Survival of Noise Traders and the Implications for Asset Prices*

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This Version: 31st March 2006
First Version: August 8, 2005

*The authors thank particularly Thorsten Hens and János Mayer for valuable input. We are grateful to Dana Bucher, Daniel Egloff, Philipp Halbherr, Stefan Reimann, Klaus Reiner Schenk-Hoppé, Mark Schindler, and Andreas Tupak for their comments. The authors welcome comments and suggestions, including references to related papers inadvertently overlooked. Financial support by the national center of competence in research “Financial Valuation and Risk Management” is gratefully acknowledged. The national centers in research are managed by the Swiss National Science Foundation on behalf of the federal authorities.

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Abstract

This paper shows that in financial markets with endogenous asset supply and demand, both rational and noise traders do co-exist in the long run. The finding implies that financial markets are neither informationally nor allocationally efficient. While rational traders have a consistently higher cash inflow from dividends, noise trader are able to speculate successfully on capital gains. Thus, investors who need a regular cash inflow should invest rationally, i.e., hold a portfolio proportional to the expected relative dividends of its assets. Investors interested in maximum capital gains may try and realize the alpha opportunities in the market. To succeed, their timing must be optimal, however. The worst strategy is 'buy and hold': Neither does it lead to a cash inflow like the rational strategy, nor does it open the chance to realize capital gains. We would like to mention that our evolutionary model does not face problems of common interactive agent models such as infinite supply of riskless assets or the lack of modelling the evolution of the investor’s wealth.

Moreover, we investigate the two most prominent puzzles related to low-frequency stock prices: The conditional volatility of price returns, and the price forecasting power of dividend yields. Using the indirect inference methodology, we estimate an version of the evolutionary stock market model with endogenous asset supply and demand. We find that our economically well founded model is able to approximate the conditional volatility, quantified with a GARCH(1,1) process, that is observed in empirical price data. To investigate the price forecasting power of dividend yields, we derive for rational markets a close form of linking dividend yields to future prices, inherently de-trending the price process. Based on this result, we show that the prices of roughly two third of the considered assets depend on the dividend yields. The prices of the other assets deviate from the rational values due to the presence of noise traders in the market, whose speculative activities leads to price bubbles. Finally, recurring to the Lyapunov characteristic exponents of the relative wealth evolution, we show that none of the traders is able to marginalize the others in the long run, i.e., to take over all wealth in the market.

JEL Classification Codes: C15, C32, C52, G11, G12

Key Words:
Indirect inference, GARCH, asset prices, stock market, endogenous asset supply, endogenous asset demand, market efficiency, co-existence, heterogeneous agents, noise traders, technical trading strategies, rational traders, Lyapunov characteristic exponents
Many market participants, especially professional traders, invest in financial markets using technical trading rules. Their behavior stands in striking contrast to economists' advice of "rational" investing, i.e., to maximize expected utilities with the objective probabilities of the state of the economy (see e.g., Lucas (1978)). According to Black (1986), technical traders act irrationally, believing that the "noise" they get from their models is actually valuable information. He calls such traders thus "Noise Traders". For a long time, Economists have come forward with the "market selection hypothesis", i.e., that traders who do not behave "rationally" in the above sense (e.g., noise traders) will eventually be driven out of the market (Alchian (1950), Friedman (1953)). Fama (1965) argued that in financial markets, traders with incorrect beliefs will lose all their money to rational traders, and finally disappear. Long run prices would therefore be determined by rational traders.

However, as already Fama recognized, there is strong empirical evidence against this conclusion (fat tails in price return distributions, excess volatilities). The discrepancy between the price characteristics that are observed in the market and those predicted by the rational equilibrium economy model has triggered a whole line of research that deviates from the homogenous agent assumption. A prominent example is the Santa Fe stock market model where heterogenous agents trade based on technical as well as fundamental information (see e.g., LeBaron, Arthur and Palmer (1999)). The prices that emerge in this computer-simulated market replicate qualitatively the empirical properties of financial time series. Due to the large amount of parameters that comes with superior flexibility, it is however impossible to estimate such models. Furthermore, these models, suffer from a rather weak economic foundation: E.g., they allow for an infinite supply of a riskless asset, and they omit to model the evolution of the investors’ wealth over time. Another example of a computer-simulated market with heterogeneous agents is Brock and Hommes (1998). They are able to show that the presence of agents with heterogeneous beliefs leads, even in very simple markets, to complicated price dynamics.

During the last decade, the market selection hypothesis has been investigated in several different model settings. De Long, Shleifer, Summers, and Waldman (1991) showed that in a partial equilibrium model of an asset market with exogenous prices, noise traders can outgrow and eventually dominate rational traders. In a general Equilibrium model with intermediate consumption however, noise traders will not survive (Sandroni (2000), Blume and Easley
The inverse may be true in dynamically incomplete markets: Beker and Chattopadhyay (2005) show that there may be asset structures in incomplete markets that lead, with probability one, to the extinction of the rational agents.

Evolutionary Finance understands financial markets as a heterogenous population of interacting strategies, which compete for market capital. Quite in analogy to Darwinian evolution theory, only the ”fittest” will survive. Markets are modeled in various ways, by employing temporary and general equilibrium models, or by using dynamical systems theory as well as game–theoretic reasoning. The behavior of some investors inhabiting the models may be triggered by expected utility maximization or genetic learning; while the action of others may only be restricted by their adaptation to the information filtration. The investment styles of the market participants vary as well, of course: There might be rational traders, noise traders, fundamentalists etc. in the market. A common feature of these models, however, is their formal and precise approach when trying to contribute to a better understanding of the dynamics in financial markets. An excellent overview of recent achievements in evolutionary finance is given in the special issue of the Journal of Mathematical Economics on Evolutionary Finance (2006), edited by T. Hens and K. R. Schenk–Hoppe. We investigate – and give strong evidence against – the EMH. Therefore, we extend the model of Evstigneev, Hens and Schenk–Hoppe (2006) to endogenous asset supply and demand. Our economic model does not face problems of common interactive agent models such as infinite supply of riskless assets or the lack of modelling the evolution of the investor’s wealth. We show that in a market where the asset supply and demand are indeed endogenous, both rational and noise traders do co–exist in the long run. The implications of this finding are stark: As argued above, if the EMH is correct, it is impossible that agents other than the representative rational investor will survive. Thus, the survival of noise traders implies that the EMH is wrong. In other words, financial markets are, even in the long run, neither informationally nor allocationally efficient. We show that the co–existence of rational– and noise traders is due to the different way how they generate revenue: On the one hand, the rational trader earns a steady income from dividend gains, as he constructs his portfolio in order to use the information content of the dividend structure in an optimal way: He invests according to the generalized rule of Kelly, i.e., proportionally to the expected relative dividends. On the other hand, noise traders make most of their money
by successfully speculating on rising asset prices, while actively shortening the supply of these assets in order to drive prices even higher. The so induced price bubbles burst, however, when the noise traders become overly greedy. Thus, they are not able to accumulate enough wealth to drive the rational traders out of the market. The consumption of the rational trader is consistently higher than the consumption of noise traders. The reason therefore lies in the fact that noise traders do speculate with part of their wealth. As they cannot use the speculative portfolio for consumption, their ability to consume is reduced in comparison with the rational trader. In situations when they have suffered losses from speculation, it gives them an edge to recover, though, as they do not have to consume an equal amount of their wealth as the rational trader. These findings lead to the following recommendations: a) Institutional traders and individuals who have to rely on a steady income flow for consumption best behave rationally. b) The existence of endogenous price bubbles may make technical trading rules interesting for speculators who are interested in maximum capital growth. However, to realize the book gains, their timing will have to beat the majority of market participants. c) The worst thing to do is 'buy and hold', as the dividend earnings are smaller than when investing rationally, and nevertheless there is no opportunity to realize capital gains. Finally, analyzing the price difference distributions that result from the model, we show that they display both the excess volatility and the fat tails that are encountered in real data. This finding hints at the fact that the presence of noise traders in the market is indeed responsible for the price "puzzles" observed in price series.

Having shown that rational investors and noise traders may co-exist in the long run within the evolutionary market model with endogenous demand and supply, we want to test empirically whether this model is capable of explaining the main stylized stock market facts that constitute a deviation from the EMH.

There are many anomalies (or stylized facts) observed in empirical price series that do not correspond to rational prices. In our paper, we focus on the two most prominent puzzles related to low frequency (quarterly) stock prices: Conditional volatility of log-price returns, and the price forecasting power of dividend yields.

The existence of conditional log price return volatility implies that returns do not fluctuate
randomly around its long term mean with constant volatility, but there are calmer periods
with few price movements followed by periods with high fluctuations. This fact is empirically
well documented for stock market returns, see e.g., Bollerslev (1986), Chou (1988) or Poterba
and Summers (1986).

The second main stylized fact contradicting with the EMH is the predictability of prices
with dividend yields. According to representative agent rational models, all relevant informa-
tion should be included in the current prices; dividend yields could thus not predict prices
(Campbell and Shiller (1987)). However, i.e., Fama and French (1988a) and Campbell and
Shiller (1988) suggest that dividends contain significant information about prices. More recent
evidence is given by Lettau and Ludvigson (2005).

In order to solve these puzzles, different approaches have been taken. Many papers are
dedicated to the question whether the Dividend Discount Model is able to explain stock price
and Cochrane (1999)). Other papers suggest models with roots in Behavioral Finance to
explain the stock price puzzles. Examples are Barberis, Shleifer and Vishny (1998), Daniel
and Subrahmanyan (1998) or Hong and Stein (1999). Yet another string of research are
models where heterogeneous agents interact. Prominent examples are the model of Brock and
Hommes (1998), Lux and Marchesi (2000) and the Santa Fe stock market model (LeBaron
et al. (1999)). In the Santa Fe stock market, heterogenous agents trade based on technical
as well as fundamental information. The prices generated in this computer-simulated market
show conditional volatility as well as several other stylized facts. However, the model suffers
from a rather weak economic foundation\(^1\), and due to the large amount of parameters, it is
impossible to estimate the model. In order to achieve a better economic foundation than
purely computational markets, recent Evolutionary Finance models emphasize the key role of
the evolution of agents’ wealth(see e.g., Evstigneev, Hens and Schenk-Hoppé (2006), EHS in
the following). The asset demand of an agent depends directly of her\(^2\) wealth. In order to
survive in the long run, she must maintain enough wealth to consume and invest. If she is

\(^1\)Examples: The supply of the riskless asset is infinite; the agent’s asset demand does not depend on her
current wealth, and the evolution of wealth over time is not modeled.

\(^2\)in the following, the masculine and feminine form will always be assumed when mentioning the
agent/investor/trader, even if not explicitly stated so
not successful in doing so, she is driven out of the market. In the first part of this paper it is shown that in the evolutionary model, extending EHS (2006) to endogenous asset supply and demand, that both rational and noise traders will co-exist in the long run. The price series resulting from his model show stylized facts like volatility clustering, price bubbles and heavy tails of the price return distribution.

In this paper, we estimate the model outlined above using the indirect inference methodology of Gouriéroux, Monfort and Renault (1993). To do so, we adapt his model to accommodate empirical dividend data. The empirical data must fulfill two conditions: a) The assets must be liquid, which avoids distortions and excessive transaction costs, and b) their dividend history must be tractable (as the rational investor relies on this information). Thus, we choose the price series of the S&P500 with the longest dividend history. We are able to show that the conditional volatility (as measured by GARCH(1,1)) of the model price index corresponds to the conditional volatility of the benchmark price index.

To investigate the forecasting power of dividend yields on prices, we achieve analytically a close form of linking today’s dividend yields to the future prices, inherently de-trending the price process. In representative agent models, stationarity has to be obtained by stationarity-generating transformations, such as first differences, filters (Hodrick-Prescott or band-pass), or error correction specifications. Such transformations lower the power of estimation. We show that the information content of dividend yields for price forecasts is maximum when only rational traders are present in the model. The introduction of noise traders into the market decreases the forecasting power for certain assets due to the speculative activities of these traders.

Finally, we investigate the asymptotic behavior of the dynamic system governing the relative wealth evolution of the rational and noise traders in the market. To do so, we calculate the Lyapunov exponents of the relative wealth of the traders over time. We find that the Lyapunov exponents are positive, confirming that no trader is able to drive the others out of the market.

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3 EHS (2006) have shown with fixed asset supply that a market is evolutionary stable if and only if all traders invest according to the generalized rule of Kelly (1956). In their model, the Kelly rule requires to hold a portfolio that distributes the agents’ wealth according to the expected relative dividends of the stocks in the market. This investment behavior is rational, as the generalized Kelly rule could also be obtained as the outcome of an idealized market with a single representative agent having rational expectations.
The paper is organized as follows: Section 1 gives an overview of the model first. Then, after the introduction of notations and definitions, the equilibrium in the market is derived. Also, the conditions for well defined prices are established. Section 2 investigates a very simple economy with only two traders and two assets, without capital inflow from outside. Section 3 extends the setting, allowing for many assets, many (differing) traders and the attraction of additional outside capital. It investigates the evolution of trader wealth, trader consumption and asset prices. Section 5 concludes and indicates directions of future research. Section 1 outlines the economic model we use to investigate the price puzzles. Section 4 assesses empirically our stock market model. The section starts with a description of empirical data that the analysis is based on. Then it motivates the use of the indirect inference methodology to estimate the heterogeneous agent model, and presents the estimation results. Finally, it investigates the price forecasting power of the dividend yields, based on the analytical result achieved in section 2, and it analyzes the wealth evolution of the traders in the market. Section 5 concludes.

1 The Evolutionary Model of the Financial Market

1.1 Overview

The model presented in this paper is a generalization of the EHS model, which is itself based on Lucas (1978)'s infinite horizon asset market model. It can be summarized as follows:

Rational and noise traders buy and sell long-lived assets. Short sales are not possible. The target of the traders is to get rich and to consume as much as possible of a perishable good. To allow the evaluation of the trading performance based on the increase in trader wealth, the consumption rate is the same for all traders and constant over time. To be able to consume, the traders must have cash. As the consumption rate is constant, the more cash a trader has, the more he consumes compared to other traders. Other than for consumption, cash can only be used for investing into assets. It cannot be used to store value. There are three sources of
cash for a trader: earning dividends, selling assets, and attracting new capital from outside the market:

- A trader earns dividends proportionally to the asset shares he holds in the portfolio.
- The cash a trader earns from selling assets corresponds to the amount of shares sold valuated at the current equilibrium prices in the market. Note that he made a capital gain and increased his wealth if he was able to sell the asset at a higher price than the purchasing price.
- Wealth that flows from outside into the market is distributed among the traders according to their past trading performance.

Only the dividend inflow and the capital inflow bring additional liquidity into the market. Capital gains of one trader are the capital losses of another. The total amount of assets in the market is fix and normalized to 1. However, both the asset demand and the asset supply are endogenous, as the traders can choose how much of an asset they want to buy, and how much of an asset hold they want to liquidate in the market.

1.2 Notations and Definitions

For the subsequent formal analysis, the following notations and definitions are used:

In discrete time $t = [1,...,T]$, a finite number of traders $i = [1...I]$ trade long-lived financial assets to increase their wealth $w_i^t$ over time. There are finitely many assets $k = [1...K]$ available for trading. The total supply of asset $k$ is denoted by $\theta^k$. It remains constant over time. Thus, it can be normalized to 1.

Each asset $k$ pays off a cash dividend at the beginning of every period, before trading starts. Let $D_i^k$ denote the total dividend paid by asset $k$ in period $t$. Note that $D_i^k \in \mathbb{R}$ and $D_i^k \geq 0$, for all $t$ and $k$. Note also that $\sum_{k=1}^{K} D_i^k > 0$. The amount of dividend that is paid depends on the state of the world $s = [1...S]$. There are finitely many states of the world.

\[\text{in the following, the masculine and feminine form will always be assumed when mentioning the investor, even if not explicitly stated so}\]
A trader earns the share of the total dividend that is relative to his share of the total asset quantity denoted by $\theta_{i,k}^{t}$. Note that $\theta_{i,k}^{t} \in \mathbb{R}$ and that $\theta_{i,k}^{t} \geq 0$.

In the beginning of every period $t$, trader $i$ holds the portfolio $\theta_{i,k}^{t-1}$, $k = [1..K]$. To make a speculative profit, he keeps the assets $\tilde{\theta}_{i,k}^{t}$ where he assumes rising prices. All other assets are sold. His supply of asset $k$ in the market becomes thus:

$$\tilde{\theta}_{i,k}^{t} = \theta_{i,k}^{t-1} - \theta_{i,k}^{t}$$ (1)

When selling $\tilde{\theta}_{i,k}^{t}$ of an asset, the trader receives cash. The amount depends on the price of the asset $p_{k}^{t}$. Prices are market clearing prices, i.e., they balance the asset supply and demand. Note that $p_{k}^{t} \in \mathbb{R}$ and that $p_{k}^{t} > 0$.

Another source of cash for trader $i$ is the outside capital $\Omega_{t}$ he is able to attract. The quantity of attracted capital depends on the trading performance $\kappa_{i}^{t}$ compared to the trading performance of the other traders in the market. The trading performance is measured in terms of the differences in total wealth $\Delta(w)^{j}$ during the previous $l = 1...L$ trading periods. Thus:

$$\kappa_{i}^{t} \equiv \kappa_{i}^{t} (\Delta(w)^{i}_{t-1}, \Delta(w)^{j}_{t-1}) ; \quad j = 1...I, j \neq i$$

The assumption that the attraction of new capital is indeed dependent on the past performance is realistic: A lot of institutional as well as individual investors use exactly this criterion when choosing an investment agent.

The total cash $\tilde{w}_{i}^{t}$ of trader $i$ in $t$ is composed of the value of assets sold, the dividend earnings and the attracted capital:

$$\tilde{w}_{i}^{t} = \sum_{k=1}^{K} \left[ \tilde{\theta}_{i,k}^{t} \cdot p_{k}^{t} + \theta_{i,k}^{t-1} \cdot D_{k}^{t} \right] + \kappa_{i}^{t} \cdot \Omega_{t}$$ (2)

The trader uses his cash for consumption and for investing in new assets. The consumption rate $c$ is the same for all traders and constant over time to allow the evaluation of the traders’ performance based on the increase in their wealth. Note that $0 < c < 1$, and $c \in \mathbb{R}$. 

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The investor spends all cash not used for consumption to buy new assets. He splits the cash among the assets according to the investment rules $\lambda_{i,k}^t$. Note that for all $t$, $\lambda_{i,k}^t \geq 0$, and $\sum_{k=1}^{K} \lambda_{i,k}^t = 1$.

Trader $i$’s demand of asset $k$ is denoted $\hat{\theta}_{i,k}^t$. It depends on the trader’s cash, on $\lambda_{i,k}^t$, on $c$, and on the asset price:

$$\hat{\theta}_{i,k}^t = \frac{\lambda_{i,k}^t \cdot (\tilde{w}_i^t (1 - c))}{p_k^t}$$

(3)

The total portfolio of the trader at the end of period $t$ corresponds to the assets he held before plus the newly purchased assets:

$$\theta_{i,k}^t = \hat{\theta}_{i,k}^t + \bar{\theta}_{i,k}^t$$

(4)

The total wealth $w_i^t$ of the trader at the beginning of period $t$, before consumption and reinvestment, consists of his cash plus the value of the speculative portfolio:

$$w_i^t = \sum_{k=1}^{K} \left[ (\hat{\theta}_{i,k}^t + \bar{\theta}_{i,k}^t) \cdot p_k^t + \theta_{i,k}^{t-1} \cdot D_k^t \right]$$

(5)

$$w_i^t = \tilde{w}_i^t + \bar{w}_i^t$$

(6)

where the speculative portfolio is valued as:

$$\bar{w}_i^t = \sum_{k=1}^{K} \left[ \theta_{i,k}^t \cdot p_k^t \right]$$

(7)

In period $t$, the total wealth of the trader allowing for consumption and reinvestment becomes:

$$w_i^t = \frac{\sum_{k=1}^{K} \left[ \theta_{i,k}^t \cdot p_k^t \right]}{1 - c}$$

$$= \frac{\sum_{k=1}^{K} \left[ (\hat{\theta}_{i,k}^t + \bar{\theta}_{i,k}^t) \cdot p_k^t \right]}{1 - c}$$

$$= \frac{\hat{w}_i^t + \bar{w}_i^t}{1 - c}$$

(8)
1.3 Equilibrium

A market is at its short term equilibrium when the asset supply and demand are equal. The equality of supply and demand is achieved when the asset prices clear the market.

Remember that the total asset supply \( \theta^k \) is constant over time and normalized to 1. Furthermore, remember from equation 1 that the asset supply of each trader is equal to the total amount of the asset in the investor’s portfolio minus the amount he decides to hold for speculation. Thus, the total market supply of asset \( k \) can be written as:

\[
\tilde{\theta}^k_t = 1 - \sum_{i=1}^{I} \tilde{\theta}^{i,k}_t
\] (11)

The market supply changes over time. However, the traders decide about their speculative portfolio in the beginning of period \( t \), before the trading starts. The market supply is thus inelastic within one period. Hence, if prices are to clear the market, they must influence the asset demand \( \hat{\theta}^{i,k}_t \) of the agents.

The total asset demand corresponds to the aggregated individual demands of the traders as defined in equation 3:

\[
\hat{\theta}^k_t = \sum_{i=1}^{I} \lambda^{i,k}_t \cdot \tilde{w}^i_t \cdot p^k_t \cdot (1 - c) \] (12)

In equilibrium, the total demand of asset \( k \) equals the total supply of the asset in the market:

\[
1 - \sum_{i=1}^{I} \tilde{\theta}^{i,k}_t = \sum_{i=1}^{I} \lambda^{i,k}_t \cdot \tilde{w}^i_t \cdot p^k_t \cdot (1 - c)
\] (13)

Remember from equation 2 that the cash of trader \( i \) in period \( t \) consists of the value of assets he sold, the dividends he earned and the external capital he was able to attract. In order to obtain an explicit solution for the market clearing prices, substitute the cash \( \tilde{w}^i_t \) in equation...
13 by its components. Regroup the resulting equation to get:

\[
\frac{1 - \sum_{i=1}^{I} \bar{\theta}_{i,k}^t}{1 - c} p_t^k = \sum_{i=1}^{I} \lambda_i^k \left( \sum_{j=1}^{K} \left( \bar{\theta}_{i,j}^t p_t^j + \theta_{i-1,j}^t D_t^j \right) + \kappa_t^i \Omega_t \right)
\]

\[
= \sum_{j=1}^{K} \sum_{i=1}^{I} \lambda_{i,k}^j \bar{\theta}_{i,j}^t p_t^j + \sum_{j=1}^{K} \sum_{i=1}^{I} \lambda_{i,k}^j \theta_{i,j}^t D_t^j + \sum_{i=1}^{I} \lambda_{i,k}^i \kappa_t^i \Omega_t \quad (14)
\]

Re-write equation 14 as:

\[
v_t^k p_t^k = \sum_{j=1}^{K} A_{t,j}^k p_t^j + b_t^k \quad (15)
\]

where:

\[
v_t^k = \frac{1 - \sum_{i=1}^{I} \bar{\theta}_{i,k}^t}{1 - c}
\]

\[
A_{t,j}^k = \sum_{i=1}^{I} \lambda_{i,k}^j \bar{\theta}_{i,j}^t
\]

\[
b_t^k = \sum_{j=1}^{K} \sum_{i=1}^{I} \lambda_{i,k}^j \theta_{i,j}^t D_t^j + \sum_{i=1}^{I} \lambda_{i,k}^i \kappa_t^i \Omega_t
\]

Extend equation 15 to all assets to obtain:

\[
\text{diag}(v_t) \cdot \vec{p}_t = A_t \cdot \vec{p}_t + \vec{b}_t \quad (16)
\]

Finally, solve equation 16 for p:

\[
[\text{diag}(v_t) - A_t] \cdot \vec{p}_t = \vec{b}_t \quad (17)
\]

\[
\therefore \quad \vec{p}_t = [\text{diag}(v_t) - A_t]^{-1} \cdot \vec{b}_t \quad (18)
\]

Assumptions A.1 and A.2 are imposed in the following to ensure that the prices \( \vec{p}_t \) are well defined in all periods:
A.1 At least one trader has a completely diversified portfolio ($\theta_{t}^{i,k} > 0$ for all $k$), and re-balances his portfolio in each trading period ($\lambda_{t}^{i,k} > 0$ for all $k$).

A.2 The consumption rate is positive and smaller than 1.

Proposition 1 As long as A.1 and A.2 hold, the prices $\tilde{p}_t$ are well defined and positive in all trading periods.

Proof of Proposition 1 The proof of proposition 1 follows, although in a different context, the argumentative line of EHS, p. 8–9.

The entries of the matrix $C \equiv [\text{diag}(v_t) - A_t]$ are, respectively on the diagonal and off-diagonal:

$$
C_{k \bar{k}} = \frac{1 - \sum_{i=1}^{I} \bar{\theta}_{i,k}^{t} - \sum_{i=1}^{I} \lambda_{t}^{i,k} \tilde{\theta}_{t}^{i,k}}{1 - c} - \sum_{i=1}^{I} \lambda_{t}^{i,k} \tilde{\theta}_{t}^{i,k}
$$

$$
C_{j \bar{k}} = -\sum_{i=1}^{I} \lambda_{t}^{i,k} \tilde{\theta}_{t}^{i,k}
$$

Regarding $C_{k \bar{k}}$, one can show:

$$
C_{k \bar{k}} > 0 \quad \therefore \quad 1 > \sum_{i=1}^{I} \bar{\theta}_{i,k}^{t} + (1 - c) \sum_{i=1}^{I} \lambda_{t}^{i,k} \tilde{\theta}_{t}^{i,k}
$$

$$
1 > \sum_{i=1}^{I} \bar{\theta}_{i,k}^{t} + \zeta \tilde{\theta}_{t}^{k}
$$

From assumption A.1 and A.2 follows that $0 < \zeta < 1$. As equation 13 holds, $C_{k \bar{k}}$ is strictly positive. Also, because of A.1, all $C_{j \bar{k}}$ are strictly negative. C is thus invertible.

Murata (1977) (Theorem 23, p. 24) ensures that $\tilde{p}_t \geq 0$ if $\tilde{b}_t \geq 0$. As A.1 holds, and at least one asset pays a strictly positive dividend, $\tilde{b}_t \geq 0$. It follows that $\tilde{p}_t > 0$, which completes the proof.
1.4 Strategies

Trading strategies are stationary, i.e., chosen by the traders in period 0 and not modified until the end of trading. They are formulated in terms of two rules: A quantity rule, and a cash rule.

The quantity rule tells what assets to use for price speculation, and the cash rule defines how to invest the cash not used for consumption.

All traders must include a cash rule into their strategy. The quantity rule, however, will come into play only for speculative traders (i.e., technical traders).

Traders can (but do not have to) choose to observe one or several of the following elements as inputs to their strategies:

- dividend history
- price history
- trading volume history

Traders can also decide about the history length of the input variables to be considered.

The quantity rule, evaluated in period $t$, indicates how much of an asset the investor is holding during this period to speculate on price gains:

$$\bar{\theta}_t^{i,k} = \bar{\theta}_t^{i,k} \left( \theta_{t-1}^k, \ D_{t-1,...,t-x}^k, \ p_{t-1,...,t-y}^k, \ \tilde{\theta}_{t-1,...,t-z}^k \right)$$ (22)

Remember that by equation 1, the quantity rule determines also the trader’s asset supply.

The cash rule specifies how a trader invests the non-consumed cash into new assets:

$$\lambda_t^{i,k} = \lambda_t^{i,k} \left( \tilde{\omega}_t^i, \ D_{t-1,...,t-x}^k, \ p_{t-1,...,t-y}^k, \ \tilde{\theta}_{t-1,...,t-z}^k \right)$$ (23)

The variables $x$, $y$ and $z$ stand for the respective length of the dividend-, price- and volume history that is taken into account by the rules.

Note that a trader with quantity rule $\bar{\theta}_t^{i,k} = 0$ cannot make net capital gains. $\bar{\theta}_t^{i,k} = 0$
for all $k = 1, \ldots, K$ implies that the trader has to sell his portfolio in period $t$. The cash rule then defines how to re-invest the received cash (after consumption). As the asset prices are not known before trading, the trader has to sell all assets and cannot limit trading to adjust for the differences of his current portfolio and the cash rule target $\lambda_t^{i,k}$. During the process of re-balancing the portfolio, the investor might very well gain on certain assets. But as the prices are clearing the market, he will loose on certain assets, too. On aggregate, the losses will exactly offset the gains.

1.5 Simulations

An analytical solution for the market’s evolutionary dynamics has yet to be achieved. Because of the explicit solution for the market equilibrium however, the dynamics of wealth, consumption and prices in the market can be investigated using computer simulations. All results presented below are robust both regarding the length of the simulation runs (i.e., for 100’000 trading periods), and regarding the repetitions of the runs (i.e., repeated several hundred times with differing random seeds). In the following, only part of the simulation runs are displayed for illustrative reasons. The robustness is assured in all results, though, even if not explicitly stated so.

In the following, a very simple market is investigated first, restricting the market participants to two traders, offering only two assets for trading, and allowing for two states of the world.

Subsequently, a extended market is investigated with many traders, many assets and many states of the world.

2 Simple Market

2.1 Setup

In this stylized market, there are two investors $i = 1, 2$ investing into two assets $k = 1, 2$. Each asset pays a dividend. The dividends depend on the states of the market. Assume that there are only two states of the world.
The dividend payoff matrix is defined, with states in the rows and assets in the columns, as:

\[
D = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}.
\]

Note that asset 2 dominates asset 1.

The two traders apply different strategies: One of them is a 'rational', the other a 'noise' trader.

The rational trader invests according to the generalized Kelly rule, called \( \lambda^* \) in the following. In other words, he distributes his wealth among the assets proportionally to their expected relative dividends. EHS show that traders who invest according to \( \lambda^* \) act rationally, as \( \lambda^* \) maximizes the expected logarithm of the growth rate of wealth in a model with endogenously determined returns. As discussed before, the fact that prices are not known in advance implies that the rational investor will never hold an asset longer than one period, as he has to re-balance his portfolio constantly to comply with \( \lambda^* \).

The cash rule and the quantity rule of the rational trader’s strategy become thus:

\[
\lambda^*_t = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} \quad \text{for } t = [1...T].
\]

\[
\tilde{\theta}^*_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } t = [1...T].
\]

The noise trader plays a very simple trend follower strategy. He does neither consider the dividend history, nor the volume history. He only considers the last observable price change. When the price of an asset has risen in \( t-1 \), he will assume that it rises further and hold the asset. If it has fallen, he will not hold it in his portfolio. The cash rule of the noise trader depends on the most recent price history, too. He distributes his liquidity equally between all assets where \( p^k_{t-1} > p^k_{t-2} \). Let \( n \) denote the total number of assets with increased prices. The quantity rule and the cash rule become:

\[
\lambda_t = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} \quad \text{for } t = [1...T].
\]

\[
\tilde{\theta}_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } t = [1...T].
\]

The noise trader plays a very simple trend follower strategy. He does neither consider the dividend history, nor the volume history. He only considers the last observable price change. When the price of an asset has risen in \( t-1 \), he will assume that it rises further and hold the asset. If it has fallen, he will not hold it in his portfolio. The cash rule of the noise trader depends on the most recent price history, too. He distributes his liquidity equally between all assets where \( p^k_{t-1} > p^k_{t-2} \). Let \( n \) denote the total number of assets with increased prices. The quantity rule and the cash rule become:

\[
\lambda_t = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} \quad \text{for } t = [1...T].
\]

\[
\tilde{\theta}_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } t = [1...T].
\]
\[ \theta_t^{2,k} = \begin{cases} 
if (p_{k-1}^k \geq p_{k-2}^k) & \Rightarrow \theta_{t-1}^{2,k} 
if (p_{k-1}^k < p_{k-2}^k) & \Rightarrow 0 
\end{cases} \text{ for } t = [1 \ldots T] \text{ and } k = 1, 2. \]

\[ \lambda_t^{2,k} = \begin{cases} 
if (p_{k-1}^k \geq p_{k-2}^k) & \Rightarrow \frac{1}{n} 
if (p_{k-1}^k < p_{k-2}^k) & \Rightarrow 0 
\end{cases} \text{ for } t = [1 \ldots T] \text{ and } k = 1, 2. \]

The only exogenous parameter in this market is the consumption rate $c$. Exemplarily, the next section looks at the evolution of a market with a consumption rate $c = 0.4$. The following section investigates then systematically the influence of the consumption rate on wealth and consumption.

### 2.2 Evolution

In simple market described above, the rational and noise trader co-exist in the long run. In other words, neither of them is able to drive the other out of the market by accumulating a share of market wealth that tends towards 1 over time.

Illustratively, consider the first 200 trading periods in such a market, given a consumption rate $c = 0.4$ (see figure 1 on page 17). Clearly, there are market cycles where one or the other trader seems to dominate. However, the opponent is always able to recover.

The survival of the noise trader relies heavily on his capital gains. This fact can be illustrated with the wealth evolution in a market where no capital gains are possible, i.e., $\theta_t^{i,k} = 0$; $i, k = 1, 2$ (see figure 2 on page 17).

Without speculation, $\lambda^*$ takes over the whole market very quickly. This is no surprise: The trend follower 'forecasts' that in $t$, the same asset will rise in price as in $t - 1$. The prices in $t$ however are largely driven by the dividend state, as the cash available for investment depends entirely on the dividends that have been payed. The trend follower believes therefore wrongly that he can 'forecast' prices, and his $\lambda_t^{2,k}$ are consistently non-optimal. $\lambda^*$, however, behaves rationally and becomes the evolutionary winner.

Consider again the simple market with speculation outlined above. The fact that the trend follower is able to survive implies that he is, at least partially, able to forecast the direction of
Figure 1. Wealth evolution with speculation: None of the traders is able to drive the opponent out of the market

Figure 2. Wealth Evolution without speculation: $\lambda^*$ drives the noise trader out of the market
the asset prices. Why is this the case?

Have a look at the evolution of the asset prices over time (figure 3 on page 19). The prices fluctuate rather irregularly, displaying bubbles and crashes. Remember from equation 17 that asset prices depend both on the demand (determined by the traders’ strategies and their cash) and on the market supply. The cash of the traders is determined by their trading success and thus not immediately influenceable. Both supply and demand of the trend follower, however, is endogenous. If the trend follower decides to speculate on an asset, he increases the demand, and at the same time reduces the supply of the asset in the market. Thus, he pushes prices actively up, acting in favor of a ‘self-fulfilling prophecy’: Chances that his price forecasts were right increase. Figure 4 on page 19 illustrates the link between speculation and higher prices: Prices rise hyper-exponentially with reduced market supply.

However, the trend follower becomes a victim of his own success: The more he was able to reduce the market supply of an asset, the bigger is the following price crash when he tries to realize his ’book gains’ (see figure 5 on page 20).

2.3 Influence of the consumption rate

The consumption rate $c$ is the only exogenous parameter to be set. Varying $c$ does not change the finding that the rational trader and the noise trader co-exist. However, it influences the quantity of wealth that the traders are able to gather, and the quantity they can consume. As a rule, the higher $c$, the less the cash available for buying assets, and the greater the price impact of the trend follower. In other words, prices are likely to be pushed higher by reduced market supply and increased demand when $\lambda^*$ has less cash available.

Have a look at figure 6 on page 21. It displays the mean and median wealth of both traders according to the consumption rate. Each data point is calculated with 100’000 observations (i.e., 50 runs with a length of 2000 subsequent trading periods). With $c$ exceeding approximately $1/3$, the average wealth of the trend follower is higher than the wealth of $\lambda^*$. The median wealth, however, remains consistently lower. This discrepancy is due to the strong increase in trend follower’s (book-)wealth during not very frequent price bubbles.

$\lambda^*$’s consumption, however, is consistently bigger than the consumption of the trend follower
Figure 3. Evolution of asset prices: Bubbles and crashes

Figure 4. Price impact of speculation: Prices rise hyper-exponentially with speculative asset holdings
(see figure 7 on page 21). Remember that only cash can be consumed. $\lambda^*$, investing rationally, maximizes the dividend returns. It has thus a steady source of cash income. The trend follower relies mainly on capital gains. These are much less frequent, though, reducing his ability to consume.

3 Extended Market

3.1 Setup

This section introduces a market that extends the settings investigated above. It has the following characteristics:

- There are $i$ rational and noise traders
- The traders invest in $k$ assets.
- These assets generate each trading period dividends $D$ according to a dividend process $\delta$.
- In addition to the dividends, there is outside capital flowing into the market.
Figure 6. Wealth according to the consumption rate

Figure 7. Consumption according to the consumption rate
This capital is distributed among the traders according to their past trading performance.

Consider the traders first. In the current implementation, there are five traders: The rational trader, three typical noise traders (trend follower, momentum trader, contrarian trader), and one 'buy-and-hold' investor. All traders use strategies containing a quantity rule and a cash rule. Note that the cash rule refers to the cash available after consumption.

The rational trader invests according to the generalized Kelly rule $\lambda^*$, i.e., the asset shares in his portfolio equal the expected relative dividends of the assets. There are as many states of the world $S$ as trading periods $T$. The probability $\rho$ of state $s$ can thus be written as $\rho(s_t)$. The trading rules of his strategy are:

$$\tilde{\theta}^{1,k}_t = [0] \quad \text{for } t = [1 \ldots T] \text{ and } k = [1 \ldots K]$$

$$\lambda^{1,k}_t = \left[ \sum_{t=1}^{T} \frac{D^k_t}{\sum_{k=1}^{K} D^k_t} \rho(s_t) \right] \quad \text{for } t = [1 \ldots T] \text{ and } k = [1 \ldots K]$$

The trend follower invests in all $n$ assets with rising prices, according to the same trading rules as in the simple market:

$$\tilde{\theta}^{2,k}_t = \begin{cases} 1 & \text{if } (p^k_{t-1} \geq p^k_{t-2}) \Rightarrow \theta^{2,k}_{t-1} \\ 0 & \text{if } (p^k_{t-1} < p^k_{t-2}) \Rightarrow 0 \end{cases} \quad \text{for } t = [1 \ldots T] \text{ and } k = [1 \ldots K].$$

$$\lambda^{2,k}_t = \begin{cases} \frac{1}{n} & \text{if } (p^k_{t-1} \geq p^k_{t-2}) \Rightarrow \frac{1}{n} \\ 0 & \text{if } (p^k_{t-1} < p^k_{t-2}) \Rightarrow 0 \end{cases} \quad \text{for } t = [1 \ldots T] \text{ and } k = [1 \ldots K]$$

The momentum trader invests similar to the trend follower. Unlike the latter though, he does not observe the prices as an input to his strategy, but the price differences $\Delta(p^k_t)$:

$$\Delta(p)_t^k = p_t^k - p_{t-1}^k.$$ 

If the difference of the price differences (the price momentum) has increased in $t - 1$, he will assume that it rises further and hold the asset. If it has fallen, he will not hold it in his portfolio. Regarding the cash rule, he distributes his liquidity equally between all $n$ assets where
\( \Delta(p)^k_{t-1} > \Delta(p)^k_{t-2} \). The quantity rule and the cash rule become:

\[
\begin{align*}
\bar{\theta}^{3,k}_{t} & = \begin{cases} 
if (\Delta(p)^k_{t-1} \geq \Delta(p)^k_{t-2}) \Rightarrow \theta^{3,k}_{t-1} \
if (\Delta(p)^k_{t-1} < \Delta(p)^k_{t-2}) \Rightarrow 0
\end{cases} 
\text{for } t = [1 \ldots T] \text{ and } k = [1 \ldots K], \\
\lambda^{3,k}_{t} & = \begin{cases} 
if (\Delta(p)^k_{t-1} \geq \Delta(p)^k_{t-2}) \Rightarrow \frac{1}{n} \
if (\Delta(p)^k_{t-1} < \Delta(p)^k_{t-2}) \Rightarrow 0
\end{cases} 
\text{for } t = [1 \ldots T] \text{ and } k = [1 \ldots K].
\end{align*}
\]

The contrarian trader believes that falling prices do eventually recover, and that it is possible to exploit this "fact". More specifically, if prices have fallen two times \((\Delta(p)^k_{t-1} < 0 \text{ AND } \Delta(p)^k_{t-2} < 0)\), he invests in all \(n\) assets that satisfy this condition according to:

\[
\begin{align*}
\bar{\theta}^{4,k}_{t} & = \begin{cases} 
if (\Delta(p)^k_{t-1} \leq 0 \text{ AND } \Delta(p)^k_{t-2} \leq 0) \Rightarrow \theta^{4,k}_{t-1} \
otherwise \Rightarrow 0
\end{cases} 
\text{for } t = [1 \ldots T] \text{ and } k = [1\ldots K], \\
\lambda^{4,k}_{t} & = \begin{cases} 
if (\Delta(p)^k_{t-1} \leq 0 \text{ AND } \Delta(p)^k_{t-2} \leq 0) \Rightarrow \frac{1}{n} \
otherwise \Rightarrow 0
\end{cases} 
\text{for } t = [1 \ldots T] \text{ and } k = [1\ldots K].
\end{align*}
\]

In case no trading rule indicates an asset as investment target, the technical traders invest their cash into a randomly chosen asset. This ensures that all traders are always invested in the market.

The 'buy-and-hold'-trader is active on the market only in the first period when he splits his wealth equally among all available assets (illusionary diversification). From period 2 on, he holds the shares bought in period one, without re-allocating any capital:

\[
\begin{align*}
\lambda^{5,k}_{t} & = \left[\frac{1}{K}\right] \text{ for } t = 1 \text{ and } k = [1 \ldots K] \\
\bar{\theta}^{5,k}_{t} & = \left[0\right] \text{ for } t = 1 \text{ and } k = [1 \ldots K] \\
\lambda^{5,k}_{t} & = \left[0\right] \text{ for } t = [2\ldots T] \text{ and } k = [1 \ldots K]
\end{align*}
\]
\[ \bar{\theta}_t^{5,k} = [\bar{\theta}_t^{5,k}] \text{ for } t = [2 \ldots T] \text{ and } k = [1 \ldots K] \]

There are \( K = 10 \) assets in the market. Each asset pays a dividend. In this simulation setup, the dividend payments are driven by the dividend process \( \delta \). \( \delta \) reflects the current economic situation as well as the performance and policies of the firm that is paying the dividend. In other words, dividends \( D \) are function of the state of the economy \( E_s \), and of the internal state of the firm \( F_s^k \) that is paying the dividend. Dividends increase according to the trend \( \tau_t \) over time. Note that \( E_s, F_s^k, \) and \( \tau \) are independent processes. \( \xi^k \) denote the sensitivity of the dividends payed by firm \( k \) to the general economic situation. The dividend process \( \delta \) becomes:

\[
D_s^k = \delta(\xi^k, E_s^k, F_s^k, \tau_t) = \left( \xi^k E_s + (1 - \xi^k) F_s^k \right) \tau_t \quad (24)
\]

Note that the dividends that are payed in time \( t \) are randomly chosen from the set of all \( s = [1 \ldots S] \). In this simulation, \( S = T \).

Additionally to the dividends, there is new capital from outside flowing into the market. It is distributed among the traders according to their trading performance in the past. The capital inflow in period \( t \) is \( \psi \) proportional to the average total dividend payments during the last \( L = 10 \) periods, and subject to some random deviations \( \phi_t \):

\[
\Omega_t = \psi \cdot \frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{K} D_{t-l}^k \cdot (1 + \phi_t) \quad (25)
\]

where:

\[
\phi_t = 1 + r_t * u d_t * \frac{1}{32} \quad (26)
\]

Note that \( r_t \) is a random number between 0 and 1, and \( u d_t = [-1,0,1] \) with the same probability.

\( ^5 \)This is no pre-condition, though. Any set of dividends can be used, as long as no dividend is negative, at least one asset pays a positive dividend each trading period, and the dividends are sufficiently different among the assets.
The additional capital is distributed proportionally to the trading performance $\kappa_i^t$ that is calculated with the average total wealth increase over the last $H = 4$ periods:

$$\kappa_i^t = \begin{cases} \text{if} & \Delta(w)_t^i \Rightarrow \frac{\Delta(w)_t^i}{\sum_{i=1}^{H} \Delta(w)_t^i} \\ \text{otherwise} & \Rightarrow 0 \end{cases} \tag{27}$$

where:

$$\Delta(w)_t^i = \begin{cases} \text{if} & \frac{\sum_{h=1}^{H} \Delta(w)_{t-h}^i}{H} > 0 \Rightarrow \frac{\sum_{h=1}^{H} \Delta(w)_{t-h}^i}{H} \\ \text{otherwise} & \Rightarrow 0 \end{cases} \tag{28}$$

### 3.2 Evolution

The extended market generalizes for $k$ assets and $i$ traders. The values of $k = 10$ and $i = 5$ serve hereby as illustrative examples. Again, the market has been extensively simulated, in various simulation lengths and run durations. All findings presented below are robust, both in terms of duration of a simulation run and in terms of run repetition, even if not explicitly stated so.

The simulations of the extended market confirm the main results of the simple market: The rational trader, the noise traders and the 'buy and hold'-trader co-exist in the long run. Look at figure 8 on page 26: None of the traders is able to gather enough wealth to drive the others out of the market. Even when accumulating a huge book wealth through speculation, the noise traders are unable marginalize the rational trader. The reason for this is twofold: The rational trader earns a) a consistent income from the dividend payments (as he invests in all assets, and at least one asset pays a positive dividend in every period), and b) he follows $\lambda^*$, i.e., he invests always the same fraction of its wealth per asset. Hence if speculation drives the price of an asset up, he will buy a correspondingly smaller quantity of that asset. Therefore, he can let speculative phases pass, waiting for the price bubbles to pop. On the other hand, the rational trader is not able to drive the noise traders out of the market, either, as the noise traders are able to recover from very low wealth by successfully speculating on capital gains. Finally, the 'buy-and-hold'-trader must survive, too: He is invested in all assets, earning thus, in every period, a minimum revenue that allows for consumption. Note that the wealth evolution of the $\lambda^*$ and the 'buy-and-hold' traders is accordingly much less volatile than of the technical
traders: Dividend payments happen more regularly than capital gains, but they are smaller in magnitude.

The co-existence of the different traders holds true however small the initial endowment of the traders is compared to the other traders. Initial wealth smaller up to a factor $10^{-8}$ has been attributed to the traders, with no qualitative difference in the long run: The under-endowed agents always recover, and after some dozens of periods, there is no difference observable to the simulations where all traders receive the same initial wealth.

Furthermore, there is no indication at all that there might exist an 'all-dominant' technical trading strategy able to grow to the detriment of the other noise trading strategies: In addition to the rational trader, different noise traders have been introduced into the market in various combinations. All traders always co-exist. In other words, there is strong evidence that the technical strategies implemented in the present model are evolutionary stable.

Finally, note that, other than in the simple market, the average wealth of the traders rises over time, driven by the dividend trend $\tau$.

The rising market wealth causes the asset prices to rise over time as well. Although the
market comprises 10 assets compared to 2 assets in the simple market, and there are as many states of the world as there are trading periods, the asset prices still exhibit bubbles and crashes (see figure 9 on page 27). Thus, bubbles and crashes are not due to an oversimplified information structure in the market, but a consequence of trader speculation. As Figure 10 on page 28 confirms, the technical traders are able to influence prices by reducing the asset supply while increasing the demand at the same time. Note that prices rise hyper-exponentially with the amount of asset speculation.

Furthermore, the noise traders beat themselves as in the simple market when trying to cash in: The smaller the quantity of an asset that remains in the market, the bigger the following price crash because of the realization of book gains (see figure 11 on page 28).

3.3 Influence of consumption rate and capital inflow

The analysis of the simple market has shown that the consumption rate $c$ plays an important role regarding the performance of the rational and noise traders: The higher $c$, the less cash is available for the traders to buy assets, and the greater the price impact of the noise trader. The influence of $c$ will be investigated by simulating 50 runs with a length of 2000 trading
**Figure 10.** Speculative shortening of asset supply drives prices up

**Figure 11.** Crashes induced by asset liquidation: The smaller the remaining asset supply in the market, the bigger the following crash
periods. The results of each value of $c$ is thus based on 100'000 observations.

Additionally to the consumption rate, there is the second exogenous parameter $\psi$, defining how much cash flows into the market additional to the dividends. As cash is the driving force for both the investment power and the consumption ability, $\psi$ is likely to have an important influence to the market. Various values of $\psi$ are thus investigated as well (using the same simulation setup as for the investigation of $c$).

Consider the total wealth of the traders in figure 12 on page 30: When $c$ exceeds approximately $\frac{1}{2}$, some noise traders and the 'buy-and-hold' trader are able to accumulate more wealth than the rational traders. A similar picture is given when $\psi$ is varied \(^6\) (see figure 13 on page 30): A higher capital inflow increases the wealth of the noise traders more than the wealth of the rational traders. Remember that the distribution of outside capital is proportional to the trading performance, i.e., to the recent increase in total wealth. Noise traders are able to grow their wealth faster than the rational traders, if they speculate successfully. They are thus able to attract a relatively larger share of new capital. This leads to higher wealth: The bigger the amount of new capital, the more 'fuel' for speculation, and the more speculative funds, the greater the chances that the speculation is successful.

Regarding consumption, a key result of the simple market is confirmed: The rational trader $\lambda^*$ is able to consume consistently the most. This finding is independent from both the consumption rate $c$ (see figure 14 on page 31) and the capital inflow factor $\psi$ (see figure 15 on page 31).

The simulation results can be summarized as follows:

- All traders survive in the market, regardless what consumption rate or capital inflow factor has been chosen
- High consumption rates and important capital inflow factors favor the noise traders, who are, under such conditions, able to accumulate more wealth than the rational trader.
- When it comes to consumption, however, the rational trader is more successful, because he optimizes the dividend earnings. The noise traders are not able to realize their book

\(^6\)Note, though, that the influence of $c$ on the wealth of the noise traders is higher than the influence of $\psi$ (exponential vs. logarithmical).
Figure 12. Total wealth according to consumption rate, $\psi=2$: Noise traders profit from higher consumption rates. (The markers indicate the average of 50 runs; the upper lines indicate the highest values of all runs, the lower lines indicate the lowest values of all runs).

Figure 13. Total wealth according to capital inflow, $c=0.2$: Noise traders profit over-proportionally. (The markers indicate the average of 50 runs; the upper lines indicate the highest values of all runs, the lower lines indicate the lowest values of all runs).
Figure 14. Consumption according to consumption rate, $\psi=2$: Lambda star consumes consistently the most. (The markers indicate the average of 50 runs; the upper lines indicate the highest values of all runs, the lower lines indicate the lowest values of all runs).

Figure 15. Consumption according to capital inflow, $c=0.2$: Lambda star consumes consistently the most. (The markers indicate the average of 50 runs; the upper lines indicate the highest values of all runs, the lower lines indicate the lowest values of all runs).
wealth without causing price crashes, and their consumption remains smaller in spite of their high average wealth.

- Each 'trader' in the simulation can be understood as the 'aggregation of all traders with the same strategy'. Thus, a noise trader with a better timing than the others in his 'strategy group' may succeed in realizing the capital gains\(^7\).

- The worst strategy is 'buy-and-hold': Meager consumption and not even the opportunity realize capital gains with a good market timing speak against this strategy.

### 3.4 Price characteristics

The prices display, unlike in models with homogenous and rational investors, characteristics that are found in *real data*. Note that it is no target of this paper to estimate the model with real data, but rather to show some of the qualitative properties of the prices that result of the simulations. They investigate the implications of surviving noise traders for asset prices using the indirect inference methodology that has been suggested by Gouriéroux et al. (1993).

A striking price characteristic is the presence of *volatility clustering*. Figure 16 on page 33) displays the volatility clustering at the example of asset 6. The volatility clustering is present in all assets.

The second of the characteristics is the presence of heavy (or fat) tails in the probability distribution of price returns. Figure 17 on page 34 displays the observed price differences in the range of [-100 100] during a simulation with 10’000 trading periods. Superposed is the theoretical distribution of the price differences under the assumption of normality. Note that the tails of the observed distributions are much heavier than the tails of the normal distribution. This indicates that ‘extreme’ price movements are much more frequent than in a i.i.d-process, a fact that is observed in real data.

A way to test for normality is to plot the price differences on a linear scale against the cumulative distribution function on a logarithmic scale. If the resulting curve is linear, the

\(^7\) A better timing implies, actually, a slight modification of the strategy. The consequence may be that innovative strategies are successful, as long as they do not accumulate to much market power. To investigate this aspect, it will be interesting to include *agent learning*, i.e., the explicit capability to modify strategies, into the model.
underlying distribution is normal. Look at figure 18 on page 34 that includes all price differences from the above simulation with 10’000 trading periods: The resulting curve is distinctively s-shaped. This is another clear indication that the prices do not follow an i.i.d-process.

4 Empirical Assessment of the stock market model with endogenous asset supply and demand

We use the same model as before except that we sample empirical dividends instead of using simulated dividends.

4.1 Asset pricing at the market equilibrium

The market is at its short term equilibrium when the asset supply and demand are equal. The equality of supply and demand is achieved when the asset prices clear the market. It was shown in a preceding section that the market clearing prices can be calculated as:

\[
\tilde{p}_t = [\text{diag}(v_t) - A_t]^{-1} \cdot \tilde{b}_t
\]  

(29)
Figure 17. Price difference distribution with $c=0.2$ and $\psi=2$: Fat tails

Figure 18. Normality test with $c=0.2$ and $\psi=2$: Tail distributions are distinctively non-normal
where:

\[ v_t^k = \frac{1 - \sum_{i=1}^I \tilde{\theta}_{t}^{i,k}}{1 - c} \]

\[ A_t^{k,j} = \sum_{i=1}^I \lambda_t^{i,k} \tilde{\theta}_{t}^{i,k} \]

\[ b_t^k = \sum_{j=1}^K \sum_{i=1}^I \lambda_t^{i,k} \theta_{t-1}^{i,j} D_{t-1}^k + \sum_{i=1}^I \lambda_t^{i,k} \eta_t^j \Omega_t \]

When only rational traders are present in the market, this solution can be simplified. In the following, we derive the equilibrium prices for such a case. This result will then be used in Section 4.5 to investigate the forecasting power of dividend yields on asset prices for both purely rational markets and markets where noise traders are present.

By the fact that the rational trader does not hold a speculative portfolio, simplify \( v_t^k \) to

\[ v_t^k = \frac{1}{1 - c} \tag{30} \]

Using equation ?? and exploiting the fact that a) \( \lambda^* \) is invariant in time and b) the total supply of all assets is, according to equation ??, normalized to 1, formulate \( A_t^{k,j} \) as:

\[ A_t^{k,j} = \lambda^* k \tag{31} \]

Aggregating over \( i \) is in the present case equivalent to substituting all traders with the one
rational trader. Exploit additionally equations ??, ?? and ?? to write:

\[ b_t^k = \lambda^k \left( \sum_{j=1}^{K} D_{j-1}^j + \Omega_t \right) \] (32)

\[ b_t^k = \lambda^k \left( \sum_{j=1}^{K} d_{s(t-1)}^j \left( \exp(\gamma t) + 100 \right) + (\psi + \epsilon_t) \left( \exp(\gamma t) + 100 \right) \right) \] (33)

\[ b_t^k = \left( \exp(\gamma t) + 100 \right) \lambda^k \left( \sum_{j=1}^{K} dy_{s(t-1)}^j + \psi + \epsilon_t \right) \] (34)

∴

\[ \bar{\theta}_t = \left[ diag(v_t) - A_t \right]^{-1} \left( \exp(\gamma t) + 100 \right) \lambda^* \left( \sum_{j=1}^{K} dy_{s(t-1)}^j + \psi + \epsilon_t \right) \] (35)

Given that \( \lambda^* \), \( c \), \( \psi \), \( \eta \) and \( \gamma \) are constant over time, we conclude that the rational prices in \( t \) are a function of the sum of all dividends in \( t-1 \) and of time \( t \). Note that the prices, corrected by the exponential time trend function, are by definition of the model stationary.

### 4.2 Strategies

Our model is inhabited by the rational trader and two noise traders (The term *noise trader* has been introduced by Black (1986), who argues that any non-rational trader would trade not on actual information, but on white (random) noise). Note that the strategies of all traders are stationary, i.e., chosen by the traders in period 0 and not modified until the end of trading. They are formulated in terms of two rules: A quantity rule, and a cash rule. The quantity rule tells what assets to use for price speculation, and the cash rule defines how to invest the cash not used for consumption.

The rational trader invests according to the generalized Kelly rule \( \lambda^* \), i.e., the asset shares in her portfolio equal the expected relative dividends of the assets. As there are as many states of the world \( S \) as there are trading periods \( T \), the probability \( \rho \) of state \( s \) can be written as \( \rho(s_t) \). The trading rules of her strategy are:

\[ \bar{\theta}_t^{1,k} = [0] \text{ for } t = [1...T] \text{ and } k = [1...K] \]
\[ \lambda_{t}^{1,k} = \left[ \sum_{i=1}^{T} \frac{D_{k}^i}{D_{t}^i} \rho(s_i) \right] \text{ for } t = [1..T] \text{ and } k = [1..K] \]

There are two types of noise traders in our model: Trend followers and Contrarian traders. Both use information contained in the price history to position themselves in the market. More specifically, the trend follower invests in all \( N \) assets where prices have been rising in the period before. The higher the price increase in percent, the bigger the investment:

\[
\theta_{t}^{2,k} = \begin{cases} 
if (p_{k-1}^t \geq p_{t-2}^k) \Rightarrow \theta_{t-1}^{2,k} \\
if (p_{k-1}^t < p_{t-2}^k) \Rightarrow 0 
\end{cases} \text{ for } t = [1..T] \text{ and } k = [1..K].
\]

\[
\lambda_{t}^{2,k} = \begin{cases} 
if (p_{k-1}^t \geq p_{t-2}^k) \Rightarrow \frac{\ln(p_{k-1}^t) - \ln(p_{t-2}^k)}{\sum_{n=1}^{N}(\ln(p_{n-1}^t) - \ln(p_{n-2}^t))} \\
if (p_{k-1}^t < p_{t-2}^k) \Rightarrow 0 
\end{cases} \text{ for } t = [1..T] \text{ and } k = [1..K].
\]

The contrarian trader believes that falling prices do eventually recover, and that it is possible to exploit this "fact". He assumes that the more prices have fallen, the stronger they recover. Thus, if the prices of \( N \) assets have fallen within the two past periods, she invests into these assets according to:

\[
\theta_{t}^{3,k} = \begin{cases} 
if (p_{k-1}^t \leq p_{t-3}^k) \Rightarrow \theta_{t-1}^{3,k} \\
if (p_{k-1}^t > p_{t-3}^k) \Rightarrow 0 
\end{cases} \text{ for } t = [1..T] \text{ and } k = [1..K].
\]

\[
\lambda_{t}^{3,k} = \begin{cases} 
if (p_{k-1}^t \leq p_{t-3}^k) \Rightarrow \frac{1}{\sum_{n=1}^{N}(\frac{1}{\exp(\ln(p_{n-1}^t) - \ln(p_{n-3}^k))})} \\
if (p_{k-1}^t > p_{t-3}^k) \Rightarrow 0 
\end{cases} \text{ for } t = [1..T] \text{ and } k = [1..K].
\]

In the beginning of the trading, when no price history is available, the noise traders invest an equal fraction of their cash into all assets. Later on, in case no trading rule indicates an asset as investment target, they choose one asset at random as the investment target. Thereby, it is ensured that all traders are always invested in the market.
4.3 Data

Estimations are conducted with quarterly stock data. Among the 100 largest stocks with respect to market capitalization in 2004, we have chosen from the FAME data base those which provide histories of at least 50 consecutive quarters of dividend payments\(^8\). The dividend data set includes data points from the first quarter 1992 to the second quarter 2004.

The behavior of the asset prices over time is summarized in an index. This index is calculated by distributing an initial index value of 100 evenly among all assets. The asset quantity remains in the following unchanged, while the price movements influence the behavior of the index. Note that as price data points, the closing prices of the last day in each quarter are utilized.

The same price index is calculated with the output prices of our model. The two indices will be used by the direct inference in the next section.

4.4 Indirect inference of the conditional volatility

The conditional volatility that is present in the benchmark price index defined above can be quantified using the following GARCH-type model, with the constant expected return \(C\) and the following stochastic volatility process:\(^9\).

\[
\sigma_t^2 = K + (\rho + \vartheta) \sigma_{t-1}^2 + \rho \sigma_{t-1}^2 (\epsilon_t^2 - 1)
\]

(36)

Note that the linear structure of this model is justified by the fact that we use low frequency returns, which are assumed to be compatible with Gaussian distributions. Table 1 summarizes the estimation of a GARCH(1,1) process based on the benchmark price index\(^10\).

We start from the null hypothesis (backed by the qualitative results of the numerical part of this paper) that the prices in the (adequately parameterized) model show conditional volatility

---

\(^8\)The names of the companies are listed in table 3.

\(^9\)The suitability of ARCH/GARCH processes to model price returns is well documented in the literature (for a survey of the large body of literature on GARCH models, see i.e., Gouriéroux (1997)).

\(^10\)Note that asymmetric conditional volatility processes such as GJR, or volatility processes based on the explicit probability distribution of the volatility innovations as EGARCH, describe the conditional volatility of this process not as good as the simpler GARCH(1,1) process.
Table 1. *GARCH (1,1) Statistics of the Benchmark index*

The benchmark index is defined as an equi-weight index of all stocks considered in the model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Benchmark Index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
</tr>
<tr>
<td>C</td>
<td>0.047</td>
</tr>
<tr>
<td>K</td>
<td>0.0012</td>
</tr>
<tr>
<td>$\rho(1)$</td>
<td>0.57</td>
</tr>
<tr>
<td>$\rho(1)$</td>
<td>0.36</td>
</tr>
</tbody>
</table>

as well. If the hypothesis is true, the price index calculated with the model output prices will show the same *GARCH*(1,1) characteristics as the benchmark price index. To test the hypothesis, it is thus necessary to estimate the parameters of the model.

It is well known that linear–quadratic dynamic rational expectations models are estimated efficiently by maximum likelihood methodology as in Hansen and Sargent (1980). Direct inference of parameters in nonlinear models such as the model described above is however not possible, because the likelihood function is not tractable. Therefore, many researchers have turned to limited information methodologies, in particular the Generalized Method of Moments (GMM). Small sample studies have shown that GMM has poor power.

Another strand of research, simulation based econometrics, has started to apply estimation schemes relying on the full numerical solution of the economic model. Here, we apply the Indirect Inference approach of Gouriéroux et al. (1993).

The dynamics of the asset prices in our model (as defined in equation 29) can be summarized by function $f$ depending on the deep parameters of the model, $\theta = (c, \psi, \gamma)$,

$$\Delta y_{t+1} = f(\Delta y_t; \theta).$$  \hspace{1cm} (37)

The estimation of the model is based on the distance between model–generated moments and the corresponding moments from the data reflected in stylized facts.

Let the stylized facts be summarized in the vector $\beta(\theta)$. Its empirical counterpart is denoted by $\beta_{emp}$. The deep parameters of the structural models are estimated by minimizing the
quadratic distance between observed and replicated moment conditions,

$$\hat{\theta} = \arg\min_\theta \hat{Q}'(y, \bar{y}, \beta) \hat{W} \hat{Q}(y, \bar{y}, \beta), \quad (38)$$

with

$$\hat{Q}(y, \bar{y}, \beta) = \beta_{emp} - \frac{1}{H} \sum_{h=1}^{H} \beta, \quad (39)$$

where \(\hat{W}\) is an arbitrary positive definite matrix that converges in probability to a deterministic positive definite matrix \(W\). Note that in our case \(H = 1\), as we estimate the GARCH(1,1) parameters based on the complete data series. Due to the central limit theorem \(\hat{\theta}\) is asymptotically normal\(^{11}\):

$$\sqrt{H} (\hat{\theta} - \theta^*) \to_d N(0, \text{Var}(\hat{\theta})) \quad (40)$$

The over-identifying restriction of the model are tested with the Wald statistic

$$\frac{H^2}{1 + H} Q'(y, \bar{y}, \beta) \hat{W} Q(y, \bar{y}, \beta) \quad (41)$$

which is asymptotically chi-square distributed with \(\text{dim}(\beta) - \text{dim}(\theta) \geq 0\) degrees of freedom.

To estimate the deep parameters of the model we use the stylized fact that asset price index is driven by the GARCH(1,1) process described above. We simulate the dynamics of the model 500 times over the same time periods as in the empirical data. In order to avoid distortions due to the initial conditioning of the system, we discard the first 10 simulated trading periods. Thus the observations 11 to 60 enter the calculations. For all observed parameters of the GARCH(1,1) process, we report the median values and the standard deviations. The states of the nature that define the dividend yields are bootstrapped, in order to generate pseudo-time series that with an average block length of 35.

The results of the Indirect Inference are summarized in Table 2 on page 41. Note that our model is able to replicate the GARCH(1,1) properties of the benchmark price index quite well. Note also that this result has not been achieved ad-hoc and potentially prone to data

\(^{11}\)Note that it is standard to assume that the parameters are estimated independently with respect to different samples.
snooping, but is based on a solid economic model with only three estimated parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Economic Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
</tr>
<tr>
<td>C</td>
<td>0.053</td>
</tr>
<tr>
<td>K</td>
<td>0.0010</td>
</tr>
<tr>
<td>( \rho(1) )</td>
<td>0.46</td>
</tr>
<tr>
<td>( \hat{\rho}(1) )</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table 2. GARCH (1,1) Statistics of the Model price index

Note that the parameter values of the model are medians of 500 simulation runs. The first ten periods of a simulation run are discarded to avoid price distortion due to initial conditioning of the system. The following data points up to the length of the empirical time series are considered in the calculations.

The adequacy of the model can be estimated by using the Wald statistic defined in 41: Its value equals 0.077. The Wald statistic follows a Chi-square distribution with a degree of freedom (number of estimated parameters minus number of model parameters). In our case, the degree of freedom is thus 1. The Chi-square distribution at \( x = 0.077 \) results in 78.14 percent. We accept thus our null hypothesis that our model adequately generates the conditional volatility with a probability of more than \( \frac{3}{4} \).

Figures 19 and 20 on page 42 show the evolution of the volatility of both the benchmark index and of a the price index that has resulted from a typical simulation run. Note the volatility clustering in both figures.

Three model parameters have been inferred: The consumption rate \( c \), the capital inflow factor \( \psi \), and the exponential dividend trend factor \( \gamma \). The values of these model parameters are:

- \( c = 0.00957 \) (consumption rate)
- \( \psi = 2.46 \) (capital inflow factor)
- \( \gamma = 0.1 \) (dividend growth)
Figure 19. Volatility of benchmark price index during from Q1 1992 to Q2 2004

Figure 20. Volatility of model price index during 50 periods in a typical simulation run
Note that the consumption rate in this context corresponds to management fees of mutual funds. The capital inflow factor scales the capital inflow vs. the dividend payments. Finally, the dividend growth is a small positive number that is compatible with the dividend policy of many firms.

This parametrization of our model will be used in the next section to draw conclusions about the price forecasting power of dividend yields.

4.5 Forecasting power of dividend yields

EHS (2006) show that in a market where traders invest according to the generalized Kelly rule, the asset pricing becomes optimal in the sense of Luenberger (1997). Gerber, Hens and Woehrmann (2005) show that log-optimal pricing is obtained if all investors have logarithmic von-Neumann – Morgenstern utilities. Hence, the generalized Kelly rule could also be obtained as the outcome of an idealized market with a single representative agent having rational expectations. We utilize this finding to define a rational benchmark case of our model, allowing only rational traders in the market.

In Section 4.1, we established equation 32 to define prices in the rational case. We now take advantage of this finding to formulate the following regression:

\[
\ln p^k_t = \alpha^k + \beta^k \ln \left( \lambda^k \sum_{j=1}^{K} dy^j_s(t-1) + \psi \right) + \phi^k \ln (\exp(\gamma t) + 100) + \varepsilon_t
\]

We use equation 42 to regress the output prices of the estimated model, and quantify the forecasting power of the dividend yields on prices with the \( R^2 \) statistic associated to these regressions. If the markets are rational, the dependence captured by the regression 42 is only disturbed by the error term \( \varepsilon_t \). The \( R^2 \)-values will be correspondingly high.

Table 3 on page 44 summarizes the \( R^2 \) values of the investigated stocks in decreasing order. While the fit is very high for certain stocks, it is rather low for others. Figure 21 on page 46
helps to evaluate the fit further. It shows the $R^2$ of the regression for all assets, both resulting from empirical observations and the rational case model. Note that the $R^2$ of roughly two third of the empirical assets lay in the boundaries of the 500 simulation runs. We conclude that the prices of these assets depend strongly on the dividend yields of the previous period.

To explain why the prices of the other third of assets do not depend equally much on the dividend yields, we start from the assumption that these prices may be driven away from their rational level by speculation of noise traders. We introduce noise traders into our model, recalculate the regressions, and compare the thus obtained $R^2$-values to the $R^2$-values of the empirical stocks. The simulations of the model economy are bootstrapped 500 times, with the

<table>
<thead>
<tr>
<th>Company</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNITED TECHNOLOGIES CORP</td>
<td>0.95</td>
</tr>
<tr>
<td>JOHNSON &amp; JOHNSON</td>
<td>0.93</td>
</tr>
<tr>
<td>AMERICAN EXPRESS CO</td>
<td>0.92</td>
</tr>
<tr>
<td>3M CO</td>
<td>0.91</td>
</tr>
<tr>
<td>MERRILL LYNCH &amp; CO INC</td>
<td>0.90</td>
</tr>
<tr>
<td>PFIZER INC</td>
<td>0.88</td>
</tr>
<tr>
<td>MICROSOFT CORP</td>
<td>0.88</td>
</tr>
<tr>
<td>INTEL CORP</td>
<td>0.88</td>
</tr>
<tr>
<td>BANK NEW YORK INC</td>
<td>0.87</td>
</tr>
<tr>
<td>GENERAL ELEC CO</td>
<td>0.87</td>
</tr>
<tr>
<td>INTERNATIONAL BUSINESS MACHS</td>
<td>0.87</td>
</tr>
<tr>
<td>HEWLETT PACKARD CO</td>
<td>0.82</td>
</tr>
<tr>
<td>J P MORGAN CHASE &amp; CO</td>
<td>0.78</td>
</tr>
<tr>
<td>MCDONALDS CORP</td>
<td>0.74</td>
</tr>
<tr>
<td>ALTRIA GROUP INC</td>
<td>0.55</td>
</tr>
<tr>
<td>GENERAL MTRS CORP</td>
<td>0.47</td>
</tr>
<tr>
<td>WACHOVIA CORP 2ND NEW</td>
<td>0.47</td>
</tr>
</tbody>
</table>

Table 3. $R^2$-values associated with the regression of the dividend yields and the time trend on asset prices, according to equation 42.
same random seed as in the indirect inference estimation. Figure 22 on page 46 shows the median $R^2$ values resulting of these simulation runs, as well as the upper and lower bounds of all runs. Overlapping, the empirically calculated $R^2$ values are drawn. Note that the empirical values are all within the upper and lower bounds interval. The fit of the two curves is, in fact, rather good. Note that our model is certainly simplistic, and that the inclusion of other types of noise traders would probably increase the explanatory power of the model further.

Economically, the finding that the presence of noise traders in the market decreases the influence of dividend yields on prices makes sense: The speculation of noise traders leads to price bubbles, and therefore to a deviation from the rational values. Our result confirms thereby the finding in the numerical part of this paper that the active shortening of asset supply in the market, combined with an increased demand of the same asset, drives the asset price up.

## 4.6 Wealth evolution

The asymptotic behavior of the wealth evolution in the model is an important characteristic, as it determines the long term survival of the traders in the model. Figure 23 on page 47 shows the logarithmical evolution of the total wealth during a typical simulation run. Note that the rational trader dominates the other two traders, but without being able to drive them out of the market. This result is consistent with the finding in the numerical part of this paper, who systematically investigates the wealth evolution of various strategies.

The asymptotic behavior of the wealth evolution can be analyzed by calculating the Lyapunov characteristic exponent of the dynamic system governing the evolution of relative wealth: If it is equal to zero for a strategy, it indicates that this strategy will accumulate all wealth in the long run. Such a finding would invalidate the model estimation, as the real model would in this case have to be composed of the dominating strategy only. A negative Lyapunov exponent indicate a stable, cyclical state, while a positive Lyapunov is a necessary but not sufficient
Figure 21. Goodness of fit ($R^2$) when regressing the lagged sum of dividend yields and the exponential time trend on the prices of the assets. In the case of rational prices, the fit is high and the same for all assets. The pricing of roughly two third of the empirical assets lies in the upper/lower bounds of the 500 simulation runs and can thus be considered rational.

Figure 22. Goodness of fit ($R^2$) when regressing the lagged sum of dividend yields and the exponential time trend on the prices of the assets. The presence of noise traders in the market decreases the $R^2$ fit gradually among the assets in the market.
condition for the presence of chaos.

The algorithm how to calculate the Lyapunov exponent is explained in the Appendix on page 51. We find that the Lyapunov characteristic exponents are positive for all three strategies. Figure 24 on page 48 shows the bootstrap distribution of 500 simulation runs. Note that the bootstrapped characteristic exponents seem to follow a lognormal distribution. The median values of the Lyapunov characteristic exponents are 0.1812, 0.2037 and 0.1879 for the relative wealth of the rational trader, the trend follower and the contrarian trader, respectively. The validity condition that no strategy is able to drive the others out of the market is thus fulfilled.

5 Conclusion

We have shown that in a financial market with endogenous asset supply and demand, both rational and noise traders do co-exist in the long run.

This result is valid in a very simple market with two traders, two assets, and only the
Figure 24. Lyapunov Characteristic Exponents related to the relative wealth of the traders
consumption rate as exogenous parameter, as well as in a more sophisticated market with more assets, more traders and additional capital inflow.

The implication of the finding that rational and noise trader co-exist is heavy: Financial markets with endogenous asset supply and demand are neither informationally nor allocationally efficient. Because noise traders survive, they keep a price impact in the long run. This confirms the finding of Kogan, Ross, Wang, and Westerfield (forthcoming) about the price impact of noise traders. However, it contradicts their second conclusion that this price impact makes recovery from heavy losses difficult. Rather, it supports the hypothesis of De Long et al. (1991) that noise traders are able to outgrow rational traders and therefore to recover. We show that the successful speculation of the noise traders is based on their active reduction of the asset’s market supply while increasing the asset demand. Combining both, noise traders are able to drive prices up and to make capital gains. However, they are not able to take over the whole market: Their attempt to realize capital gains by liquidating the asset in the market leads to price crashes and reduces their wealth.

Rational traders are able to consume consistently more than noise traders. Thus, in terms of policy, investors who need a regular cash flow should invest rationally, i.e., hold a portfolio relative to the expected relative dividends of the assets in the portfolio. Investors interested in maximum capital gains may try and beat the market. In fact, there are time-restricted alpha opportunities in the market, but timing is crucial: Noise traders must realize capital gains before the majority of the other noise traders intends to do so, but without sacrificing to much of the alpha potential. Finally, the worst thing to do is 'buy and hold': Neither does this strategy achieve the same cash inflow like the rational strategy, nor does it stand the chance to realize capital gains like noise traders do.

Future research may incorporate active learning in the model. Agent learning may be based on Parse Tree Evolution (Bucher (2005)), Genetic Algorithms (Holland (1975)) or Genetic Pro-
gramming (Koza (1992))\textsuperscript{12}. Model implicit learning may provide further insights about what noise trading strategies are likely to be the most successful under what market conditions.

The target of the empirical part of this paper was to investigate the two main stylized facts of low-frequency price series: The conditional volatility of log-price differences, and the forecasting power of dividend yields on the stock prices.

To do so, we adapted the model in the numerical part of this paper to encompass empirical dividend yield data. Subsequently, we estimated the model using indirect inference. The probability that the GARCH(1,1) process generated by our model coincides with the GARCH(1,1) process of the market benchmark is over 75 percent. Our economically well founded model is thus able to generate a price dynamic that shows the conditional volatility observed in empirical data.

In order to investigate the forecasting power of dividend yields on asset prices, we derived for rational markets a close form of linking today’s dividend yields to tomorrow’s stock prices. Log-linearizing the analytical result, we quantified this influence with a regression. Investigating the S&P500 stocks with the longest quarterly dividend yield history, we found that two third of the considered assets prices are, in the upper and lower bound interval of the simulations, determined by the dividend yields. The remaining assets deviate more or less strongly from the rational price.

To explain the pricing of these assets, we introduced, in addition to the rational trader, two noise traders into our stock market model: Trend followers and contrarian traders. We recalculated the regression, and found that the $R^2$-values of all empirical assets lies within the upper and lower bound interval of the simulations. Thus, we conclude that the presence of noise traders in the market reduces the influence of dividend yields on stock prices.

\textsuperscript{12}For a recent attempt to use Genetic Programming as a learning mechanism in foreign exchange markets, see Lux and Schornstein (2005)
Finally, we examined the wealth evolution of the traders, calculating the Lyapunov characteristic exponent of the dynamic system governing the relative wealth. All Lyapunov characteristic exponents are positive, which implies that none of the trader would either take over the market in the long run or vanish.

6 Appendix

Consider a low \( m \)-dimensional dynamical system \( x_{t+1} = f(x_t) \) with an embedding vector \( x_t = (r_t, \ldots, r_{t+m-1}) \) and \( t = 1, \ldots, T - m + 1 \). Tests, wether these systems are stable, were performed on the basis of Lyapunov exponents. Lyapunov exponents measure stability of dynamical systems by mean exponential divergence of it’s neighborhood trajectories. Hence the spectrum of all Lyapunov exponents \( \lambda = (\lambda_1, \ldots, \lambda_n) \) can be derived analytically as follows:\(^\text{13}\)

\[
\lambda = \lim_{T \to \infty} \frac{1}{T} \ln \left( \prod_{t=1}^{T} Df(x(t)) \right) \tag{43}
\]

Physicists developed various numerical methods, e.g. the Wolf-algorithm, in order to estimate Lyapunov exponents.\(^\text{14}\)

1. Using the algorithm of Wolf a reference trajectory \( y \) is chosen and starting with \( y_0 \) on it a point \( z_0(t_0) \) on the neighborhood trajectory \( z_0 \) with \( |y_0 - z_0(t_0)| \leq \varepsilon \) is selected.

2. Further both trajectories are pursued until the distance between them is larger than \( \varepsilon' \).

3. Then another neighborhood trajectory \( z_1 \) with \( |z_1(t_1) - z_2(t_2)| < \varepsilon'' \) is chosen and the procedure described above is repeated until end of trajectory \( y \) is reached.

\(^\text{13}\)Note that \( D \) stands for the Jacobian-operator.

\(^\text{14}\)Note that we have enough data points to apply the algorithm of Wolf since we want to calculate the largest Lyapunov exponent with respect to the simulations.
Thus, applying the Wolf algorithm the largest Lyapunov $\lambda_1$ exponent can be computed as

$$\lambda_1 = \frac{1}{T \Delta t} \sum_{i=0}^{M-1} \ln \frac{H_i'}{H_i}$$

with $M$ iterations of length $\Delta t$. Especially examining short time series the main problem is the determination of $\varepsilon, \varepsilon', \varepsilon''$.

References


