On the design of coordinating contracts

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A B S T R A C T

We concentrate on a principal and agent system often seen in supply chain management studies. Under reasonable conditions, we show that a nearly systematic three-step method can be used to find coordination contracts for the system. The number of terms involved in our contract is in some sense positively correlated with the degree of information asymmetry between the principal and agent. Several known contract types for the supplier–retailer newsvendor system can be viewed as special cases resulting from applications of this method. Following our three-step approach, we succeed in identifying coordinating contracts for a supply chain involving two products and unobservable retailer efforts.

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1. Introduction

In the business world, it can often be observed that two parties would collaborate in such a manner that one party, dubbed the agent, would accomplish a task for a fee for another party, dubbed the principal. The structuring of the fee plays the role of re-aligning incentives for the two parties and hence, influencing both parties' decision processes. Due to the different interests of the two parties, the best one could hope for is the so-called Pareto-optimal fee structure, one such that, neither party could be better off without hurting the other under a different fee contract than the current one.

In the economics literature on the principal–agent theory, research has been focused on the case where the accomplishment of the agent is a certain prize which can be divided between the two parties. One may refer to Mirrlees (1974), Harris and Raviv (1978), Shavell (1979), Holmström (1979), and Grossman and Hart (1983) for works done on different aspects of the theory. Plambeck and Zenios (2000) introduced a multi-period principal–agent model with some special features tailored to OM applications, and provided a two-step dynamic programming method for its analysis. Marketing is one area where the economics theory has been successfully applied. For instance, Basu et al. (1985) used results of Holmström (1979) to explain differences in compensation plans seen in sales forces in different industries. Rao (1990) studied compensation plans made up solely of quotas, payments, and commissions for a heterogeneous sales force.

As we aim at applications to supply chain management problems, our focus is different from that of the traditional economics approach. We deal with a principal and agent problem in which both parties are risk neutral, and hence system coordination implies Pareto-optimality. On the other hand, unlike in the traditional theory, we do not assume that a dividable prize can be identified as the agent’s accomplishment. In particular, we assume that the agent may have incentives other than earning the transfer fee to expend his effort.

Much research has been done on supply chain competitions and coordinations. Pasternack (1985) introduced a simple buyback contract which can enable a self-interested pair of supplier and retailer to behave optimally for the entire chain, thus achieving channel coordination. Cachon (1999) and Lariviere (1999) gave reviews on supplier–retailer coordinations, with the former focusing on a multi-period setting where both parties employ base

Corbett et al. (2004) studied the value, to a supplier, of information about the cost structure of a retailer she faces, who in turn faces a deterministic demand quantity that is linear in the retail price. Through the adoption of retailer-and price-dependent price-discounting-and-buyback contracts, Bernstein and Federgruen (2005) achieved channel coordination in the face of price-sensitive demands and multiple retailers. Bernstein et al. (2006) identified conditions under which simple pricing contracts can coordinate a supply chain system containing one retailer and multiple retailers. Coordination problems also arise in other applications, such as the call center problem studied by Ren and Zhou (2008).

Implications of and remedies for unobservable effort have lately become hotly pursued topics. Taylor (2002) proposed a buyback and target rebate contract to coordinate a supplier–retailer system in which the retailer effort is unobservable. Krishnan et al. (2004) treated the case where retailer effort is made after demand realization. Chen (2005) studied coordination in the presence of both unobservable sales force effort and asymmetric marketing information. Cachon and Lariviere (2005) analyzed the strengths and limitations of revenue sharing contracts in both single- and multi-retailer settings. They pointed out that these contracts do not coordinate systems with unobservable retailer effort. Lee and Schwarz (2007) presented payment schemes that could be used by a principal to induce an agent to achieve desirable leadtime levels for an (Q, r) inventory system.

While various coordinating contracts have been discovered, their design and formation mostly still result from case-dependent efforts. In this paper, we attempt to uncover some general underlying principles in the design of coordinating contracts. Specifically, for our principal and agent problem, we propose a nearly systematic method for obtaining a coordinating contract under conditions that the agent’s utility function is concave in his effort, and that the random outcome of his effort is of a linear form. The method boils down to:

\begin{itemize}
  \item \textbf{Step 1:} identifying concave and linear terms in the agent’s utility function;
  \item \textbf{Step 2:} introducing additional concave terms;
  \item \textbf{Step 3:} deciding coefficients for aforementioned terms in the transfer fee contract using linear programming or linear algebra techniques.
\end{itemize}

The one “un-systematic” step of this approach is Step 2. Still, our method seems less ad hoc than the traditional approach, which is based mostly on experience and business acumen. The end result from applying our method will be a specific contract containing nonlinear terms. Once the terms are determined, we have a set procedure to decide the coefficients that go with these terms.

To facilitate the presentation of our general framework and demonstrate the usefulness of our method, we introduce an example involving a supplier and retailer pair as well as multiple effort and product types, which is essentially a generalization of Taylor’s (2002) system. In the example, the retailer’s marketing effort is not observable to the supplier, but the final demand realizations are.

Among contracts that coordinate the well-studied supplier–retailer newsvendor system, we show that the wholesale price, buyback, and revenue sharing contracts can all be viewed as special cases resulting from applications of our method. These contracts differ in Step 2 of the method, the selection of terms to be included in the transfer fee. We also show that, as a specialized version of the one involved in the multi-product-effort example, the system studied by Taylor (2002) can be coordinated using contracts derived from our method. We go further to show that our method can be used to identify coordinating contracts for a supply chain involving two products and unobservable retailer efforts. These contracts can arbitrarily carve up the total profit between the supplier and retailer. Due primarily to its nonconcave nature, we find that it is hard to extend Taylor’s approach to this problem.

The rest of the paper is organized as follows. In Section 2, we precede the more formal introduction of our method by a re-examination of Taylor (2002)’s problem employing ideas embedded in the method. In Section 3, we formulate the general problem; and in Section 4, we present the main method for finding coordinating contracts. We show how our method can be applied to a multi-product/effort problem involving private effort levels in Section 5, and specialize our discussion to a two-product system in Section 6. We conclude the paper in Section 7.

2. A quantity and effort model

Taylor (2002) studied the following problem. A make-to-order supplier is charged with supplying one product type to a retailer at a unit production cost c. The retailer will receive p for each unit sold and s for every remaining unit; and, in the absence of any other contract, he will pay the supplier a nominal wholesale price w0 for every unit ordered. The retailer decides the quantity \( \tilde{q} \) to order from the supplier and the marketing effort \( \tilde{e} \), known only to himself, to expend. Effort level \( \tilde{e} \) will incur cost \( V(\tilde{e}) \) to the retailer, with \( V(\cdot) \) being an increasing and convex function, and will result in a random demand \( D = Me \) for some random variable \( M \).

For this model, the retailer payoff function is

\begin{equation}
  f(\tilde{q}, \tilde{e}) = (p - w0) \cdot \tilde{q} + (p - s) \cdot (-E(\tilde{q} - M\tilde{e})^+) - V(\tilde{e}),
\end{equation}

where the first term is what the retailer expects to earn if there were no left-over items, the second term describes
his loss due to potential over-ordering, and the last term comes from the cost of his sales effort. After observing the realized demand level \(d\), the supplier payoff function is
\[
g^0(q, d) = (w^0 - c) \cdot q,
\]
where the first term is what the supplier will earn from the retailer, and the second term reflects her production cost. Note that we have purposely expressed \(g^0\)'s silent dependence on \(d\) to make our notation consistent with the more general cases to be introduced later. For systemwide payoff, we have
\[
(f + g)(\tilde{q}, \tilde{e}) = f(\tilde{q}, \tilde{e}) + E[g^0(\tilde{q}, M\tilde{e})] = (p - c) \cdot \tilde{q} + (p - s) \cdot (-E(\tilde{q} - M\tilde{e})^+) - V(\tilde{e}).
\]
(3)

Apparently, both \(f\) and \(g\) are concave functions. For the entire system, the optimal solution \((\tilde{q}^0, \tilde{e}^0)\) can be obtained by solving the following:
\[
\begin{align*}
\tilde{c}_q(f(\tilde{q}^0, \tilde{e}^0) + g(\tilde{q}^0, \tilde{e}^0)) & \leq 0, \\
\tilde{c}_q(f(\tilde{q}^0, \tilde{e}^0) + g(\tilde{q}^0, \tilde{e}^0)) & \leq 0.
\end{align*}
\]
(4)

But when left to his own device, the retailer will decide solution \((q^0, e^0)\) that is best to himself, by solving the following:
\[
\begin{align*}
\tilde{c}_q(f(q^0, e^0)) & \leq 0, \\
\tilde{c}_q(f(q^0, e^0)) & \leq 0,
\end{align*}
\]
(5)

Solution \((q^0, e^0)\) is most likely sub-optimal when the system-wide benefit is of concern.

We may use a contract to induce the retailer to make more considerate decisions. The essence of the contract is the transfer of some wealth \(t\), from the supplier to the retailer, that is contingent on observable outcomes from the retailer's decision. As the supplier can only observe \(\tilde{q}\) and \(d\), the contract to be sought after should be in the form of \((\tilde{q}, d)\). Ideally, the contract should coordinate the system, in the sense that its resulting average payoff \(f(\tilde{q}, \tilde{e}) + E[t(\tilde{q}, M\tilde{e})]\) to the retailer would lure him to make the system-optimal decision \((q^*, e^*)\).

Taylor proposed the following returns and rebate contract:
\[
t(\tilde{q}, d) = (w^0 - w) \cdot \tilde{q} + ((p - b) \cdot \tilde{q} - p + s)
\]
\[
\times ((\tilde{q} - d)^+) + u \cdot \min(\tilde{q}, d) - T^+ \\
= (w^0 - w) \cdot \tilde{q} + (b - s) \cdot \tilde{q} - d^+ \\
+ u \cdot \min(\tilde{q}, d) - T^+.
\]
(6)

where \(w\) is the actual wholesale price, \(b \in [s, p]\) is the return credit for each unsold unit, \(u\) is the unit rebate rate for achieving target level, and \(T\) is the rebate target. Except for the last target rebate form which is not concave in \((q, d)\), this contract is almost in the form we are to propose.

As for our alternative contract, we adopt the following general guideline to obtain it. First, we decompose the retailer payoff function \(f(\tilde{q}, \tilde{e})\) so that
\[
f(\tilde{q}, \tilde{e}) = E[\tilde{f}_1(\tilde{q}, M\tilde{e})] + \cdots + E[\tilde{f}_s(\tilde{q}, M\tilde{e})] + f^R(\tilde{q}, \tilde{e}),
\]
(7)

with \(\tilde{f}_1(q, d), \ldots, \tilde{f}_s(q, d)\) all being concave in \((q, d)\) and \(f^R(\tilde{q}, \tilde{e})\) being concave in \((\tilde{q}, \tilde{e})\). Then, we design/invent some other \(r - a\) concave terms \(\tilde{f}_{a+1}(\tilde{q}, d), \ldots, \tilde{f}_r(\tilde{q}, d)\) for which it is easy to find business meanings, and consider contract \(t(\tilde{q}, d)\) in the following fashion:
\[
t(\tilde{q}, d) = (\gamma_1^1 + 1) \cdot \tilde{f}_1(\tilde{q}, d) + \cdots + (\gamma_r^r + 1) \cdot \tilde{f}_r(\tilde{q}, d)
\]
\[
+ \gamma_{a+1}^+ \cdot \tilde{f}_{a+1}(\tilde{q}, d) + \cdots + \gamma_{r+1}^+ \cdot \tilde{f}_{r+1}(\tilde{q}, d) + \gamma_{r+2}^+ \cdot \tilde{q}
\]
\[
+ \gamma_{r+3} \cdot d + \gamma_{r+3}^+.
\]
(8)

where \(\gamma_1^1, \ldots, \gamma_r^r\) are positive constants, while \(\gamma_{r+1}^+, \gamma_{r+2}^+\), and \(\gamma_{r+3}\) are mere constants. The contract form (8) guarantees the concavity of the ultimate retailer payoff \(f(\tilde{q}, \tilde{e}) + E[t(\tilde{q}, M\tilde{e})]\), as
\[
f(\tilde{q}, \tilde{e}) + E[t(\tilde{q}, M\tilde{e})]
\]
\[
= \gamma_1^1 \cdot E[\tilde{f}_1(\tilde{q}, M\tilde{e})] + \cdots + \gamma_r^r \cdot E[\tilde{f}_r(\tilde{q}, M\tilde{e})]
\]
\[
+ f^R(\tilde{q}, \tilde{e}) + \gamma_{r+1}^+ \cdot \tilde{q} + \gamma_{r+2}^+ \cdot E[M\tilde{e}] + \gamma_{r+3}^+.
\]
(9)

Finally, we settle on positive parameters \(\gamma_1^1, \ldots, \gamma_r^r\) and parameters \(\gamma_{r+1}^+, \gamma_{r+2}^+, \gamma_{r+3}\) by solving
\[
\begin{align*}
\tilde{c}_q(f(q^0, e^0) + E[t(q^0, M^e)]) & \leq 0, \\
\tilde{c}_q(f(q^0, e^0) + E[t(q^0, M^e)]) & \leq 0,
\end{align*}
\]
(10)

as well as selecting a proper allocation of profits between the supplier and the retailer. Because all terms are concave, the above first-order optimality conditions will suffice for the retailer to adopt the system-optimal solutions out of his own interest. We will show that the last step of parameter identification can be achieved through solving certain linear programs.

We note that a forcing contract is within our framework: \(t(\tilde{q}, d) = -M \cdot |\tilde{q} - \tilde{q}^*|\) is a concave function, and when \(M\) is greater than the largest possible marginal utility/disutility of a unit product, the retailer’s best ordering quantity under this contract will be the coordinating \(\tilde{q}^*\). We caution, however, that ordering quantities are often not contractible; hence, they often cannot be forced.

Let us resume with the particular example. From (1), we identify concave term \(\tilde{f}_1(q, d) = -(p - s) \cdot (\tilde{q} - d)^+\) and residue term \(f^R(\tilde{q}, \tilde{e}) = (p - w^0) \cdot \tilde{q} - V(\tilde{e})\) for our current \(f(\tilde{q}, \tilde{e})\). We now invent concave term \(\tilde{f}_2(q, d) = -(\tilde{q} - (1 + \varepsilon)^2)^2\), which represents the penalty to the retailer when his ordering quantity strays too much away from the level \((1 + \varepsilon)^2\) for some fixed \(\varepsilon > 0\), and concave term \(\tilde{f}_2(q, d) = -(\tilde{q} - d)^2\), which represents the penalty to the retailer when he ends up with too much leftover. When \(\varepsilon\) is small enough, the concerned term will almost be for punishing deviations from the optimal ordering quantity.

After (22) later, it will be clear that the strict positivity of \(\varepsilon\) is needed for the arbitrary allocation of profits between the supplier and the retailer. The alternative term \(\tilde{f}_2(q, d) = -(\tilde{q} - d)^2\) after forcing \(\varepsilon = 0\) certainly seems more straightforward than our choice of \(\tilde{f}_2(q, d)\). The former will indeed still induce coordination. However, the payoff allocation under this coordination will be inflexible and unattractive to the supplier, as under it she will always incur a net loss. Although the real reason lies behind the linear program (20) to be introduced, we stress that this is related to the need for a term, penalizing the
manufacturer for ordering too little, to balance out terms \( \ell_1 \) and \( \ell_2 \), which all penalize the manufacturer for ordering too much.

Note that
\[
\ell_2(\tilde{q}, d) = -(\tilde{q} - (1 + \epsilon)\tilde{q})^2
= -\epsilon^2(\tilde{q}^2) + 2\alpha \tilde{q} \cdot (\tilde{q} - \tilde{q})^2.
\]
(11)

Therefore, an alternative interpretation for \( \ell_2 \) is that it is really a combination of three terms. The first term is a transfer of \( \epsilon^2(\tilde{q}^2) \) in fixed fees from the retailer to the supplier; the second term is a price discount of \( 2\alpha \tilde{q} \) per unit to the retailer; and the third term is a deviation penalty in the more conventional sense.

Following some re-definitions of parameters for ease of presentation, we express the contract as follows:
\[
t(\tilde{q}, d) = (\gamma_1^+ - (p - s)) \cdot (-(\tilde{q} - d)^+) + \gamma_2^+ \cdot ((\tilde{q} - (1 + \epsilon)\tilde{q})^2)
+ \gamma_3^+ \cdot ((\tilde{q} - d)^+)^2 + (w^d - (-\gamma_4)) \cdot \tilde{q},
\]
(12)

where we have omitted the constant term in (8) which is not needed here. Using this contract, we may accommodate the possibility that even \( d \) is not fully observable unless when it is below \( \tilde{q} \).

In the contract, \( \gamma_1^+ - (p - s) \) stands for the extra penalty on top of (reduction in penalty from) the price–salvage difference for every unit of over-ordering, when it is positive (negative); \( \gamma_2^+ \) is essentially for punishing under-ordering; \( \gamma_3^+ \) is for punishing over-ordering, and it being strictly positive will make the overall penalty convex rather than linear in the over-ordering level; and, \( -\gamma_4 \) is the actual wholesale price (negative of the subsidy for a unit order), when it is positive (negative). An alternative way to understand \( \gamma_4^+ \) is that, when \( \gamma_4^+ \in [0, p - s) \), the term \( p - \gamma_4^+ \), much like the constant \( b \) in the returns/buyback contract in Taylor, is the return to the retailer on each unsold product unit, which the supplier takes back and salvages herself; when \( \gamma_4^+ \in [p - s, +\infty) \), the term \( \gamma_4^+ - (p - s) \) is the penalty charged to the retailer for each unsold product unit.

Under this contract, we have
\[
f(\tilde{q}, \tilde{e}) + E[t(\tilde{q}, M\tilde{e})] = \gamma_1^+ \cdot (-E[(\tilde{q} - M\tilde{e})^+])
+ \gamma_2^+ \cdot ((\tilde{q} - (1 + \epsilon)\tilde{q})^2)
+ \gamma_3^+ \cdot (-E[(\tilde{q} - M\tilde{e})^+])
+ (p + \gamma_4) \cdot \tilde{q} - V(\tilde{e}),
\]
(13)
while (10) will translate into the following:
\[
\begin{align*}
&P[M \leq \tilde{q} / \tilde{e}] \cdot \gamma_1^+ + 2\alpha \tilde{q} \cdot \gamma_2^+ \\
&= -2 \cdot E[(\tilde{q} - M\tilde{e})^+] \cdot \gamma_3^+ + \gamma_4 = -p.
\end{align*}
\]

As in Taylor (2002), we may assume that \( V(\tilde{e}) = a \cdot \tilde{e}^2 / 2 \) for some positive constant \( a \) and that \( M \) is uniformly distributed between 0 and 1. Using a method based on linear programming, we can find suitable parameters \( \gamma_1^+, \gamma_2^+, \gamma_3^+ \geq 0 \) and \( \gamma_4 \) that satisfy (14). The method can also help us fine tune the allocation of profits between the two parties.

According to Taylor, it is true that
\[
\tilde{e}^* = \frac{(p - c)^2}{2a \cdot (p - s)} \quad \text{and} \quad \tilde{e}^* = \frac{(p - c)^3}{2a \cdot (p - s)^2}.
\]
(15)

Plugging these into (3), we obtain the total system payoff as
\[
(f + g)(\tilde{q}^*, \tilde{e}^*) = \frac{(p - c)^4}{8a \cdot (p - s)^3}.
\]
(16)

By (13), we have
\[
f(\tilde{q}^*, \tilde{e}^*) + E[t(\tilde{q}^*, M\tilde{e}^*)] = Z^2(\gamma_1^+, \gamma_2^+, \gamma_3^+, \gamma_4) = \frac{(p - c)^3 \cdot (3p + c)}{8a \cdot (p - s)^2}.
\]
(17)

where
\[
Z^2(\gamma_1^+, \gamma_2^+, \gamma_3^+, \gamma_4) = \left[ \begin{array}{c}
-p \cdot \gamma_1^+ / (p - s) + \epsilon(\gamma_1^+ - (p - s)) \cdot \gamma_2^+ / (a \cdot (p - s)^2) \\
-p \cdot \gamma_1^+ / (2a \cdot (p - s)^2) + \gamma_4 = -p, \\
(p - c)^4 \cdot \gamma_3^+ / (2 \cdot (p - s)^3) + (p - c)^2 \cdot \gamma_3^+ / (6a \cdot (p - s)^4)
\end{array} \right].
\]
(18)

By (14), we also have
\[
\left\{ \begin{array}{l}
-(p - c) \cdot \gamma_1^+ / (p - s) + (4a \cdot (p - s)^3) \cdot \gamma_2^+ / (a \cdot (p - s)^2)
\end{array} \right.
\]= (p - c)^2 / (2 \cdot (p - s)).
\]
(19)

Suppose \( p = 10, c = 6, s = 2, \) and \( a = 1 \). Then, we have that the system-optimal effort level \( \tilde{e}^* = 1 \), the system-optimal order quantity \( \tilde{q}^* = \frac{3}{2} \), and the system-optimal payoff \( (f + g)(\tilde{q}^*, \tilde{e}^*) = \frac{1}{2} \). Now we can identify suitable contract parameters by investigating the following linear programs, whose objective comes from (18) and constraints come from (19):
\[
\begin{align*}
\frac{Z_{\text{min/max}} = \text{min/(max)}}{s.t.} \quad & -\gamma_1^+ / 8 - \epsilon^2 \gamma_2^+ / 4 - \gamma_4^+ / 24 + \gamma_4 / 24 = -10, \\
\quad & \gamma_1^+ / 8 + \gamma_2^+ / 24 + \gamma_4 = 1, \\
\end{align*}
\]
\[
\gamma_2^+ \quad \geq 0, \\
\gamma_4^+ \quad \geq 0.
\]
(20)

We can check that the maximization portion of (20) has multiple solutions and the minimization portion is unbounded. Indeed, we have
\[
\gamma_1^+ = 8 - \frac{\gamma_2^+}{3} \quad \text{and} \quad \gamma_4 = -6 - \epsilon^2 \gamma_2^+ + \frac{\gamma_2^+}{12}.
\]
(21)

Plugging (21) into (18) and (17), we obtain retailer payoff
\[
f(\tilde{q}^*, \tilde{e}^*) + E[t(\tilde{q}^*, M\tilde{e}^*)] = \left( -4 - \frac{3\gamma_2^+}{4} + \frac{\gamma_2^+}{24} \right) + \frac{9}{2}
= \frac{1}{2} - \left( -\epsilon^2 / 4 + \frac{\epsilon}{2} \right) \gamma_2^+ + \frac{1}{24} \gamma_4^+.
\]
(22)

Hence, the supplier’s payoff under this contract is \( (\epsilon^2 / 4 + \epsilon / 2) \gamma_2^+ / 124 \). For \( \gamma_2^+ \) to be positive, we need \( \gamma_2^+ \in [0, 24] \), while for both parties to make money, we need to impose that \( 6a(\epsilon + 2) \gamma_2^+ / 124 \in [0, 12] \). Due to the
latter requirement, we see that the $\gamma^2_x \cdot (\tilde{q} - (1 + \varepsilon)\tilde{q})^2$ term in the contract cannot be dispensed of without hampering the contract’s ability to arbitrarily allocate profits between the supplier and the retailer.

Further analysis of this problem has been presented in Appendix A. In summary, we have used Taylor’s method to illustrate the following three main steps of our method: concave-term decomposition, new-term introduction, and term-parameter identification. We are now in a position to more formally describe the method in a more general setting, which we will achieve in the next three sections. We will also compare this method with Taylor’s approach on a two-product system in Section 6.

3. The general formulation

We consider a system made up of one principal–agent pair. There is a decision set $X$. At the time of decision making, there is a random factor $\Omega$ (drawn from some probability space $(\Omega, \mathcal{F}, P)$) whose distribution is commonly known while realization is equally unknown. When $x \in X$ is taken, the agent will try to solve the following problem:

$$\max_{x \in X} f(x) + g(x)$$

(23)

where we have let $g(x) = E[f^0(h(x, \Omega))]$. Under the realization $\omega$ of $\Omega$ and decision $x$, what is observable by the principal is $y = h(x, \omega)$ for some fixed function $h$. Under observable $y$, the principal gets real-valued payoff $g^0(y)$.

For the example in the previous section, the supplier is the principal, the passive taker of the other party’s decision after the signing of the contract, and the retailer is the agent. We may treat the retailer’s order–effort pair $(\tilde{q}, \tilde{e})$ there as the decision $x$ here, the random multiplier $M$ there as the random factor $\Omega$ here, the order–demand pair $(\tilde{q}, \tilde{d})$ as the observable $y$ here, and the map $(\tilde{q}, \tilde{e}) \to (\tilde{q}, \tilde{m})$ as the action-to-observable map $h(x, \omega)$ here.

For the entire system’s total benefit, the optimization problem for finding the best decision $x^*$ is as follows:

$$z^* = \max_{x \in X} f(x) + g(x)$$

(24)

where we have let $g(x) = E[f^0(h(x, \Omega))]$. When only the agent is allowed to decide for $x$, while the principal can only accept the consequence of the agent’s decision, the former will solve the following problem:

$$z^0 = \max_{x \in X} f(x)$$

(25)

of choosing the $x$ that maximizes $f$ while ignoring the valuation of $g$.

To counter the self-serving inclination of the agent, people design contracts that force the transfer of wealth $t(y)$ from the principal to the agent. This way, the agent will try to solve

$$\hat{z}^i = \max_{x \in X} f(x) + E[t(h(x, \Omega))]$$

(26)

while the principal will on average receive $g(x) - E[t(h(x, \Omega))]$ for whatever decision $x$ the agent makes.

Given decision set $X$, agent payoff function $f$, principal payoff function $g^0$, random factor $\Omega$, and observation function $h$, we say that contract $t$ coordinates the system $(X, f, g^0, \Omega, h)$, when

$$\arg\max_{x \in X} \{f(x) + g(x)\} \cap \arg\max_{x \in X} \{f(x) + E[t(h(x, \Omega))]\} \neq \emptyset.$$

(27)

That is, $t$ will be considered a coordinating contract when there exists some decision $x^*$ that is simultaneously optimal for the entire system and for the agent when he is to receive wealth transfer in the form of $t(y)$. To see how our setting can fit into the principal–agent framework more familiar to economists, the interested reader may refer to Appendix B.

A special case of our principal and agent problem is one in which there is no randomness in the problem, or effectively, $\Omega$ can take only one value, say some $\omega_0$ ($\hat{\Omega} = \{\omega_0\}$). Contract $t(y)$ will be coordinating when $f(x) + t(h(x, \omega_0))$ reaches its maximum at the system-wide optimal solution $x^*$. We note that the partnership (PART) contract proposed by Ren and Zhou (2008), between a call center (the agent) and a user (the principal), is of this type when what is observable to the user is the call center’s expected problem resolution probability $p(e)$, but not its effort $e$. Our $x$ corresponds to this paper’s $(s, e)$, where $s$ is the call center’s staffing level, and our $h(x, \omega_0)$ corresponds to their map from $(s, e)$ to $(s, p(e))$.

More special is the case where furthermore, the decision–observable transition function $h(\cdot, \omega_0)$ is known to be invertible, and hence observable $y$ can be treated in the same fashion as decision $x$. For this case, we can express the contract directly in terms of $x$. For $x \in (0, 1]$, consider the contract with a transfer fee in the form of

$$t(x) = -(1 - x) \cdot f(x) + x \cdot g(x).$$

(28)

It is coordinating since under it, we have

$$\begin{align*}
\min_{x \in [0, 1]} & f(x) + t(x) = x \cdot f(x) + g(x), \\
& g(x) - t(x) = (1 - x) \cdot f(x) + g(x),
\end{align*}$$

and hence the agent receiving the transfer fee $t$ will act optimally for the entire system. This contract lets the agent and principal share the total profit, in respective portions of $x$ and $1 - x$. The pay-per-call-resolved plus cost-sharing (PPCR+CS) contract proposed by Ren and Zhou (2008), under the circumstance when the call center’s effort is observable by the user, falls into this category.

4. The concave-function decomposition method

It is very common that the decision set $X$ is an $n$-dimensional closed interval $[x_1, x_2] \times \cdots \times [x_{n_1}, x_{n_2}]$, where for $i = 1, \ldots, n, x_i$ may be $-\infty$ while $x_i$ may be $+\infty$, that observable $y = y = (y_1, \ldots, y_m)^T$ is an $m$-dimensional vector, and that both $f$ and $f + g$ are concave. For this case, by the Karush–Kuhn–Tucker (KKT) condition, we know that $x^* = (x^*_1, \ldots, x^*_n)^T \in X$ will be an optimal solution to (23) if and only if, for $i = 1, \ldots, n$,

$$\begin{align*}
\partial_x f(x^*) + \partial_x g(x^*) & \begin{cases} 
0 & \text{when } x^*_i = x_i, \\
= 0 & \text{when } x^*_i \in (x_i, x_i), \\
\geq 0 & \text{when } x^*_i = x_i.
\end{cases}
\end{align*}$$

(29)
Suppose \( t(h(x, \omega)) \) is concave in \( x \). Then, this \( x^* \) will also be an optimal solution to (23) if and only if, for \( i = 1, \ldots, n \),
\[
E(\{dy(t(h(x^*, \Omega))\}^T \cdot \partial_x h(x^*, \Omega))
\]
\[
\leq -\partial_x f(x^*) \quad \text{when } x_i^* = x_i,
\]
\[
= -\partial_x f(x^*) = \partial_x g(x^*) \quad \text{when } x_i^* \in (x_i, x_i),
\]
\[
\geq -\partial_x f(x^*) \quad \text{when } x_i^* = x_i,
\]
where \( dy(t(y), \ldots, \partial_x g(t(y))^T \) and \( \partial_x h(x, \omega) = (\partial_x h_1(x, \omega), \ldots, \partial_x h_m(x, \omega))^T \).

We consider a more special and yet still fairly common case. In this case, we further assume the following:

(1) \( x^* \) is an interior solution to (23);
(2) the random matrix \( \Omega = (\Pi, \Sigma) \), where \( \Pi \) is an \( m \times n \) random matrix and \( \Sigma \) an \( m \)-dimensional random vector, with \( E[\Pi] = (p_1, \ldots, p_m)^T \) and \( E[\Sigma] = (s_1, \ldots, s_m)^T \) for \( n \)-dimensional vectors \( p_1, \ldots, p_m \) and real constants \( s_1, \ldots, s_m \);
(3) the observation function \( h(x, \omega) = h(x, \pi, \sigma) = \pi x + \sigma \);
(4) there are \( r \) concave functions \( t_1, \ldots, t_r \), from \( \mathbb{R}^m \) to \( \mathbb{R} \) and one concave function \( f^R \) from \( \mathbb{R}^m \) to \( \mathbb{R} \), such that
\[
f(x) = E[f_1(\Pi x + \Sigma)] + \cdots + E[f_r(\Pi x + \Sigma)] + f^R(x).
\]

For this case, we seek contract \( t \) of the following form:
\[
t(y) = (\gamma_i - 1) \cdot t_1(y) + \cdots + (\gamma_i - 1) \cdot t_k(y)
\]
\[
+ \gamma_{i+1} \cdot t_{i+1}(y) + \cdots + \gamma_{i+r} \cdot t_i(y)
\]
\[
+ \gamma_{r+1} \cdot y_1 + \cdots + \gamma_{r+m} \cdot y_m + \gamma_{r+m+1}
\]
(32)
where \( r \) is an integer constant above \( a \), \( t_{a+1}, \ldots, t_r \) are concave functions from \( \mathbb{R}^m \) to \( \mathbb{R} \), \( \gamma_i^+ \), \( \gamma_i^- \) are positive parameters, and \( \gamma_{r+1}, \ldots, \gamma_{r+m}, \gamma_{r+m+1} \) are real parameters. Under this contract form, the agent’s objective is
\[
f(x) + E[t(\Pi x + \Sigma)]
\]
\[
= \gamma_1^+ \cdot E[t_1(\Pi x + \Sigma)] + \cdots + \gamma_r^+ \cdot E[t_r(\Pi x + \Sigma)]
\]
\[
+ \gamma_{r+1} \cdot (p_1^T x + s_1) + \cdots + \gamma_{r+m} \cdot (p_m^T x + s_m)
\]
\[
+ \gamma_{r+m+1} + f^R(x),
\]
which is guaranteed to be concave.

This contract has a reasonable interpretation. For \( j = 1, \ldots, a \), when \( \gamma_j^+ \in [0, 1] \), it is as if the agent will still retain a \((1 - \gamma_j^+)\) portion of his earning on the term \( E[t_j(\Pi x + \Sigma)] \) to the principal; while when \( \gamma_j^+ > 1 \), it is as though the agent will earn an additional \((\gamma_j^+ - 1)\) portion of his \( E[t_j(\Pi x + \Sigma)] \) term. For \( j = a + 1, \ldots, r \), if \( t_j(y) \) is explicitly a term belonging to \( g^0(y) \), it is as if the principal will give \( \gamma_j^+ \) times of what she earns on this term to the agent; otherwise, \( \gamma_j^+ \cdot t_j(y) \) is merely a portion of the transfer fee from the principal to the agent. For \( j = r + 1, \ldots, m + r \), if \( \gamma_j > 0 \), then \( \gamma_j y_{r+j} \) is what the principal pays to the agent for realizing \( y_{r+j} \); otherwise, \( -\gamma_j y_{r+j} \) is what the agent pays to the principal for causing \( y_{r+j} \). Finally, \( \gamma_{r+m+1} \) represents a fixed fee transferred between the supplier and the retailer.

From our earlier assumptions and (30), we know that contract \( t \) defined through (32) is coordinating if and only if
\[
\gamma_1^+ t_1 + \cdots + \gamma_r^+ t_r = g = \gamma_{r+1} p_1 + \cdots + \gamma_{r+m} p_m,
\]
where we have let
\[
\begin{align*}
\{ g = -d(x^*) \},
\end{align*}
\[
\begin{align*}
t_j = E[t_j(\Pi x + \Sigma)], \quad \forall j = 1, \ldots, r.
\end{align*}
\]

In view of (33), we see that the agent’s payoff under contract \( t \) is given by the following:
\[
f(x) + E[t(\Pi x + \Sigma)] = t_1^+ + \cdots + t_r^+ t_i + p_1^+ y_{r+1}
\]
\[
+ \cdots + p_m^+ y_{r+m} + f^R(x),
\]
where \( t_j = E[t_j(\Pi x + \Sigma)] \) for \( j = 1, \ldots, r \), and \( p_i^+ = p_i^T x + s_i \) for \( i = 1, \ldots, m \).

When the principal’s willingness to enter the contract is ignored, we can obtain the range of agent payoff \( z^+_{\min} \) from \( \gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_{r+m} \) by solving the following two linear programming problems:
\[
z^+_{\min} = \min \{ \gamma_1^+ + \cdots + \gamma_r^+ t_i + p_1^+ y_{r+1}
\]
\[
+ \cdots + p_m^+ y_{r+m} + f^R(x),
\]
s.t. \( t_1^+ + \cdots + t_r^+ t_i + p_1^+ y_{r+1}
\]
\[
+ \cdots + p_m^+ y_{r+m} = s_1,
\]
\[
\gamma_1^+ \geq 0,
\]
\[
\gamma_r^+ \geq 0.
\]

A natural condition for the principal to be willing to enter into the contract is \( g^0(x^*) \geq g^0(x^*) \), where as was mentioned before, \( x^* \) is a solution for (24). Since under decision \( x^* \), the system-wide payoff is \( f(x^*) + g(x^*) \), the principal’s share of it will be more than \( g^0(x^*) \) if and only if the agent’s share is less than \( f(x^*) + g(x^*) - g^0(x^*) \). Under the natural condition, the range of agent payoff will be \( z_{\min}, z_{\max} \), where \( z_{\min} = (z_{\min} + z_{\max}) + f^R(x^*) \) and \( f^R(x^*) \) is the decomposed function of \( f(x) \) as in (31);

Step 1: the identification of the concave terms \( t_1, \ldots, t_r \) and \( f^R(x) \) that result in the decomposition of \( f(x) \); and

Step 2: the design/invention of additional \( r - a \) concave terms \( t_{a+1}, \ldots, t_r \); and

Step 3: the determination of positive coefficients \( \gamma_{r+1}, \ldots, \gamma_{r+m} \) and coefficients \( \gamma_{r+1}, \ldots, \gamma_{r+m} \) through solving the linear programs (37) and going over ensuing steps if necessary.

For Step 2, one would most often try to exhaust the decomposition effort on \( g^0(y) \) before going on to "invent" other concave terms, as terms coming from the former means normally have better interpretations. Based on our
own experiences, we believe that terms that discourage the retailer from over-ordering should in general be balanced out with those that discourage him from under-ordering. From (37), we see that most likely, coordination will be more achievable when \( r \) becomes larger.

We can reduce the linear programs (37) to the ones whose all variables come from a subspace of \( \mathbb{R}^n \). The reduction also provides us a key insight into the relation between system coordination and information loss in observation.

Given vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n \), we use \( C(\mathbf{v}_1, \ldots, \mathbf{v}_k) \) to denote the cone containing all positive linear combinations of the given vectors, and use \( S(\mathbf{v}_1, \ldots, \mathbf{v}_k) \) to denote the linear subspace spanned by these vectors. From (34), we see that there will be a coordinating contract when cone \( C(t_1, \ldots, t_k) \) intersects with affine subspace \( \mathbf{g} + S(p_1, \ldots, p_m) \).

Suppose in \( \mathbb{R}^n \), we are given vector \( \mathbf{v} \), subset \( T \), and subspace \( S \). We use \( \mathbf{w}(S) \) to denote the projection of \( \mathbf{v} \) to \( S \), use \( (T)S \) to denote the projection of \( T \) to \( S \), and use \( \mathbb{R}^n/S \) to denote the orthogonal subspace of \( S \).

**Proposition 1.** \( C(t_1, \ldots, t_k) \cap (\mathbb{R}^n/S(p_1, \ldots, p_m)) \neq \emptyset \) if and only if \( (\mathbb{R}^n/S(p_1, \ldots, p_m)) \in C(t_1, \ldots, t_k) \).

**Proof.** For convenience, we use \( C \) for \( C(t_1, \ldots, t_k) \), \( S \) for \( S(p_1, \ldots, p_m) \), and further \( S' \) for \( \mathbb{R}^n/S \). We have \( (\mathbb{R}^n/S(p_1, \ldots, p_m)) \in C(t_1, \ldots, t_k) \) if and only if there exists \( \mathbf{x} \in S \) so that \( \mathbf{x} + (\mathbb{R}^n/S(p_1, \ldots, p_m)) \in C \).

Let \( \mathbf{y} = \mathbf{x} - (\mathbb{R}^n/S(p_1, \ldots, p_m)) \). Then (38) means that \( \mathbf{g} = \mathbf{g} + \mathbf{y} \in C \).

Since both \( \mathbf{x} \) and \( (\mathbb{R}^n/S(p_1, \ldots, p_m)) \) are in \( S \), we know that \( \mathbf{y} \) is in \( S \), and hence \( \mathbf{g} \) is in \( \mathbb{R}^n/S \).

Now, we have realized that the question of whether or not there are required variables to satisfy (34) is equivalent to that of whether or not \( (\mathbb{R}^n/S(p_1, \ldots, p_m)) \in C(t_1, \ldots, t_k) \).

\[
(\mathbb{R}^n/S(p_1, \ldots, p_m)) \in C(t_1, \ldots, t_k) \quad \text{if and only if} \quad (\mathbb{R}^n/S(p_1, \ldots, p_m)) \in C(t_1, \ldots, t_k). 
\]

Let \( q \) be the dimensionality of \( S' = \mathbb{R}^n/S(p_1, \ldots, p_m) \), which is necessarily between \((n - m)\) and \( n \). When \( q = 0 \), we have (40) being true automatically. When \( q = 1, \ldots, n \), we suppose that \( e_1, \ldots, e_q \) form an orthonormal basis of \( S' \). Now for \( q' = 1, \ldots, q \) and \( r' = 1, \ldots, r \), let \( \mathbf{g}_q = e_{q'} \mathbf{g} \) and \( \mathbf{t}_{r'} = e_{r'} \mathbf{t} \). Then we have

\[
\begin{cases}
\mathbf{g} = \mathbf{g}_q e_1 + \cdots + \mathbf{g}_q e_q, \\
\mathbf{t}_{r'} = \mathbf{t}_{r'} e_1 + \cdots + \mathbf{t}_{r'} e_q, \quad \forall r' = 1, \ldots, r.
\end{cases}
\]

The question concerning (40) can now be translated into whether or not there exists positive vector \( \mathbf{g}^+ = (\gamma_1^+, \ldots, \gamma_r^+) \in \mathbb{R}^r \), so that

\[
\begin{align*}
\mathbf{g}_1 &= \mathbf{t}_{11} \gamma_1^+ + \cdots + \mathbf{t}_{1r} \gamma_r^+, \\
\cdots &= \cdots, \\
\mathbf{g}_q &= \mathbf{t}_{q1} \gamma_1^+ + \cdots + \mathbf{t}_{qr} \gamma_r^+.
\end{align*}
\]

We can answer the above question by solving a standard-form linear programming problem with \( \mathbf{g}^+ \in \mathbb{R}^r \) as the decision vector, \( \mathbf{0} \) as the objective vector, and the \( q \) equations in (42) as equality constraints.

Suppose the coefficient matrix of the above linear program is fully ranked. Then, for an arbitrary \( \mathbf{g} \), there will be a positive chance for a feasible solution of the problem to exist when \( r \geq q \), while there will be zero chance when \( r < q - 1 \). Note that \( r \) is the number of \( t_{ij} \) terms, and \( q \) being \( n \) minus the rank of \( E[I] \), measures the loss of information in the transition from decision \( \mathbf{x} \) to observable \( \mathbf{y} \). Therefore, we may argue that, system coordination hinges on whether or not the variety of terms in the concerned contract more than compensates the loss of information in observation.

For the supplier–retailer newsvendor system often studied in the literature, we can think of the wholesale price, buyback, and revenue sharing contracts as resulting from applying the above method to the concerned problem. Appendix C explains how this interpretation can materialize.

Following the method above, we will propose in Section 5 the contract forms for a supply chain system involving multiple resource types, product types, and unobservable retailer effort types. And, in Section 6, we will completely solve the coordination problem for a two-product special case. Solving this two-product problem is of certain significance in its own right, as it is itself an extension of the problem studied by Taylor (2002), it does not seem to have been solved in the literature, and the contract we shall propose for it has terms that are novel and yet meaningful.

### 5. A multi-product/effort problem with private effort levels

We first describe the concerned system as follows. A supplier is charged with supplying \( m \) different products for a retailer. In total \( l \) resources on the supplier side and \( n \) activities on the retailer side are involved in the production and sales of these products. For instance, we may think of the supplier as a brand-name fashion goods manufacturer and the retailer as a department store. Product types may include various categories of clothes, handbags, accessories, etc.; resource types may include raw fabrics, electric power supplies, labor inputs, etc., that go into the manufacturing of the products; at the same time, sales activities may include TV commercials, newspaper advertisements, sales events, etc., that are run by the store.

For resource \( i = 1, \ldots, l \) and product \( j = 1, \ldots, m \), the supplier will consume \( h_{ij} \) units of resource \( i \) to produce one unit of product \( j \). For the consumption of resource \( i \), the supplier faces a convex cost function \( C_i \). To express the relation between the retailer’s marketing efforts and sales results, we introduce \( m \times n \) random matrix \( \mathbf{M} = (M_{11}, \ldots, M_{1m}) \) with \( M_{1j} = (M_{1j1}, \ldots, M_{1jm}) \) for each \( j = 1, \ldots, m \), and \( m \)-dimensional random vector \( \mathbf{A} = (A_1, \ldots, A_m) \). When the retailer’s marketing-effort vector is \( \mathbf{e} = (c_1, \ldots, c_n) \), he will face a random demand vector \( \mathbf{D} = (D_1, \ldots, D_m) \) that is determined by

\[
\mathbf{D} = \mathbf{M} \mathbf{e} + \mathbf{A}.
\]

(43)
We suppose that effort \( \hat{e} \) costs \( V(\hat{e}) \), where \( V(\cdot) \) is increasing in all components and jointly convex.

Each product \( j \) is associated with a sales price \( p_j \), salvage value \( s_j \), and a nominal wholesale price \( w^0_j \). We let

(a) \( \mathbf{h}_i = (h_{i1}, \ldots, h_{im})^T \) for \( i = 1, \ldots, l \), with each \( \mathbf{h}_i \) conveys information about consumptions of resource \( i \) of various products;

(b) \( \mathbf{p} = (p_1, \ldots, p_m)^T \), which conveys information about unit sales prices of various products;

(c) \( \mathbf{s} = (s_1, \ldots, s_m)^T \), which conveys information about salvage values of various products;

(d) \( \mathbf{w}^0 = (w^0_1, \ldots, w^0_m)^T \), which conveys information about nominal wholesale prices of various products.

The retailer gets to decide his ordering-quantity vector \( \mathbf{q} = (q_1, \ldots, q_m)^T \) and sales-effort vector \( \mathbf{e} \). His payoff function \( f(\mathbf{q}, \mathbf{e}) \) is given by

\[
f(\mathbf{q}, \mathbf{e}) = (\mathbf{p} - \mathbf{w}^0)^T \mathbf{q} + \sum_{j=1}^{m} (p_j - s_j) \times \left(-E(\tilde{q}_j - (\mathbf{M}_j^T \mathbf{e} + A_j))^+\right) - V(\mathbf{e}). \tag{44}
\]

To the supplier, only \( \mathbf{q} \) and the realization \( \mathbf{d} \) of the random demand \( \mathbf{D} \), but not \( \mathbf{e} \), are observable. Without a contract the supplier will passively accept payoff

\[
g^0(\mathbf{q}, \mathbf{d}) = (\mathbf{w}^0)^T \mathbf{q} - \sum_{i=1}^{l} C_i(h_i^T \mathbf{q}). \tag{45}
\]

Now we let \( g(\mathbf{q}, \mathbf{e}) = E[g^0(\mathbf{q}, \mathbf{M} \hat{e} + A)]. \)

Apparently, both \( f \) and \( g \) are concave in \( \mathbb{R}^{m+n} \). For the entire system, the optimal solution \((\mathbf{q}^*, \mathbf{e}^*)\) can be obtained by solving

\[
\begin{align*}
\hat{e}_i(f(\mathbf{q}^*, \mathbf{e}^*) + g(\mathbf{q}^*, \mathbf{e}^*)) & \leq 0, \\
\hat{e}_i(f(\mathbf{q}^*, \mathbf{e}^*) + g(\mathbf{q}^*, \mathbf{e}^*)) & \leq 0, \\
(\mathbf{q}^*)^T \hat{e}_i(f(\mathbf{q}^*, \mathbf{e}^*) + g(\mathbf{q}^*, \mathbf{e}^*)) & \leq 0, \\
(\mathbf{q}^*)^T \hat{e}_i(f(\mathbf{q}^*, \mathbf{e}^*) + g(\mathbf{q}^*, \mathbf{e}^*)) & = 0.
\end{align*}
\tag{46}
\]

We now apply the general method introduced in the previous section to the design of a coordinating contract for this system. This contract should be in the form of a wealth transfer \( t(\mathbf{q}, \mathbf{d}) \) from the supplier to the retailer, such that its resulting average payoff \( f(\mathbf{q}, \mathbf{e}) + E[t(\mathbf{q}, \mathbf{M} \hat{e} + A)] \) to the retailer would lure him to make the system-optimal decision \((\mathbf{q}^*, \mathbf{e}^*)\).

From (44) and (45), we see that

\[
f(\mathbf{q}, \mathbf{e}) + g(\mathbf{q}, \mathbf{e}) \\
= \mathbf{p}^T \mathbf{q} + \sum_{j=1}^{m} (p_j - s_j) \times \left(-E(\tilde{q}_j - (\mathbf{M}_j^T \mathbf{e} + A_j))^+\right) \\
- \sum_{i=1}^{l} C_i(h_i^T \mathbf{q}) - V(\mathbf{e}). \tag{47}
\]

Recall that the first term is the sales revenue, the second term reflects the monetary loss due to supply shortage, the third term is the material cost charged to the supplier, and the last term is the marketing cost shouldered by the retailer. Assuming \( \mathbf{q}^* > 0 \) and \( \mathbf{e}^* > 0 \), (46) will translate into

\[
\begin{align*}
&\left\{ p_j - (p_j - s_j) \cdot P(\mathbf{M}_j^T \hat{e} + A_j \leq \tilde{q}_j) \right. \\
&- \sum_{j=1}^{l} h_{ij} \cdot d_{ik} C_i(h_i^T \mathbf{q}^*) = 0, \quad \forall j = 1, \ldots, m, \\
&\sum_{j=1}^{m} (p_j - s_j) \cdot E[M_{jk} \cdot 1(\mathbf{M}_j^T \mathbf{e}^* + A_j \leq \tilde{q}_j)] \\
&- \bar{c}_k \cdot V(\mathbf{e}^*) = 0, \quad \forall k = 1, \ldots, n.
\end{align*}
\tag{48}
\]

We shall solve for the \( m + n \) unknowns \( q^*_1, \ldots, q^*_m, \hat{e}^*_1, \ldots, \hat{e}^*_n \) from the above \( m + n \) equations.

In view of the current business environment, we may seek the following contract \( t(\mathbf{q}, \mathbf{d}) \), which, in accordance with the form proposed in (32), includes not only concave terms already associated with \( f(\mathbf{q}, \mathbf{e}) \), but also other invented concave terms:

\[
t(\mathbf{q}, \mathbf{d}) = \sum_{j=1}^{m} \left( \gamma^*_j + p_j + s_j \right) \cdot (-\tilde{q}_j - d_j)^+ \\
+ \sum_{i=1}^{l} \gamma^+_m \cdot (-\bar{c}_i(h_i^T \mathbf{q})) \\
+ \sum_{i=1}^{l} \gamma^+_m \cdot \left( -\sum_{j=1}^{m} \gamma_j \cdot (\tilde{q}_j - d_j)^2 \right)^+ \\
+ \sum_{j=1}^{m} \gamma^+_m \cdot \tilde{q}_j + \sum_{j=1}^{m} \gamma^+_m \cdot \tilde{q}^+_m \cdot d_j, \tag{49}
\]

where \( b \) is a pre-determined positive integer and \( \gamma_1, \ldots, \gamma_m \) are pre-determined positive constants, while \( \gamma^+_m, \ldots, \gamma^+_m \) are positive parameters and \( \gamma^+_m, \ldots, \gamma^+_m \) are real parameters. Note that we have omitted the constant term in (32) which is most likely not needed.

When \( \gamma^*_j \in [0, p_j - s_j] \), we may think of \( p_j - \gamma^*_j \), much like the constant \( b \) in the returns/buyback contract in Taylor (2002), as the return to the retailer on each unsold unit of product \( j \), which the supplier takes back and salvages herself; when \( \gamma^*_j \in [p_j - s_j, +\infty) \), we may think of \( \gamma^*_j \) as the penalty charged to the retailer for each unsold unit of product \( j \). Also, we use concave terms \(-\bar{c}_i(h_i^T \mathbf{q})\) of \( g^0(\mathbf{q}, \mathbf{d}) \) along with concave terms \(-\sum_{j=1}^{m} \gamma_j \cdot (\tilde{q}_j - d_j)^2 \) as the “free” terms \( t_{\varepsilon_1}(y), \ldots, t_{\varepsilon_2}(y) \) in (32). These terms give the retailer the incentive to rein in the differences between order quantities \( q \), and demand levels \( d \). We may use trial and error to decide how big \( b \) should be to render the system coordinate-able. If necessary, we may add fixed payoff \( \gamma_0 \) to the contract to achieve desirable profit distributions.

It is often common in practice that even the demand vector \( \mathbf{d} \) is not observable, but the leftover vector \((\tilde{q}_1 - d_1)^+, \ldots, (\tilde{q}_m - d_m)^+\) \( \mathbb{R}^m \) is. If so, we may consider the following contract that is quite similar to (49):

\[
t(\mathbf{q}, \mathbf{d}) = \bar{R}(\mathbf{q}, (\tilde{q}_1 - d_1)^+, \ldots, (\tilde{q}_m - d_m)^+) \\
+ \sum_{j=1}^{m} \gamma^+_m \cdot (-\bar{c}_i(h_i^T \mathbf{q}))
\]
where, besides other constants that have already appeared in (49), \( c \) is a positive integer, and \( \beta_1, \ldots, \beta_m \) are the predetermined constants. The analysis that follows will be similar.

6. A two-product specialization

We now consider a single-resource (the number of resource types is not really a complicating factor), two-product, and two-effort special case. We also assume that only multiplicative factors appear in the effort–demand relation, that the relation is uncorrelated between the two products, and that production costs are linear. Due to these assumptions, we use random matrix \( M = (M_1, 0)^T.0, M_2)^T \) to denote the multiplicative factor, and use \((c_1, c_2)^T\) to represent the unit production cost vector. Following (44), we have, for retailer payoff, that

\[
\begin{align*}
& f(q_1, q_2, e_1, e_2) \\
& = (p_1 - w_1^0) \cdot q_1 + (p_2 - w_2^0) \cdot q_2 \\
& + (p_1 - s_1) \cdot ((-E(q_1 - M_1e_1)))) \\
& + (p_2 - s_2) \cdot ((-E(q_2 - M_2e_2))) - V(e_1, e_2).
\end{align*}
\]

Here \( \hat{e}_1 \) is the level of effort exerted solely for promoting the sales of product-1, and \( \hat{e}_2 \) is the level of effort exerted solely for promoting the sales of product-2.

Following (47), we have, for system-wide payoff, that

\[
\begin{align*}
& (f + g)(q_1, q_2, e_1, e_2) \\
& = (p_1 - c_1) \cdot q_1 + (p_2 - c_2) \cdot q_2 + (p_1 - s_1) \\
& \cdot ((-E(q_1 - M_1e_1)))) \\
& + (p_2 - s_2) \cdot ((-E(q_2 - M_2e_2))) - V(e_1, e_2).
\end{align*}
\]

Suppose we know that \( \hat{q}_1, \hat{q}_2, \hat{e}_1, \hat{e}_2 > 0 \), then they must satisfy the following equations in the spirit of (48):

\[
\begin{align*}
& P[M_1 \leq \hat{q}_1/c_1] = (p_1 - c_1)/(p_1 - s_1), \\
& P[M_2 \leq \hat{q}_2/c_2] = (p_2 - c_2)/(p_2 - s_2), \\
& (p_1 - s_1) \cdot E[M_1 \cdot 1(M_1 \leq \hat{q}_1/c_1)] = \hat{e}_1 V(\hat{e}_1, \hat{e}_2), \\
& (p_2 - s_2) \cdot E[M_2 \cdot 1(M_2 \leq \hat{q}_2/c_2)] = \hat{e}_2 V(\hat{e}_1, \hat{e}_2).
\end{align*}
\]

Following (50), we propose the following contract form:

\[
\begin{align*}
& t(q_1, q_2, d_1, d_2) = (\gamma_1^1 \cdot p_1 - s_1) \cdot ((-\hat{q}_1 - d_1) + \\
& + (\gamma_2^1 \cdot p_2 - s_2) \cdot ((-\hat{q}_2 - d_2) + \\
& + \gamma_3^1 \cdot ((-\hat{q}_1 - 2d_1)^2) + \gamma_4^1 \cdot ((-\hat{q}_2 - 2d_2)^2) \\
& + \gamma_5^1 \cdot ((-\hat{q}_1 - d_1)^2) + \gamma_6^1 \cdot ((-\hat{q}_2 - d_2)^2) \\
& + w_1^0 \cdot (-\gamma_7) \cdot \hat{q}_1 + w_2^0 \cdot (-\gamma_8) \cdot \hat{q}_2),
\end{align*}
\]

where \( \gamma_1^1, \gamma_2^1, \gamma_3^1, \gamma_4^1, \gamma_5^1, \gamma_6^1, \gamma_7, \gamma_8 \) are positive parameters, and \( \gamma_7, \gamma_8 \) are real parameters. Note that we have let \( b, c, \beta_1, \) and \( \beta_2 \) in (50) be 1, 1, \((1 + \hat{e}_1)\hat{q}_1, \) and \((1 + \hat{e}_2)\hat{q}_2, \)

respectively, and furthermore, have made the simple choice of \( \hat{e}_1 = \hat{e}_2 = 1 \). We can give all terms interpretations similar to those offered by the discussion after (12).

Now, we have

\[
\begin{align*}
& f(q_1, q_2, e_1, e_2) + E[t(q_1, q_2, M_1e_1, M_2e_2)] \\
& = \gamma_1^1 \cdot (-E(q_1 - M_1e_1)) + \gamma_2^1 \cdot (-E(q_2 - M_2e_2)) \\
& + \gamma_3^1 \cdot ((-\hat{q}_1 - 2d_1)^2) + \gamma_4^1 \cdot ((-\hat{q}_2 - 2d_2)^2) \\
& + \gamma_5^1 \cdot ((-\hat{q}_1 - d_1)^2) + \gamma_6^1 \cdot ((-\hat{q}_2 - d_2)^2) \\
& + \gamma_7 \cdot \hat{q}_1 + \gamma_8 \cdot \hat{q}_2 - V(e_1, e_2).
\end{align*}
\]
identify a slew of coordinating contracts that satisfy $E$ to linear, programs when adapting Taylor's approach to over-ordering, and charge the retailer a unit wholesale price of $37/6$, slightly above her production cost of $6$.

7. Concluding remarks

We proposed a method for finding coordinating contracts for a type of principal and agent problems that are often seen in the supply chain management context. This method is versatile enough to, on the one hand, produce well-known contract types for the supplier-retailer newsvendor system, and on the other hand, lead to alternative contract forms for supply chain systems, including for example, the one studied by Taylor (2002) involving unobservable retailer effort. Finally, we have to emphasize that contract design is more a business decision than the solving of a mathematical formulation. When our approach is being used, those invented terms need to have reasonable business meanings.

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Appendix A. Further analysis of the one-product problem

For ease of presentation, we choose $\varepsilon = 1$. Now consider contract $(26/3, 6, -37/6)$. We can check that it is coordinating and allows the two parties to equally share the total profit. Its meaning is that, the supplier will subsidize the retailer 2 for every unit leftover, or equivalently, she will buy back from the retailer every unsold unit at the price of 4 and salvage it herself; the supplier will also ask the retailer to pay a $2/3$ penalty for every unit square deviation between his actual order quantity and twice the system-optimal order quantity, penalize the retailer 6 for every unit square over-ordering, and charge the retailer a unit wholesale price of $37/6$, slightly above her production cost of 6.

From the above analysis, it is clear that we can afford to set $\gamma_2^+=0$ and consider the following simpler contract instead:

$$
\begin{align*}
&-x_1^2 \beta_1^2 \cdot \gamma_3^+ / A^2 - x_2^2 \beta_2^2 \cdot \gamma_4^+ / A^2 \\
&-x_1^2 \beta_1^2 \cdot \gamma_5^+ (3A^2) - x_2^2 \beta_2^2 \cdot \gamma_6^+ (3A^2) \\
&+ x_1 \beta_1 \cdot \gamma_7 / A + x_2 \beta_2 \cdot \gamma_8 / A, \\
&\text{while}
&f(q_1^*, q_2^*, \varepsilon^2) + E[t(q_1^*, q_2^*, M_1 \varepsilon^1, M_2 \varepsilon^2)]
\end{align*}
$$

\begin{align*}
&= \mathcal{Z}'(\gamma_1^+, \gamma_2^+, \gamma_3^+, \gamma_4^+, \gamma_5^+, \gamma_6^+, \gamma_7, \gamma_8) \\
&+ x_1 \beta_1 \cdot p_1 / A + x_2 \beta_2 \cdot p_2 / A - a_{11} \cdot \beta_1^2 / (2A^2) \\
&- a_{12} \cdot \beta_1 \beta_2 / A^2 - a_{22} \cdot \beta_2^2 / (2A^2).
\end{align*}

By (56), we also have

$$
\begin{align*}
&\begin{cases}
-x_1 \cdot \gamma_1^+ + 2x_1 \beta_1 \cdot \gamma_3^+ / A - x_2^2 \beta_1 \cdot \gamma_5^+ / A + \gamma_7 = -p_1, \\
-x_2 \cdot \gamma_2^+ + 2x_2 \beta_2 \cdot \gamma_4^+ / A - x_2^2 \beta_2 \cdot \gamma_6^+ / A + \gamma_8 = -p_2, \\
x_1^2 \cdot \gamma_1^+ / 2 + x_2^2 \beta_1 \cdot \gamma_5^+ / (3A) = \delta_1, \\
x_2^2 \cdot \gamma_2^+ / 2 + x_2^2 \beta_2 \cdot \gamma_6^+ / (3A) = \delta_2.
\end{cases}
\end{align*}
$$

(63)

Suppose $p_1 = 10, c_1 = 6, s_1 = 2, p_2 = 15, c_2 = 7, s_2 = 3, a_{11} = 2, a_{12} = 1, a_{22} = 3$. Then, we can get $A = 5$, and by (58) and (57), that $\delta_1 = 1, \delta_2 = 8/3, x_1 = 1/2, x_2 = 2/3, \beta_1 = 1/3$, and $\beta_2 = 13/3$. Hence, by (59), we have $\gamma_1^+ = 1/15, \gamma_2^+ = 13/15, q_1^+ = 1/30$, and $q_2^+ = 26/45$. By (60), we know that the system-optimal payoff $(f + g)(q_1^*, q_2^*, \varepsilon^1, \varepsilon^2) = 107/90$. By (61) and (63), linear programs (37) now become

$$
\begin{align*}
&z_{\text{max(min)}}^* \\
&= \text{min(max)} \begin{cases}
\gamma_1^+ / 120, \\
- \gamma_3^+ / 900, \\
- \gamma_5^+ / 5400, \\
+ \gamma_7 / 30 \\
-26 \gamma_2^+ / 135, \\
-676 \gamma_4^+ / 2025, \\
-1352 \gamma_6^+ / 18225, \\
+ 26 \gamma_8 / 45
\end{cases} \\
\text{s.t.} \\
\begin{cases}
-x_1^2 / 2, \\
+ \gamma_3^+ / 15, \\
- \gamma_5^+ / 60, \\
+ \gamma_7, \\
-2 \gamma_2^+ / 3, \\
+ 52 \gamma_4^+ / 45, \\
- 52 \gamma_6^+ / 135, \\
+ \gamma_8 / 360, \\
2 \gamma_1^+ / 9, \\
104 \gamma_2^+ / 1215, \\
+ \gamma_3^+ / 45, \\
+ \gamma_5^+ / 255, \\
+ \gamma_6^+ \\
\gamma_7, \\
\gamma_8 / 360, \\
\gamma_2^+, \gamma_4^+ \\
\gamma_5^+, \gamma_6^+ \\
\gamma_7, \gamma_8
\end{cases}
\end{align*}
$$

(64)

If a solution with objective value $z^*$ has been adopted as a contract, then according to (62), the retailer will earn $z^* + 703/90$ under this contract. By studying (64), we can identify a slew of coordinating contracts that satisfy varieties of participation constraints. We have relegated further analysis to Appendix D.

Let us come back to examine Taylor's approach. Note that the target rebate term $E[\text{min}(q, M_2 - T)]$ in this approach contains the target level $T$ as a parameter, and in addition, is not jointly concave in $(q, \varepsilon)$. Due to these, we will have to solve nonlinear, as opposed to linear, programs when adapting Taylor's approach to multi-product systems; in addition, the number of constraints in each of these nonlinear programs will grow exponentially with the number of products involved. We have left details on the above two-product system to Appendix E.
This time, we have
\[ \gamma_1^* = 8 \quad \text{and} \quad \gamma_4 = -6 - \gamma_2^*. \] (66)
and
\[ f(q^*, c^*) + E[t(q^*, Mc^*)] = \frac{1}{2} - \frac{3\gamma_2^*}{4}. \] (67)

For both parties to make money, we must impose that \( \gamma_2^* \in [0, 2/3] \). For instance, it can be checked that contract \( (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4) = (8, 1/3, 0, -19/3) \) is coordinating and it lets the two parties equally share the total profit. Its meaning is that, the supplier will ask the retailer to pay a 1/3 penalty for every unit square deviation between his actual order quantity and twice the supply chain’s optimal order quantity, and charge the retailer a unit wholesale price of 19/3, slightly above its production cost of 6. Of course, the whole contract can be viewed as a nonlinear pricing scheme.

An even simpler contract is one with \( \gamma_1^* = 8, \gamma_2^* = \gamma_3^* = 0, \) and \( \gamma_4 = -6 \), where the supplier charges the retailer the wholesale price which is exactly her production cost. Unless a fixed fee is introduced into it, this contract will leave all profits to the retailer and none to the supplier.

Appendix B. That our model is a special principal and agent system

In the better known literature of the principal and agent theory, what the agent desires is a large \( f(x) = E[V(t(h(x, \Omega), x)) \), with the agent utility function \( V(t, x) \) being increasing and concave in the transfer fee \( t \) and decreasing in the effort level \( x \); at the same time, what the principal desires is a large \( g(x) = E[U(g^0(h(x, \Omega)) - t(h(x, \Omega))) \), with the principal utility function \( U(w) \) being increasing and concave in her wealth level \( w \) (which is equal to \( g^0(y) - t \), the payoff out of the outcome \( y \) minus the transfer fee). Solution \( x^* \) will be considered Pareto-optimal when no other \( x \in X \) could simultaneously reach \( f(x^*) > f(x^2) \) and \( g(x^*) > g(x^2) \).

To fit our setting into the above framework, we need to let \( V(t, x) = f(x) + t \) and \( U(w) = w \). But, our \( f(x) \) is not necessarily decreasing in \( x \), and hence neither will the resultant \( V(t, x) \) be decreasing in \( x \). So the earlier theory cannot be readily applied to our problem. On the other hand, the total utility in our case satisfies

\[ f(x) + g(x) = E[V(t(h(x, \Omega), x)) + t(h(x, \Omega))] + E[U(g^0(h(x, \Omega)) - t(h(x, \Omega)))] = f(x) + E[th(h(x, \Omega))] + E[g^0(h(x, \Omega)) - t(h(x, \Omega))] = f(x) + g(x). \] (68)

For this, a coordinating solution \( x^* \) that maximizes \( f(x) + g(x) \) must be Pareto-optimal.

Appendix C. Contracts of the newsvendor system

In the supplier–retailer newsvendor system, the supplier is the principal and the retailer the agent. The retailer’s ordering decision \( q \) is observable to both parties.

The retailer faces random demand \( D \) with distribution function \( F \) and mean \( \mu \). The retail price is \( p \), the supplier’s production cost per unit is \( c \), and the nominal wholesale price is \( w^0 \in (c, p) \). For each unit of unsold item, the retailer earns salvage value \( s \), where \( s < c \).

We have

\[ \begin{align*}
  f(q^*) &= (p - w^0) \cdot q^* + (p - s) \cdot (-E[(q^* - D)^+]), \\
  g(q^*) &= (w^0 - c) \cdot q^*.
\end{align*} \] (69)

For this, we have

\[ f(q^*) + g(q^*) = -c \cdot q^* + p \cdot E[\min(\hat{q}, D)] + s \cdot E[(q^* - D)^+], \] (70)

and the solution \( q^* \) for (23) is

\[ \hat{q}^* = F^{-1} \left( \frac{p - c}{(p - c) + (c - s)} \right), \] (71)

while the solution \( q^0 \) for (24) is

\[ \hat{q}^0 = F^{-1} \left( \frac{p - w^0}{(p - w^0) + (w^0 - s)} \right) < \hat{q}^*. \] (72)

The goal of system coordination is to design contract \( t \) so that the solution \( \hat{q}^t \) for (25) is equal to \( \hat{q}^* \).

For the wholesale price contract, we are effectively considering

\[ f(q^*) + t(q^*) = (p - w^0) \cdot q^* + (p - s) \cdot (-E[(q^* - D)^+]), \] (73)

where \( w \) is the contractual wholesale price. Here, \( t(y) = (w^0 - w) \cdot y \) (observable \( y \) is decision \( \hat{q} \)) in this special example contains only one linear term of the observable. The making \( q^* = \hat{q}^0 \) is actually \( c \). But if so, the supplier will make no profit.

For the buyback contract, we are effectively considering

\[ f(q^*) + t(q^*) = (p - w) \cdot q^* + (p - b) \cdot (-E[(q^* - D)^+]), \] (74)

where \( w \) is the contractual wholesale price, and \( b \in [s, p] \) is the amount the supplier pays to the retailer for every unsold unit, which she takes back and salvages herself. Here, \( t(y) = (w^0 - w) \cdot y + ((p - b) - (p - s) \cdot (-E[(y - D)^+])) \), where \( (p - b)/(p - s) \in [0, 1] \), contains one linear term and one concave term.

For the revenue sharing contract, we are effectively considering

\[ f(q^*) + t(q^*) = (p - w) \cdot q^* + \psi(p - s) \cdot (-E[(q^* - D)^+]), \] (75)

where \( w \) is the contractual wholesale price, and \( \psi \) is the fraction of supply chain revenue the retailer keeps. Here, \( t(y) = (w^0 - w) \cdot y + ((p - s) - (E[(y - D)^+])) \) contains one linear term and one concave term.

On the other hand, the quantity flexibility contract, the sales rebate contract, or the quantity-discount contract is not the type of contract we considered earlier.

Appendix D. Further analysis of the two-product problem

In the language of Section 4, it is true that \( t_1 = (-1/2, 0, 1/8, 0)^T, \quad t_2 = (0, -2/3, 0, 2/9)^T, \quad t_3 = (1/15, 0, 1/8, 0)^T \).
Among the terms used in the above contract, we note that terms associated with $\gamma^1_2$, $\gamma^2_2$, $\gamma^7_2$, and $\gamma^8_2$ have appeared frequently in the literature; terms associated with $\gamma^3_2$ and $\gamma^4_2$ merely turn the penalty on the retailer's over-ordering into a convex function of the over-ordering level; only terms associated with $\gamma^1_2$ and $\gamma^2_2$ are slightly extraordinary. But being convex and two-sided penalties on deviations of ordering quantities from certain target levels, even these terms can be easily enforced between two business partners.

In the constraints of (64), it is true that $\gamma^1_2$ appears only when $\gamma^1_2$ appears, and $\gamma^2_2$ appears only when $\gamma^2_2$ appears. One may naturally want to let the linear terms "absorb" the square penalty terms. However, we may note that coefficients for $\gamma^1_2$ and $\gamma^2_2$ are of opposite signs in the objective function of (64) while of the same sign in its constraints; and the same phenomenon applies to $\gamma^3_2$ and $\gamma^4_2$. So, we may still achieve system coordination without $\gamma^3_2$ or $\gamma^4_2$, but at the expense of not being able to reasonably control the partition of the profit between the supplier and the retailer.

When the requirement $\gamma^1_2 = \gamma^2_2 = 0$ is imposed for the minimization portion of (64), we will again obtain the earlier maximization solution $\gamma^0$. Therefore, the only coordinating contract in the form of (54) with $\gamma^4_2 = \gamma^3_2 = 0$ that leaves a nonnegative profit to the supplier is one allowing exactly a zero profit for her. For this example, $\gamma^4_2$ behaves as a knob that can be used to fine tune the profit allocation between the two parties.

On the other hand, the choice of $2\tilde{q}_i$ is only idiiosyncratic. It can be easily replaced by $(1 + \varepsilon)\tilde{q}_i^2$ for any $\varepsilon > 0$, with a proper adjustment of the corresponding parameter. An interpretation for any of these $-(\tilde{q}_i - ((1 + \varepsilon)\tilde{q}_i^2)$ terms is that, the supplier, though knowing that the system-optimal ordering quantity for product $i$ is $\tilde{q}_i$, will deliberately penalize the retailer for deviating from the "suggested" ordering quantity $(1 + \varepsilon)\tilde{q}_i^2$ for this product type.

Appendix E. Taking Taylor's approach to the two-product system

Following (6), a straightforward extension of Taylor's approach to the two-product example would produce the following contract $t'$:

$$
t'(\tilde{q}_1, \tilde{q}_2, d_1, d_2) = (w_1^0 - w_1) \cdot q_1 + (w_2^0 - w_2) \cdot q_2 + (b_1 - s_1) \cdot (q_1 - d_1)^+ + (b_2 - s_2) \cdot (q_2 - d_2)^+ + u_1 \cdot (\min(\tilde{q}_1, d_1) - T_1)^+ + u_2 \cdot (\min(\tilde{q}_2, d_2) - T_2)^+. \quad (76)
$$

In consultation with (51), this will lead to

$$
f(\tilde{q}, \tilde{q}_2, \tilde{e}_1, \tilde{e}_2) + E[t(\tilde{q}_1, \tilde{q}_2, M_1\tilde{e}_1, M_2\tilde{e}_2)] = (p_1 - b_1) \cdot (\tilde{q}_1 - M_1\tilde{e}_1)^+) + (p_2 - b_2) \cdot (\tilde{q}_2 - M_2\tilde{e}_2)^+) + u_1 \cdot E(\min(\tilde{q}_1, \tilde{e}_1) - T_1)^+ + u_2 \cdot E(\min(\tilde{q}_2, \tilde{e}_2) - T_2)^+ + (p_1 - w_1) \cdot \tilde{q}_1 + (p_2 - w_2) \cdot \tilde{q}_2 - V(\tilde{e}_1, \tilde{e}_2). \quad (77)
$$
When either $T_1 \geq q_1^*$ or $T_2 \geq q_2^*$, one of the target rebate terms will disappear at system optimality. So, meaningful ranges for the target levels $T_1$ and $T_2$ are $[0,q_1^*]$ and $[0,q_2^*]$, respectively. Suppose $T_1$ and $T_2$ are within their ranges. For contract $t'$ to be coordinating, the following first-order conditions must be true:

$$
\begin{align*}
-PM_1 & = q_1^*/e_1' \cdot (p_1 - b_1) + PM_1 \geq q_1^*/e_1' \cdot u_1, \\
-w_1 & = -p_1, \\
-PM_2 & \geq q_2^*/e_2' \cdot (p_2 - b_2) + PM_2 \geq q_2^*/e_2' \cdot u_2, \\
-w_2 & = -p_2, \\
EM_1 & \cdot (1)M_1 \leq q_1^*/e_1' \cdot (p_1 - b_1) \\
+EM_1 \cdot (1)T_1 \leq q_1^*/e_1' \cdot u_1, \\
EM_2 & \cdot (1)M_2 \leq q_2^*/e_2' \cdot (p_2 - b_2) \\
+EM_2 \cdot (1)T_2 \leq (M_1 \leq q_2^*/e_2') \cdot u_2, \\
& = \tilde{e}_0 V(C_1, e_2').
\end{align*}
$$

(78)

Under the special assumptions revolving around (57) to (59), we will derive from (77) that

$$
f(q_1^*, q_2^*, e_1, e_2) + EF(t(q_1^*, q_2^*, M_1 e_1', M_2 e_2')]
= z'(b_1, b_2, u_1, u_2, T_1, T_2, w_1, w_2)
+ (x_1 \beta_1 + x_2 \beta_2 - x_2 (1/2) A^2) \cdot p_1
+ (x_2 \beta_2 - x_2 (1/2) A^2) \cdot p_2
- x_1 \beta_1 \cdot w_1 - x_2 \beta_2 \cdot w_2 / A.
$$

(79)

where

$$
z'(b_1, b_2, u_1, u_2, T_1, T_2, w_1, w_2)
= x_1 \beta_1 \cdot b_1 / (2A) + x_2 \beta_2 / (2A) + A \cdot u_1 T_1^2 / (2 \beta_1)
+ A \cdot u_2 T_2^2 / (2 \beta_2) - u_1 T_1 - u_2 T_2
+ (x_1 \beta_1 + x_2 \beta_2 - x_2 (1/2) A^2) \cdot u_1
+ (x_2 \beta_2 - x_2 (1/2) A^2) \cdot u_2
- x_1 \beta_1 \cdot w_1 - x_2 \beta_2 \cdot w_2 / A.
$$

(80)

At the same time, (78) will become

$$
\begin{align*}
-x_1 \beta_1 \cdot b_1 + (1 - x_1) \cdot u_1 - w_1 & = -(1 - x_1) \cdot p_1, \\
x_2 \beta_2 + (1 - x_2) \cdot u_2 - w_2 & = -(1 - x_2) \cdot p_2, \\
x_1 \beta_1 / 2 + x_2 \beta_2 / 2 - A^2 \cdot u_1 T_1^2 / (2 \beta_1) & = \delta_1 - x_1 \beta_1 / 2, \\
x_2 \beta_2 / 2 + x_2 \beta_2 / 2 - A^2 \cdot u_2 T_2^2 / (2 \beta_2) & = \delta_2 - x_2 \beta_2 / 2.
\end{align*}
$$

(81)

This time, however, solving nonlinear programs with $b_1, b_2, u_1, u_2, T_1, T_2, w_1, w_2$ as decision variables, $z'$ as the objective function, and (81) as primary constraints will not be sufficient to identify the coordinating parameters. The reason is that, due to the target rebate terms, function $E[F(t(q_1^*, q_2^*, M_1 e_1', M_2 e_2')]]$ is not jointly concave in $(q_1^*, q_2^*, e_1, e_2)$. Even to guarantee that the retailer will regard the point $(q_1^*, q_2^*, e_1, e_2)$ as a local maximum for his under-contract payoff $f(q_1^*, q_2^*, e_1, e_2) + E[F(t(q_1^*, q_2^*, M_1 e_1', M_2 e_2')]]$, we need to impose second-order conditions that reflect the negative semi-definiteness of the payoff function's Hessian $H$ at the given point.

After some calculation, we find that,

$$
\begin{align*}
H_{q_1^*} & = A \cdot (b_1 - u_1 - p_1) / \beta_1, \\
H_{q_2^*} & = A \cdot (b_2 - u_2 - p_2) / \beta_2, \\
H_{q_1^* q_2^*} & = H_{q_2^* q_1^*} = A \cdot (u_1 - b_1 + p_1) / \beta_1, \\
H_{q_1^* q_2^*} & = A \cdot (u_2 - b_2 + p_2) / \beta_2, \\
H_{q_1^*} & = H_{q_2^*} = H_{q_1^* q_2^*} = H_{q_2^* q_1^*} = H_{q_1^* q_2^*} = H_{q_2^* q_1^*} = H_{q_1^* q_2^*} = 0, \\
H_{q_1^* q_2^*} & = A^2 \cdot (b_1 - u_1 - p_1) / (4 \beta_1^3) + A^3 \cdot u_1 T_1^2 / (4 \beta_1^3) - \alpha_{11}, \\
H_{q_1^* q_2^*} & = -\alpha_{12}.
\end{align*}
$$

(82)

From the negative semi-definiteness of the $4 \times 4$ matrix $H$, we will obtain in total $1 + 4 + 6 + 4 = 15$ inequalities. Solving the aforementioned nonlinear programs with these 15 inequalities as additional constraints will help identify potential coordinating contracts in the form of $t'$.

For the same numerical example introduced right after (63), we obtain that

$$
f(q_1^*, q_2^*, e_1, e_2) + EF(t(q_1^*, q_2^*, M_1 e_1', M_2 e_2')]
= z'(b_1, b_2, u_1, u_2, T_1, T_2, w_1, w_2) + 871/180.
$$

(83)

$$
z'(b_1, b_2, u_1, u_2, T_1, T_2, w_1, w_2)
= b_1 / 120 + 26b_2 / 135 + 15u_1 T_1^2 / 2 + 15u_2 T_2^2 / 26
- u_1 T_1 - u_2 T_2 + u_1 / 40
+ 52u_2 / 135 - w_1 / 30 - 26w_2 / 45.
$$

(84)

Eq. (81) will become

$$
\begin{align*}
& [b_1 / 2 + u_1 / 2 - w_1 = -5, \\
& 2b_2 / 3 + u_2 / 3 - w_2 = -5, \\
& -b_1 / 8 + u_1 / 8 - 225u_1 T_1^2 / 2 = -1 / 4, \\
& -2b_2 / 9 + 2u_2 / 9 - 225u_2 T_2^2 / 338 = -2 / 3
\end{align*}
$$

and

$$
H = \begin{pmatrix}
\mu_1(b_1, u_1) & 0 & -\mu_1(b_1, u_1) / 2 & 0 \\
0 & \mu_2(b_2, u_2) & 0 & -2\mu_2(b_2, u_2) / 3 \\
-\mu_1(b_1, u_1) / 2 & 0 & v_1(b_1, u_1, T_1) & -1 \\
0 & -2\mu_2(b_2, u_2) / 3 & v_2(b_2, u_2, T_2)
\end{pmatrix}
$$

(86)

where

$$
\begin{align*}
\mu_1(b_1, u_1) & = 15b_1 - 15u_1 - 150, \\
\mu_2(b_2, u_2) & = 15b_2 / 13 - 15u_2 / 13 - 225 / 13, \\
v_1(b_1, u_1, T_1) & = 15b_1 / 16 - 15u_1 / 16 + 3375u_1 T_1^2 / 4 - 91 / 8, \\
v_2(b_2, u_2, T_2) & = 5b_2 / 39 - 5u_2 / 39 + 3375u_2 T_2^2 / 8788 - 64 / 13.
\end{align*}
$$

(87)
After removing redundant ones out of the 15 inequalities, we get

\[
\begin{align*}
\mu_1(b_1, u_1) &= 15b_1 - 15u_1 - 150 \leq 0, \\
\mu_2(b_2, u_2) &= 15b_2/13 - 15u_2/13 - 225/13 \leq 0, \\
v_1(b_1, u_1, T_1) &= 15b_1/16 - 15u_1/16 + 3375u_1T_1^2/4 - 91/8 \leq 0, \\
v_2(b_2, u_2, T_2) &= 5b_2/39 - 5u_2/39 + 3375u_2T_2^2/8788 - 64/13 \leq 0, \\
v_1(b_1, u_1, T_1) - \mu_1(b_1, u_1)/4 &= 45u_1/16 - 45b_1/16 + 3375u_1T_1^2/4 + 209/8 \leq 0, \\
v_2(b_2, u_2, T_2) - 4\mu_2(b_2, u_2)/9 &= 5u_2/13 - 5b_2/13 + 3375u_2T_2^2/8788 + 36/13 \leq 0, \\
&(v_1(b_1, u_1, T_1) - \mu_1(b_1, u_1)/4) - (v_2(b_2, u_2, T_2) - 4\mu_2(b_2, u_2)/9) - 1 = 0, \\
&= (45u_1/16 - 45b_1/16 + 3375u_1T_1^2/4 + 209/8) - (5u_2/13 - 5b_2/13 + 3375u_2T_2^2/8788 + 36/13) - 1 \geq 0. \\
\end{align*}
\]

(88)

We may identify potential coordinating parameters by \(\min(\max) z^c\) with constraints (85) and (88), as well as conditions \(2 \leq b_1 \leq 10, 3 \leq b_2 \leq 15, u_1 \geq 0, u_2 \geq 0, 0 \leq T_1 \leq 1/30 \) and \(0 \leq T_2 \leq 26/45\).

Using a commercial solver, we found that the above nonlinear programs are infeasible. This indicates that, for the current numerical example, there does not exist any returns and target rebate contract in the fashion of Taylor.

We may also see that, to use Taylor’s approach to tackle an \(m\)-product system, one will have to solve nonlinear programs with numbers of constraints in the order of \(2^m\). In contrast, our method will require solving linear programs with numbers of constraints in the order of \(m\).

References


