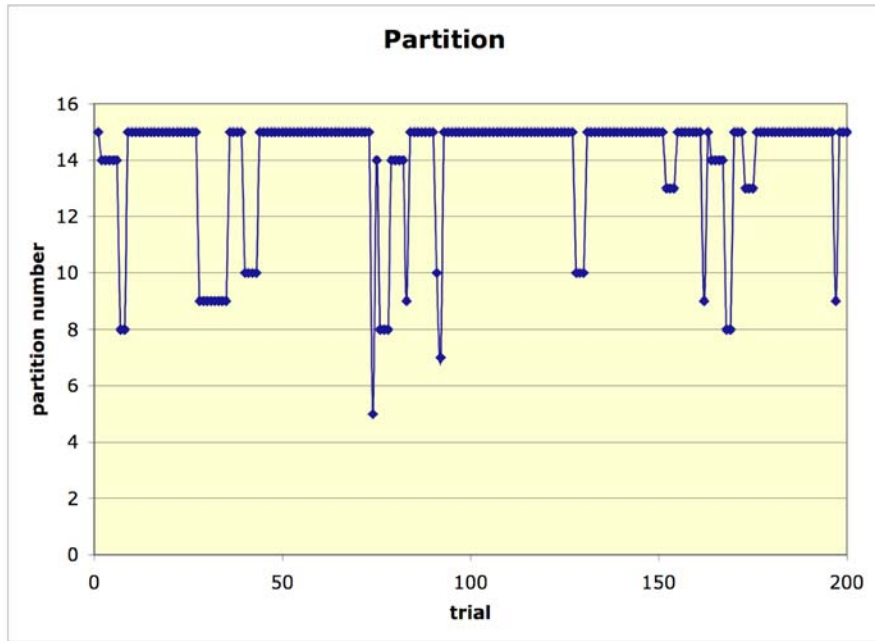


Figure 10. Partitions Observed in a Naïve Simulation of Game 10

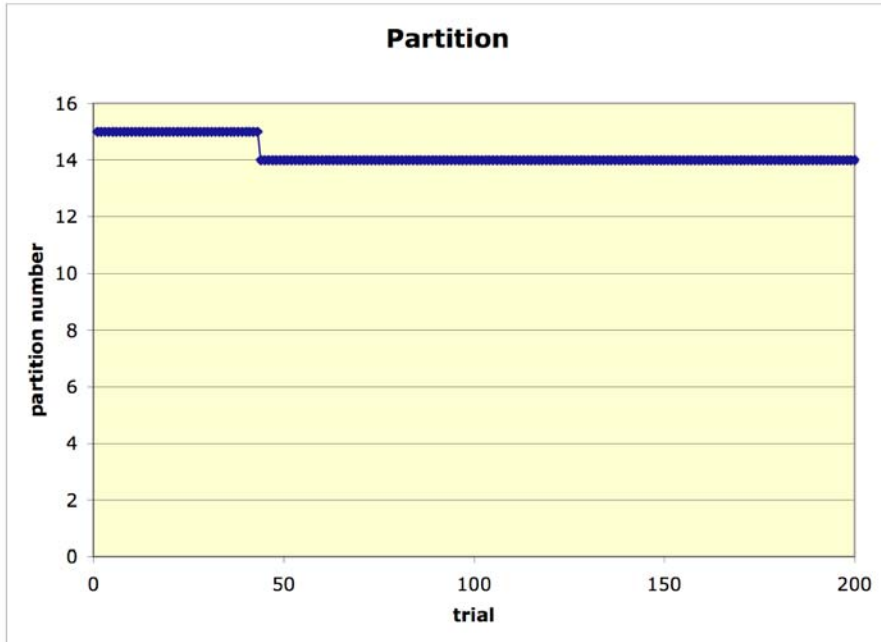


Figure 11. Partitions Observed in a Naïve Simulation of Game 9

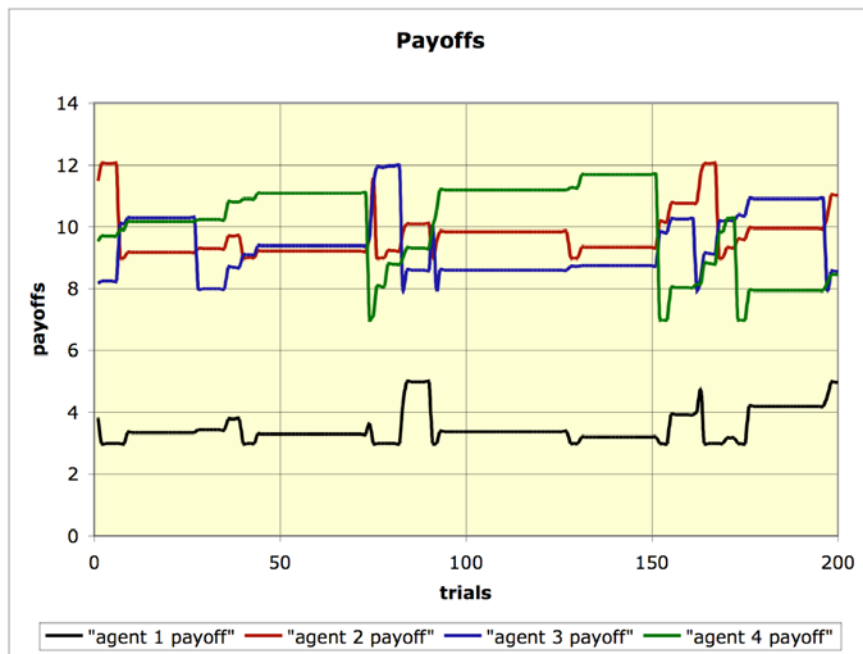


Figure 12. Payoffs in a Naïve Simulation of Game 10

Table 16. Partitions Observed in Simulations of Games 9 and 10

partition	Game 9				Game 10			
	naive		farsighted		naive		farsighted	
	average	S D	average	S D	average	S D	average	S D
1	0.33	0.71	0.50	0.53	0.56	0.73	0.38	0.52
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	3.22	2.59	47.13	65.45
5	0.33	1.00	0.13	0.35	3.78	4.82	0.88	1.36
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.11	0.33	0.13	0.35	5.33	3.08	16.38	28.72
8	0.11	0.33	0.00	0.00	6.78	3.70	47.88	60.59
9	0.00	0.00	0.38	1.06	6.44	3.50	0.25	0.71
10	0.00	0.00	0.00	0.00	6.78	1.56	10.88	28.35
11	0.11	0.33	0.00	0.00	5.44	4.95	2.25	3.41
12	0.11	0.33	0.00	0.00	4.89	3.59	17.38	38.93
13	0.00	0.00	0.00	0.00	5.11	2.57	1.75	2.31
14	170.89	27.23	133.63	66.59	45.78	25.13	22.25	14.68
15	28.00	27.47	65.25	67.11	105.89	26.01	32.63	14.97

h. A John Bates Clark Game

As Aumann and Dreze say, “‘acting together’ and sharing the proceeds may change the nature of the game.” According to John Bates Clark, the formation of a coalition for purposes of production of goods and services changes the nature of the game in that it implies the need for a specialist in the entrepreneurial function of coordination. Game 11 is suggested by that idea. In Game 11, player 1 is a pure coordinator, who produces nothing in a singleton coalition. The other players are craftsmen who can produce positive amounts working as individual craftsmen in singleton coalition. However, a coalition of two or more of agents 2, 3, 4 produces nothing unless player 1 is also a member (since it would be uncoordinated) but produces an amount that depends only on the number of non-entrepreneur members, supposing player 1 is present. In this game there are increasing returns to employment scale, with production (coalition value) of 5 in a coalition containing one craftsman, 15 in a coalition containing 2 craftsmen, or

26 in the grand coalition of three craftsmen together with the entrepreneur. As a result, the grand coalition is the only efficient partition in this game. Nevertheless, the game is not superadditive, and the complexity of its nonsuperadditive structure is one reason for its interest. Partitions 2-14 are all dominated by *both* partition 1, the fine partition, and by partition 15, the grand coalition, so there can be no intermediate stages.

Game 11. A John Bates Clark Game

1	{ 1 } { 2 } { 3 } { 4 }	0, 9, 8, 7
2	{ 1 2 } { 3 } { 4 }	5, 8, 7
3	{ 1 3 } { 2 } { 4 }	5, 9, 7
4	{ 1 } { 2 3 } { 4 }	0, 0, 7
5	{ 1 2 3 } { 4 }	15, 7
6	{ 1 4 } { 2 } { 3 }	5, 9, 8
7	{ 1 } { 2 4 } { 3 }	0, 0, 8
8	{ 1 } { 2 } { 3 4 }	0, 9, 0
9	{ 1 2 4 } { 3 }	15, 8
10	{ 1 2 } { 3 4 }	5, 0
11	{ 1 3 4 } { 2 }	15, 8
12	{ 1 3 } { 2 4 }	5, 0
13	{ 1 4 } { 2 3 }	5, 0
14	{ 1 } { 2 3 4 }	0, 0
15*	{ 1 2 3 4 }	26

The key result is that both naïve and farsighted simulations move swiftly to the efficient grand coalition and remain there. The complex nonsuperadditivity of this model is not an obstacle to efficiency in these simulations. Table 17 shows the frequency with which different partitions are observed in the simulations of Game 11. Payoffs are shown in Table 18. We see that the entrepreneur, indispensable as he is, makes very little.

Table 17. Partitions Observed in Simulations of Game 11

partition	naïve		farsighted	
	average	S D	average	S D
1	1.00	1.00	0.88	0.99
2	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00
11	0.00	0.00	0.00	0.00
12	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00
15	199.00	1.00	199.13	0.99

Table 18. Payoffs in Simulations of Game 11

naïve agent	1	2	3	4	overall
Average	0.35	9.58	8.56	7.50	6.50
SD	0.17	0.27	0.25	0.24	3.68
Farsighted agent					
Farsighted agent	1	2	3	4	overall
Average	0.39	9.54	8.53	7.53	6.50
SD	0.16	0.26	0.28	0.27	3.67

4. Conclusions

The simulations reported here were based on a variety of three and four-person games. In games that incorporate the usual simplifying assumptions of cooperative game theory (that are superadditive and expressible in coalition function form) the simulations move rapidly to efficient partitions and to remain there, very much as conventional cooperative game theory might predict. More generally, when an efficient partition is strongly dominant and unique, it is a very strong attractor, and positive and negative

externalities do not seem to create problems. When there are dominance cycles, however, they can be highly disruptive of convergence and stabilization. Naïve simulations tend to return often to an efficient partition, so, although they do not stabilize, efficiency can be a predictor of the relative frequency with which particular partitions are observed. In most cases, farsighted simulations stabilize when there are dominance cycles. When externalities and dominance cycles are combined, it was fairly common for the farsighted simulations to stabilize on inefficient partitions, perhaps a “realistic” result. This should be considered tentative and relies very much on the concept of farsightedness postulated here. The main conclusion to be drawn from these simulations is that externalities are more likely to lead to inefficiency if they are linked with dominance cycles or weak dominance, and may be far less important otherwise.

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