Connections between Finite- and Infinite-player Games:
Normal- and Extended-form Analyses

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Abstract

We establish two links between finite-player games and their corresponding nonatomic games (NGs). In a normal-form game setting, we show that an NG’s equilibrium can generate near equilibrium payoffs for large finite games whose player types are sampled from the NG’s signature distribution. In an extended-form game setting involving individual player states that bridge players’ past actions with their future gains, we show that large finite games can use an equilibrium of the NG counterpart to reap close to best payoffs. The action plan recommended by the NG equilibrium is prominent in that it is blind to the current player’s observation of its immediate surrounding. These links especially the latter one have potential to help simplify the analysis of competitive situations with many players.

Keywords: Nonatomic Game; Empirical Distribution; Idiosyncratic Shock
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1 Introduction

We explore links of the following nature between nonatomic games (NGs) and their finite-player counterparts: in a certain sense, a given equilibrium of an NG can be plugged back to the game’s close-by finite-player games and still be approximate equilibria. Our focus is not on establishing equilibrium existence; rather, we take the existence question as settled in the affirmative, and then move on to tackle the aforementioned connections.

NGs are often easier to analyze than their finite counterparts, because in them, the action of an individual player has no impact on payoffs of the other players. Therefore, they are often used as proxies of real competitive systems in economic studies. For instance, Aumann (1964) established the equivalence between an exchange economy’s core and its set of competitive equilibria when there is a continuum of nonatomic traders. Many other researchers used NGs as approximations of reality; see, e.g., Reny and Perry (2006). Our results hence has the potential to aid in the search of approximate equilibria for complex competitive situations.

Our finding is lower hemi-continuity in nature: we show the asymptotic rationality of an equilibrium developed for the NG setting when it is plugged back to finite-player settings. For a normal-form game setting, we show that an NG’s equilibrium can generate near equilibrium payoffs for large finite games whose player types are sampled from the NG’s signature distribution. In an extended-form game setting involving individual player states that are influenced by current actions and in turn influence future payoffs, we show that large finite games can use an equilibrium of the NG counterpart to reap close to best payoffs. The action plan recommended by the NG equilibrium is prominent in that it is blind to the current player’s observation of its immediate surrounding. In the latter setting, we also take advantage of the added benefit of an NG that its aggregate distribution of player states evolves over a deterministic trajectory.

Systematic research on games involving a continuum of nonatomic players started with Schmeidler (1973). When the action space is finite, Schmeidler established the existence of pure equilibria when the game becomes anonymous (so that opponents affect a given player through “what was done collectively”, but not in “who did what”). A finite-player counter-
part to Schmeidler was obtained by Rashid (1983), who did not resort to any relationship between finite- and infinite-player games. Mas-Colell (1984) achieved a result similar to Schmeidler’s using different representations of games and equilibria. Khan and Sun (1995) generalized Schmeidler’s result to the case with a countable compact metric action space.

Green (1984) and Housman (1988) introduced different frameworks under which finite games and their NG counterparts can be studied simultaneously. Khan, Rath, and Sun (1997) identified a certain limit to which Schmeidler’s result can be extended. Carmona (2004) showed that every pure equilibrium of an anonymous NG is in some sense a limit point of a sequence of approximate pure equilibria of suitably chosen anonymous games. Kalai (2004) studied large semi-anonymous games defined on finite type and action spaces, and found very strong robustness properties of these games’ equilibria. Al-Najjar (2008) studied discrete large games on the basis of finitely additive probabilities, and established that restrictions of an equilibrium of the discrete large game offer ε-equilibria for its finite counterparts. This result is similar to our result in the norm-form game setting, albeit that it is achieved on a pre-determined sequence of finite-player games.

For multi-period games without individual states that allow past actions to impact future gains, Green (1980), Sabourian (1990), and Al-Najjar and Smorodinsky (2001) showed that equilibria for large games are nearly myopic. For a case with these states, we demonstrate that the central theme will shift from myopicity to observation-blindness. This is indeed expected since as the player population grows, the distribution of player states will become less random and hence the information about itself will be less valuable. We contribute on formalizing this idea, so that exploitation of it in real situations is one step closer.

Weintraub, Benkard, and van Roy (2008) recently proposed the concept of oblivious equilibrium (OE) for large dynamic games exhibiting stationary traits. An OE is a profile of firm strategies that allows each firm to achieve the best outcome at every juncture of the game when, besides its own state, the firm is only aware of the long-run average industry state rather than the exact present state of the industry. When ever more firms participate, authors showed the asymptotic rationality of OE even when firms have access to real-time industry information. OE’s connections with other solution concepts were probed in Weintraub, Benkard, and van Roy (2010). We caution that OE is not suitable for our extended-form
game portion. Our setting does not insist on the existence of a long-run average system state, and hence is transient in general. Even in the NG limit, every player has to optimally respond to a central time-varying system-state trajectory.

Our basic tools are the strong and weak Laws of Large Numbers (LLN) for empirical distributions based on separable metric spaces. There is a sizable literature on LLN in game-theoretic contexts. But they mostly concentrated on providing remedies for the failure, in the usual sense, of LLN on a continuum of samples (Feldman and Gilles, 1985, and Judd 1985); see, e.g., Páscoa (1998) and Al-Najjar (2004).

In the remainder of the paper, we present the single-period normal-form game setting in Section 2 and the multi-period extended-form game setting in Section 3. The paper is concluded in Section 4.

2 The Single-period Model

Through the single-period model, the reader may get familiarized with the notation, setting, and methods used in the more complex multi-period model. All proofs in the paper have been relegated to appendices.

2.1 Technical Preparation

Throughout, we shall use $N$ for the set of natural numbers and $R$ for the real line. Given a separable metric space $A$, we use $d_A$ to denote its metric, $\mathcal{B}(A)$ its Borel $\sigma$-field, and $P(A)$ the set of all probability measures (distributions) on the measurable space $(A, \mathcal{B}(A))$. The space $P(A)$ is metrized by the Prohorov metric $\rho_A$, and the resultant metric space is separable too. One definition for $\rho_A$ is such that, for any distributions $\pi, \pi' \in P(A)$,

$$\rho_A(\pi, \pi') = \inf(\epsilon > 0 \mid \pi'((A')^\epsilon) + \epsilon \geq \pi(A') \text{ for all } A' \in \mathcal{B}(A)), \quad (1)$$

where

$$(A')^\epsilon = \{a \in A \mid d_A(a, a') < \epsilon \text{ for some } a' \in A'\}. \quad (2)$$

The metric $\rho_A$ is known to generate the weak topology for $P(A)$. For $a \in A$, we use $1_a$ to denote the singleton probability measure with $1_a(\{a\}) = 1$. For $a = (a_1, \ldots, a_n) \in A^n$, 

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though, we use $1_a$ for $\sum_{m=1}^{n} 1_{a_m}/n$. The two uses are consistent. We also use $P_n(A)$ to denote the space of probability measures of the type $1_a$ for $a \in A^n$, i.e., the space of empirical distributions generated from $n$ samples.

We use $(A^n, \mathcal{B}^n(A))$ to denote the product measurable space that houses $n$-long sample sequences and $(A^\infty, \mathcal{B}^\infty(A))$ the product measurable space that houses infinitely-long sample sequences. Given $\pi \in P(A)$, we use $\pi^n$ to denote the product measure on $(A^n, \mathcal{B}^n(A))$ and $\pi^\infty$ the product measure on $(A^\infty, \mathcal{B}^\infty(A))$. Given $a = (a_1, a_2, \ldots) \in A^\infty$, we use $a^n$ to denote the first-$n$ cutoff $(a_1, \ldots, a_n)$. Given $a = (a_1, \ldots, a_n) \in A^n$ and $m = 1, 2, \ldots, n$, we use $a_{-m}$ to represent $(a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_n)$. For $a = (a_1, a_2, \ldots) \in A^\infty$, note that $a^n_{-m}$ now stands for $(a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_n)$.

Given separable metric spaces $A$ and $B$, we use $M(A, B)$ to represent all measurable functions from $A$ to $B$, i.e., functions $y : A \to B$ such that, for any $B' \in \mathcal{B}(B)$, the set $y^{-1}(B') = \{a \in A \mid y(a) \in B'\}$ is a member of $\mathcal{B}(A)$. Given $\pi \in P(A)$, one should understand $\pi y^{-1}$ as a member of $P(B)$, such that $\pi y^{-1}(B') = \pi(y^{-1}(B'))$ for any $B' \in \mathcal{B}(B)$.

According to Parthasarathy (2005, Theorem II.7.1), the strong LLN applies to the empirical distribution under the weak topology, and hence under the Prohorov metric.

**Proposition 1** Given separable metric spaces $A$ and $B$, suppose $\pi \in P(A)$ and $y \in M(A, B)$. Then, there exists $A' \in \mathcal{B}^\infty(A)$ with $\pi^\infty(A') = 1$, such that, for any $a = (a_1, a_2, \ldots) \in A'$ and $\epsilon > 0$, as long as $n$ is large enough,

$$\rho_B(1_{a^n} y^{-1}, \pi y^{-1}) < \epsilon.$$  

For $a \in A$ and $\pi \in P_{n-1}(A)$, we use $(a, \pi)_n$ to represent the member of $P_n(A)$ that has an additional $1/n$ weight on the point $a$, but with probability masses in $\pi$ being reduced to $(n - 1)/n$ times of their original values. For $a \in A^n$ and $m = 1, \ldots, n$, we have $(a_m, 1_{a_{-m}})_n = 1_a$. The following is a useful observation.

**Proposition 2** Let $A$ be a separable metric space. Then, for any $n = 2, 3, \ldots$, $a \in A$, and $\pi \in P_{n-1}(A)$,

$$\rho_A((a, \pi)_n, \pi) \leq \frac{1}{n}.$$
2.2 Formulation

We assume that player types form a separable metric space \( Q \). Players, regardless of their types, have a common separable metric action space \( X \). For each \( n \in \mathbb{N} \), we let some function \( f_n : Q \times X \times P_{n-1}(X) \to \mathbb{R} \) be used in our \( n \)-player games. For any type vector \( q \in Q^n \), we define an anonymous \( n \)-person game \( \Phi_n(1_q) \) with player-type distribution \( 1_q \).

In this game, the player set is \( \{1, 2, ..., n\} \); under strategy profile \( x = (x_1, ..., x_n) \in X^n \), the payoff to player \( m \) is \( f_n(q_m, x_m, 1_{x_{-m}}) \). The anonymity of \( \Phi_n(1_q) \) is reflected in that a player’s payoff depends on himself only through his own type and action, and on other players only through what they did in aggregation.

We can apply usual equilibrium concepts to this game. For instance, for \( \epsilon \geq 0 \), we say \( x^* = (x^*_1, ..., x^*_n) \in X^n \) forms an \( \epsilon \)-Nash equilibrium for this game, if and only if, for any \( m = 1, 2, ..., n \) and any \( x_m \in X \),

\[
f_n(q_m, x^*_m, 1_{x^*_{-m}}) \geq f_n(q_m, x_m, 1_{x^*_{-m}}) - \epsilon.
\] (3)

We let some function \( f : Q \times X \times P(X) \to \mathbb{R} \) be used in our NGs. Given type distribution \( p \in P(Q) \), we define our NG \( \Phi(p) \) so that, \( f(q, x, py^{-1}) \) is the payoff to a type-\( q \) player, who takes action \( x \) and faces the multitude with type distribution \( p \) and action profile \( y = (y(q) \mid q \in Q) \in M(Q, X) \). Note that \( py^{-1} \) is the action distribution every player prepares to face, when it is anticipated that a type-\( q \) player will take action \( y(q) \).

We say that \( x^* \in M(Q, X) \) forms an NG equilibrium for \( \Phi(p) \), if and only if for any \( q \in Q \) and any \( x \in X \),

\[
f(q, x^*(q), p(x^*)^{-1}) \geq f(q, x, p(x^*)^{-1}).
\] (4)

The above reflects that \( x^*(q) \) is the best response for a type-\( q \) player to action distribution \( p(x^*)^{-1} \) generated by the type distribution \( p \) and action plan \( x^* \). So, \( x^* \) is an equilibrium if and only if the anticipated action profile offers a best response to itself for every player type. The nonatomicity of the game is reflected not in the nonatomicity of the distribution \( p \); rather, it is reflected in (4): no single player has any sway over the common environment faced by all players. For this reason, we believe it more fitting to call each \( q \in Q \) a type than a name, as in the NG, there can be “many” players of the same type.
2.3 A Convergence Result

Let a type distribution $p \in P(Q)$ be given. For any $n \in N$, we may treat the projection map $(q_1, q_2, \ldots) \rightarrow q_n$ in $M(Q^\infty, Q)$ as a random variable $\Theta_n$ that is defined between the base probability space $(Q^\infty, B^\infty(Q), p^\infty)$ and range measurable space $(Q, B(Q))$. Now, $\Phi_n(1_{\Theta_n})$, where $\Theta^n = (\Theta_1, \ldots, \Theta_n)$, stands for the randomly generated game such that, for any $Q' \in B^n(Q)$, there is a $p^n(Q')$ chance that the realized game will be $\Phi_n(1_q)$ for some $q = (q_1, \ldots, q_n) \in Q'$. We shall investigate the sense in which the sequence of randomly generated games $(\Phi_n(1_{\Theta_n}) \mid n \in N)$ converges to the NG $\Phi(p)$.

Let us make the following assumptions:

(S1) The payoff function $f(q, x, \cdot)$ is equi-continuous in the following fashion: for any $\epsilon > 0$, there exists $\delta > 0$, such that, for any $q \in Q$, $x \in X$, and $r^1, r^2 \in P(X)$ with $\rho_X(r^1, r^2) < \delta$,

$$| f(q, x, r^1) - f(q, x, r^2) | < \epsilon.$$  

(S2) The payoff function $f_n$ converges to $f$ uniformly. That is, for any $\epsilon > 0$, there exists $\bar{n} \in N$, such that, for any $n \geq \bar{n}$, $q \in Q$, $x \in X$, and $r \in P_{n-1}(X),$

$$| f_n(q, x, r) - f(q, x, r) | < \epsilon.$$  

Our single-period result says that an equilibrium of the NG $\Phi(p)$ can almost surely provide an $\epsilon$-Nash equilibrium for all randomly generated games $\Phi_n(1_{\Theta_n})$ for $n$’s greater than a certain threshold $\bar{n}$, and we can let $\epsilon$ approach 0 when $\bar{n}$ goes to $+\infty$.

**Theorem 1**  Suppose $p \in P(Q)$ is a type distribution and $x^* \in M(Q, X)$ is an equilibrium of the NG $\Phi(p)$. Then, $x^*$ will be an asymptotic Nash equilibrium for the randomly generated game sequence $(\Phi_n(1_{\Theta_n}) \mid n \in N)$ in the following sense: there exists $Q' \in B^\infty(Q)$ with $p^\infty(Q') = 1$, such that, for any $q = (q_1, q_2, \ldots) \in Q'$ and $\epsilon > 0$, there exists $\bar{n} \in N$, so that for any $n \geq \bar{n}$, $x = (x_1, \ldots, x_n) \in X^n$, and $m = 1, 2, \ldots, n$,

$$f_n(q_m, x^*(q_m), 1_{q_{-m}}(x^*)^{-1}) > f_n(q_m, x_m, 1_{q_{-m}}(x^*)^{-1}) - \epsilon.$$  

This theorem says that, when player types are independently drawn from distribution $p$ and the number of players is huge, a player may as well adopt the type-specific equilibrium.
action initially intended for the \(p\)-defined NG. When all other players do the same, the player will be virtually guaranteed to gain not much less than the maximum he could have accomplished. In Appendices C and D, we also show that the claim made in Theorem 1 need not be true when either (S1) or (S2) is slightly violated.

2.4 Discussion

Carmona (2004) showed that a strategy profile is a pure equilibrium of an NG only if its restricted versions on any given sequence of finite games provide approximate pure equilibria for these games. Al-Najjar (2008) studied discrete large games on the basis of finitely additive probabilities, and established that restrictions of an equilibrium of the discrete large game offer \(\epsilon\)-equilibria for its finite counterparts. Our Theorem 1 is certainly similar to both results in spirit. However, the earlier results are achieved on pre-determined sequences of finite games. In contrast, we have let random sampling take care of player-type selections.

For a multi-period situation involving idiosyncratic (local) shocks but without time-evolving individual states that carry information about past actions, Green (1980), Sabourian (1990), and Al-Najjar and Smorodinsky (2001) showed that equilibria for large games are nearly myopic. We can apply Theorem 1 to this situation to show something almost reciprocal—that myopic profiles can be nearly equilibrium for certain large games; that is, when the profiles are generated from equilibria of an NG correspondent to the given large games. We omit present this result due to its marginal value.

On the other hand, a more complex multi-period situation should warrant more attention. In this situation, players are associated with time-evolving individual states (as opposed to pre-endowed types) that connect past actions with future payoffs. Think of firms engaged in price competition to sell pre-endowed inventories of goods, where idiosyncratic shocks are independent potential demands these firms may face, and each firm’s state in a period is made up of its pricing action in the previous period and its current inventory level. The quantity the firm can sell in the current period, which influences the depletion of its inventory, depends on both the price it charges and on prices charged by all other firms. Yet, each firm is unaware of other firms’ inventory levels.

Therefore, an additional feature of this situation is that, in a finite game, a player may
have no or only partial knowledge on other players’ states. This makes the analysis of such a
dynamic game next to impossible. Our key intuition is, when the number of players increases,
the aggregate environment of the system will evolve in a more and more predictable way, to
the effect that the real-time knowledge of it will be less and less valuable to rational players.
It will not be surprising that an observation-blind equilibrium of the NG limit shall offer
reasonably good action plans for large finite games. Though the above is quite intuitive, we
believe a formal treatment has not been provided. Our next section shall concentrate on the
filling of such a gap.

3 The Multi-period Model

3.1 More Technical Issues

Let \( A \) and \( B \) be two separable metric spaces. We use \( K(A,B) \subset M(A,B) \) to represent
all continuous functions from \( A \) to \( B \). Given \( \pi_A \in P(A) \) and \( \pi_B \in P(B) \), the product
distribution \( \pi_A \times \pi_B \) is certainly the member of \( P(A \times B) \) defined for the measurable space
\( (A \times B, \mathcal{B}(A \times B)) \), such that \( (\pi_A \times \pi_B)(A' \times B') = \pi_A(A') \cdot \pi_B(B') \) for every \( A' \in \mathcal{B}(A) \) and \( B' \in \mathcal{B}(B) \).

The following two results will be important for showing the near-trajectory evolution of
aggregate environments in large multi-period games.

**Proposition 3** Given separable metric spaces \( A \), as well as complete separable metric spaces
\( B \) and \( C \), suppose \( y_n \in M(A^n, B^n) \) for every \( n \in N \), \( \pi_A \in P(A) \), \( \pi_B \in P(B) \), and \( \pi_C \in P(C) \). If
\[
(\pi_A)^n(\{a \in A^n \mid \rho_B(1_{y_n(a)}, \pi_B) < \epsilon\}) > 1 - \epsilon,
\]
for any \( \epsilon > 0 \) and any \( n \) large enough, then
\[
(\pi_A \times \pi_C)^n(\{(a, c) \in (A \times C)^n \mid \rho_{B \times C}(1_{y_n(a)c}, \pi_B \times \pi_C) < \epsilon\}) > 1 - \epsilon,
\]
for any \( \epsilon > 0 \) and any \( n \) large enough.

Because the equivalence between tightness and relative compactness of a collection of
probability measures is indirectly related to the proof of Proposition 3, we require $B$ and $C$

to be complete separable metric spaces.

**Proposition 4** Given separable metric spaces $A$, $B$, and $C$, suppose $y_n \in M(A^n, B^n)$ for
every $n \in N$, $\pi_A \in P(A)$, $\pi_B \in P(B)$, and $z \in K(B, C)$. If

$$(\pi_A)^n(\{a \in A^n \mid \rho_B(1_{y_n(a)}, \pi_B) < \epsilon\}) > 1 - \epsilon,$$

for any $\epsilon > 0$ and any $n$ large enough, then

$$(\pi_A)^n(\{a \in A^n \mid \rho_C(1_{y_n(a)}z^{-1}, \pi_Bz^{-1}) < \epsilon\}) > 1 - \epsilon,$$

for any $\epsilon > 0$ and any $n$ large enough.

The requirement $z \in K(B, C)$ arises out of our need to utilize the continuous mapping
theorem.

### 3.2 The Setup

We still use separable metric space $X$ for player actions. Instead of the type space $Q$, we let
complete separable metric space $S$ be the space of individual states. In certain applications,
each state may contain information about the most immediate action, i.e., $S = L \times X$ for
some other space $L$. This way, we may succinctly model players’ influences on other players
through their states without doubly expressing them through player actions again. The extra
space $L$ may in turn be in the form of $Q \times V$, where $Q$ contains some time-invariant type
like location and/or weight and $V$ contains some time-varying condition like inventory level.
We also let complete separable metric space $I$ contain idiosyncratic shocks experienced by
players. Idiosyncratic shocks follow a distribution $\iota \in P(I)$.

Each player will feel an aggregate environment in the form of the empirical distribution
of individual states of all other players. Thus, we use $\sigma \in P(S)$ to describe an aggregate
environment. To express the phenomenon that a player in an initial state $s \in S$ under
environment $\sigma \in P(S)$ will turn into a new state under action $x \in X$ and shock $i \in I$, we
introduce mapping $\bar{s} : S \times P(S) \times X \times I \to S$. We require that $\bar{s}(\cdot, \sigma, \cdot, \cdot) \in M(S \times X \times I, S)$
at every $\sigma \in P(S)$. Both here and immediately afterwards, there is no need to introduce a
function’s dependence on some \( r \in P(X) \) due to the potential action-encompassing nature of states.

In addition, we suppose the same first three factors, \( s \in S, \sigma \in P(S) \), and \( x \in X \), are responsible for a normalized single-period payoff function \( f : S \times P(S) \times X \to [0, 1] \). We require that \( f(\cdot, \cdot, \cdot) \in M(S \times X, [0, 1]) \) for every \( \sigma \in P(S) \). Note that the function may itself be an average over different shocks \( i \). In Section 2, we have brought out the point that payoff functions for finite games and the NG may be different for asymptotic results to be sustained. Here, knowing that further generalization is not difficult, we shall work with the simpler case where both types of models use the same single-period payoff function.

We let time indexes form the set \( \{0, 1, \ldots, \bar{t}\} \) and let \( \bar{\alpha} \in [0, 1] \) stand for a per-period discount factor for payoffs. We shall consider both nonatomic games \( \Gamma(\sigma_0) \) where \( \sigma_0 \in P(S) \) are given initial environments, and finite games \( \Gamma_n(\sigma_0) \), whereas in each of them, the number of players \( n \in N \) and the initial environment of players \( \sigma_0 \in P_n(S) \) are given. In each of these games, suppose a player has just made his decision \( x_t \in X \) for period \( t = 1, 2, \ldots, \bar{t} \), when he starts the period with state \( s_{t-1} \) and outside environment \( \sigma_{t-1} \). Then, he will experience his own period-\( t \) idiosyncratic shock \( i_t \), which will propel his state into

\[
s_t = \bar{s}(s_{t-1}, \sigma_{t-1}, x_t, i_t).
\]

On average, his period-\( t \) payoff will be

\[
f_t = f(s_{t-1}, \sigma_{t-1}, x_t).
\]

For simplicity, we assume a zero terminal payoff after period \( \bar{t} \). What the player’s decision depends on and what becomes of the period-\( t \) environment \( \sigma_t \) will depend on more model specifics, which we shall lay out in the sequel.

### 3.3 Convergence of Aggregate Environments

For a nonatomic game \( \Gamma(\sigma_0) \), we may use \( x = (x_t \mid t = 1, 2, \ldots, \bar{t}) \in (M(S, X))^\bar{t} \) to denote a policy profile. Here, each \( x_t \in M(S, X) \) is a map from a player’s state to the player’s action. Along with the given initial environment \( \sigma_0 \), this profile will help generate a deterministic environment trajectory \( \sigma = (\sigma_t \mid t = 0, 1, \ldots, \bar{t}) \in (P(S))^\bar{t}+1 \). This allows a player’s policy to
be observation-blind; that is, what portion of $\sigma_{t-1}$ is observable to the player in each period $t$ is not an issue. In the following, we discuss how the deterministic trajectory can be formed. Let $t = 1, 2, ..., \bar{t}$ be given. For a player with starting state $s_{t-1}$, his action will be $x_t(s_{t-1})$. When the player further experiences shock $i$, his new state will become

$$s_t = \bar{s}(s_{t-1}, \sigma_{t-1}, x_t(s_{t-1}), i).$$

(7)

We may define $T(x_t)$ as the operator defined on $P(S)$ that converts $\sigma_{t-1}$ into $\sigma_t$ under the given $x_t \in M(S, X)$. Since $s_{t-1}$ is distributed according to $\sigma_{t-1}$, $i$ is distributed according to $\iota$, and the two are independent of each other, we have, for any $S' \in \mathcal{B}(S)$,

$$\sigma_t(S') = [T(x_t) \circ \sigma_{t-1}](S') = \int \sigma_{t-1}(\{s_{t-1} \in S \mid \bar{s}(s_{t-1}, \sigma_{t-1}, x_t(s_{t-1}), i) \in S'\}) \cdot \iota(di).$$

(8)

The environment trajectory alluded to earlier is merely $\sigma = (T^t(x_{[1\bar{t}]}) \circ \sigma_0 \mid t = 0, 1, ..., \bar{t})$, where $x_{[\cdot \bar{t}]}$ stands for $(x_t, x_{t+1}, ..., x_{\bar{t}})$, $T^0$ is the identity map, and each $T^{t+1}(x_{[1\bar{t}+1]}) = T(x_{t+1}) \circ T^t(x_{[1\bar{t}]})$.

In an $n$-player game $\Gamma_n(\sigma_0)$, the initial environment $\sigma_0 \in P_n(S)$ must be of the form $1_{s_0} = 1_{(s_{01}, s_{02}, ..., s_{0n})}$, where each $s_{0m}$ may be understood as player $m$’s initial state. We use $x = (x_t \mid t = 1, 2, ..., \bar{t}) \in (M(S, X))^\bar{t}$ to denote an observation-blind policy adopted by all $n$ players. As our focus is on studying how well an observation-blind equilibrium of a nonatomic game can fare in a finite game like the current one, we have no need to deal with observation-conscious policy profiles, though working with the latter is slightly more involved. For the current game, the symmetric profile generated by $x$ will not help generate a deterministic environment trajectory. To help describe the stochastic environment process, for $i = (i_1, i_2, ..., i_n) \in I^n$, we may define $T_n(x_t, i)$ as the operator on $P_n(S)$ that converts $1_{s_{t-1}}$ into its corresponding $1_{s_t}$. Here, $1_{s_t} = T_n(x_t, i) \circ 1_{s_{t-1}}$ is such that

$$s_{tm} = \bar{s}(s_{t-1, m}, 1_{(s_{t-1})_{\bar{m}}}, x_t(s_{t-1, m}), i_m), \quad \forall m = 1, 2, ..., n.$$

(9)

Again, we may define $T^0_n$ as the identity map and each $T^{t+1}_n(x_{[1\bar{t}+1]}, i_{[1\bar{t}+1]}) = T_n(x_{t+1}, i_{t+1}) \circ T^t_n(x_{[1\bar{t}]}, i_{[1\bar{t}]})$.

Even before the discussion of cumulative payoffs and equilibria, we introduce in Theorem 2 an interesting link between finite games and their NG counterpart. It reflects that
stochastic environment paths experienced by large finite games converge to the NG’s deterministic environment trajectory. To achieve this result, we need the following assumption on \( \tilde{s} \):

(M1) \( \tilde{s} \) is uniformly continuous in \((s, \sigma, x, i)\) in the sense that, for any \( \epsilon > 0 \), there are \( \delta_S, \delta_X, \delta_I, \delta_I > 0 \), so that for any \( s^1, s^2 \in S \) satisfying \( d_S(s^1, s^2) < \delta_S \), \( \sigma^1, \sigma^2 \in P(S) \) satisfying \( \rho_S(\sigma^1, \sigma^2) < \delta_X \), \( x^1, x^2 \in X \) satisfying \( d_X(x^1, x^2) < \delta_X \), and \( i^1, i^2 \in I \) satisfying \( d_I(i^1, i^2) < \delta_I \),

\[
d_S(\tilde{s}(s^1, \sigma^1, x^1, i^1), \tilde{s}(s^2, \sigma^2, x^2, i^2)) < \epsilon.
\]

With this assumption, we can prove the following key lemma, which is useful in the proof of the later Theorem 3 as well. This lemma calls upon Proposition 3.

**Lemma 1** Given probability space \((A, \mathcal{B}(A), \pi)\) and environment \( \sigma \in P(S) \), suppose \( s_n \) for each \( n \in N \) is a member of \( M(A^n, S^n) \), and \( 1_{s_n(a)} \) converges to \( \pi \) in probability, to the effect that

\[
\pi^n(\{a \in A^n \mid \rho_S(1_{s_n(a)}, \sigma) < \epsilon\}) > 1 - \epsilon,
\]

for any \( \epsilon > 0 \) and any \( n \) large enough. Then, \( T_n(x, i) \circ 1_{s_n(a)} \) will converge to \( T(x, i) \circ \sigma \) in probability for any continuous \( x \) too. That is, for any \( x \in K(S, X) \),

\[
(\pi \times \iota)^n(\{(a, i) \in (A \times I)^n \mid \rho_S(T_n(x, i) \circ 1_{s_n(a)}, T(x) \circ \sigma) < \epsilon\}) > 1 - \epsilon,
\]

for any \( \epsilon > 0 \) and any \( n \) large enough.

Here comes our main result of this subsection.

**Theorem 2** Let \( x_{[t]} \in (K(S, X))^{\bar{t}-t+1} \) be given. When we sample \( s_{t-1} = (s_{t-1,1}, s_{t-1,2}, ..., s_{t-1,n}) \) from a given \( \sigma_{t-1} \), the sequence \( (T_n^\tau(x_{[t,\tau+t-1]}), i_{[t,\tau+t-1]})) \circ 1_{s_{t-1}} \mid \tau = 0, 1, ..., \bar{t} - t \) will converge to \( (T^\tau(x_{[t,\tau+t-1]})) \circ \sigma_{t-1} \mid \tau = 0, 1, ..., \bar{t} - t \) in probability. That is, for any \( \epsilon > 0 \) and any \( n \) large enough,

\[
(\sigma_{t-1} \times \iota^{\bar{t}-t})^n(\tilde{A}_n(\epsilon)) > 1 - \epsilon,
\]

where \( \tilde{A}_n(\epsilon) \in \mathcal{B}^n(S \times I^{\bar{t}-t}) \) is such that, for any \( (s_{t-1}, i_{[t,\bar{t}-1]}) \in \tilde{A}_n(\epsilon) \),

\[
\rho_S(T_n^\tau(x_{[t,\tau+t-1]}), i_{[t,\tau+t-1]})) \circ 1_{s_{t-1}}, T^\tau(x_{[t,\tau+t-1]})) \circ \sigma_{t-1} < \epsilon, \quad \forall \tau = 0, 1, ..., \bar{t} - t.
\]
3.4 NG and Finite-game Equilibria

In defining $\Gamma(\sigma_0)$’s equilibria, we subject a candidate policy profile to the one-time deviation of a single player, who is by default infinitesimal in influence, rather than one-time deviations made by players of the same state who may make up a substantial body. Hence, our definition has nothing to do with whether or not the concerned distributions $\sigma_t$ are nonatomic; also, the deviation will not alter the environment trajectory corresponding to the candidate profile. With this understanding, we may define $v_t(s_{t-1}, \sigma_{t-1}, x_{[t]}^i)$ as the total discounted expected payoff a player can make from time $t$ to $\bar{t}$, when he starts with state $s_{t-1} \in S$ and outside environment $\sigma_{t-1} \in P(S)$, and players’ policy profile from $t$ to $\bar{t}$ is given by $x_{[t]} = (x_t, x_{t+1}, \ldots, x_{\bar{t}}) \in (M(S, X))^{\bar{t}-t+1}$. As a terminal condition, we certainly have

$$v_{t+1}(s_t, \sigma_t) = 0. \quad (10)$$

For $t = \bar{t}, \bar{t}-1, \ldots, 1$, we have the recursive relationship

$$v_t(s_{t-1}, \sigma_{t-1}, x_{[t]}^i) = f(s_{t-1}, \sigma_{t-1}, x_t(s_{t-1})) + \alpha \cdot \int_t v_{t+1}(\tilde{s}(s_{t-1}, \sigma_{t-1}, x_t(s_{t-1}), i), T(x_t) \circ \sigma_{t-1}, x_{[t+1], i}) \cdot \omega(di). \quad (11)$$

Given policy $y \in M(S, X)$, we may define $v'_t(s_{t-1}, \sigma_{t-1}, x_{[t]}^i, y)$ as the total discounted expected payoff similar to the just defined entity, with the only difference that, in period $t$, the current player is to adopt policy $y$ instead of $x_t$. For $t = 1, 2, \ldots, \bar{t}$, we have

$$v'_t(s_{t-1}, \sigma_{t-1}, x_{[t]}^i, y) = f(s_{t-1}, \sigma_{t-1}, y(s_{t-1})) + \alpha \cdot \int_t v_{t+1}(\tilde{s}(s_{t-1}, \sigma_{t-1}, y(s_{t-1}), i), T(x_t) \circ \sigma_{t-1}, x_{[t+1], i}) \cdot \omega(di), \quad (12)$$

and

$$v_t(s_{t-1}, \sigma_{t-1}, x_{[t]}^i) = v'_t(s_{t-1}, \sigma_{t-1}, x_{[t]}^i, x_t). \quad (13)$$

Now, we deem $x^* = (x^*_t | t = 1, 2, \ldots, \bar{t}) \in (M(S, X))^\bar{t}$ a Markov equilibrium for the game $\Gamma(\sigma_0)$ when, for every $t = 1, 2, \ldots, \bar{t}$ and $y \in M(S, X)$,

$$\int_S v_t(s_{t-1}, T^{t-1}(x^*_t) \circ \sigma_0, x^*_t) \cdot [T^{t-1}(x^*_t) \circ \sigma_0](ds_{t-1}) \geq \int_S v'_t(s_{t-1}, T^{t-1}(x^*_t) \circ \sigma_0, x^*_t, y) \cdot [T^{t-1}(x^*_t) \circ \sigma_0](ds_{t-1}). \quad (14)$$

For an $n$-player game, let $v_{nt}(s_{t-1,1}, 1_{(s_{t-1})^n_{-1}}, x_{[t]}^i)$ be the total discounted expected payoff player 1 can make from $t$ to $\bar{t}$, when he starts with state $s_{t-1,1} \in S$, other players’ initial
environments can be described by their aggregate state empirical distribution \( 1_{(s_{t-1})^n_1} = 1_{(s_{t-1},\ldots,s_{t-1,n})} \), and all players adopt the policy \( x_{[t]} = (x_t, x_{t+1}, \ldots, x_t) \in (M(S,X))^{f-t+1} \) from period \( t \) to period \( \tilde{t} \). As a terminal condition, we have

\[
v_{n,\tilde{t}+1}(s_{\tilde{t}1}, 1_{(s_t)^n_1}) = 0.
\]

For \( t = \tilde{t}, \tilde{t}-1, \ldots, 1 \), we have the recursive relationship

\[
v_{nt}(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x_{[t]}) = f(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x_t(s_{t-1,1})) + \bar{\alpha} \cdot \int_{I^t} \nu^n(di) \times \\
x_{n,t+1}(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x_t(s_{t-1,1}), i_1), [T_n(x_t, i^n_{t-1}) \circ (s_{t-1,1}, 1_{(s_{t-1})^n_1})]^n_{t-1}, x_{[t+1,\tilde{t}]})
\]

By \([T_n(x_t, i^n_{t-1}) \circ (s_{t-1,1}, 1_{(s_{t-1})^n_1})]^n_{t-1}, x_{[t+1,\tilde{t}]})\), we mean \( 1_{s^n_{t-1}} \) when \( 1_{s^n_{t}} \) is used to denote \( T_n(x_t, i) \circ (s_{t-1,1}, 1_{(s_{t-1})^n_1}) \). This entity depends on \( i \) only through \( i^n_{t-1} = (i_2, i_3, \ldots, i_n) \) and has nothing to do with whether action rule \( x_t \) has been taken with player 1’s state \( s_{t-1,1} \). On the other hand, \((s_{t-1,1}, 1_{(s_{t-1})^n_1}, x_t(s_{t-1,1}), i_1)\) and \([T_n(x_t, i^n_{t-1}) \circ (s_{t-1,1}, 1_{(s_{t-1})^n_1})]^n_{t-1}\) shall together form \( T_n(x_t, i) \circ (s_{t-1,1}, 1_{(s_{t-1})^n_1}) \), which will determine future environments to come.

Given policy \( y \in M(S,X) \), let \( v_{nt}'(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x_{[t]}, y) \) be the total discounted expected payoff player 1 similar to the just defined entity, with the only difference that, in period \( t \), player 1 is to adopt policy \( y \). For \( t = 1, 2, \ldots, \tilde{t} \), we have

\[
v_{nt}'(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x_{[t]}, y) = f(s_{t-1,1}, 1_{(s_{t-1})^n_1}, y(s_{t-1,1})) + \bar{\alpha} \cdot \int_{I^t} \nu^n(di) \times \\
x_{n,t+1}(s_{t-1,1}, 1_{(s_{t-1})^n_1}, y(s_{t-1,1}), i_1), [T_n(x_t, i^n_{t-1}) \circ (s_{t-1,1}, 1_{(s_{t-1})^n_1})]^n_{t-1}, x_{[t+1,\tilde{t}]})
\]

and

\[
v_{nt}(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x_{[t]}) = v_{nt}'(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x_{[t]}, x_t).
\]

Let \( \sigma = (\sigma_t | t = 0, 1, \ldots, \tilde{t}-1) \in (P(S))^\tilde{t} \) be a sequence of environments. For \( \epsilon \geq 0 \), we deem \( x^* = (x^*_t | t = 1, 2, \ldots, \tilde{t}) \in (M(S,X))^{\tilde{t}} \) an \( \epsilon \)-Markov equilibrium for the game family \((\Gamma_n(1_{s_0}) | s_0 \in S^n)\) in the sense of \( \sigma \) when, for every \( t = 1, 2, \ldots, \tilde{t} \) and \( y \in M(S,X) \),

\[
\int_{S^n} v_{nt}(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x^*_t, ds_t) \cdot \sigma^n_{t-1}(ds_{t-1}) \\
\geq \int_{S^n} v_{nt}'(s_{t-1,1}, 1_{(s_{t-1})^n_1}, x^*_t, y, ds_t) \cdot \sigma^n_{t-1}(ds_{t-1}) - \epsilon.
\]

### 3.5 The Main Result

We will eventually establish the main result Theorem 3, that an equilibrium from the NG will serve as an ever more accurate approximate equilibrium for ever larger finite games.
First, we need to assume that the single-period function $f$ is continuous:

$$(M2) f$$ is uniformly continuous in $(s, \sigma, x)$ in the sense that, for any $\epsilon > 0$, there is $\delta_{S}, \delta_{\Sigma}, \delta_{X} > 0$, so that for any $s^{1}, s^{2} \in S$ satisfying $d_{S}(s^{1}, s^{2}) < \delta_{S}$, $\sigma^{1}, \sigma^{2} \in P(S)$ satisfying $\rho_{S}(\sigma^{1}, \sigma^{2}) < \delta_{\Sigma}$, and $x^{1}, x^{2} \in X$ satisfying $d_{X}(x^{1}, x^{2}) < \delta_{X}$,

$$| f(s^{1}, \sigma^{1}, x^{1}) - f(s^{2}, \sigma^{2}, x^{2}) | < \epsilon.$$

The following are a couple more technical lemmas.

**Lemma 2** $v_{t}(s_{t-1}, \sigma_{t-1}, x_{[t]})$ is continuous in $s_{t-1}$ under continuous $x_{t}$’s.

**Lemma 3** Given $x_{[t]} \in (K(S, X))^{t-t+1}$. Suppose $1_{\sigma_{n, t-1}}$ converges to $\sigma_{t-1}$, then the value $v_{n}(s_{t-1, 1}, (1_{\sigma_{n, t-1}})^{n-1}_{t-1}, x_{[t]})$ will converge to $v_{t}(s_{t-1, 1}, \sigma_{t-1}, x_{[t]})$.

We now present the main result.

**Theorem 3** For some $\sigma_{0} \in P(S)$, suppose $x^{*} = (x^{*}_{t} \mid t = 1, 2, \ldots, \bar{t}) \in (K(S, X))^{\bar{t}}$ is a continuous Markov equilibrium of the nonatomic game $\Gamma(\sigma_{0})$. Then, for any $\epsilon > 0$, there is $\bar{n} \in N$ such that, for any $n = \bar{n}, \bar{n} + 1, \ldots$, this $x^{*}$ is also an $\epsilon$-Markov equilibrium for the game family $(\Gamma_{n}(1_{s_{0}}) \mid s_{0} \in S^{n})$ in the sense of $(T^{t}(x^{*}_{[t]})) \circ \sigma_{0} \mid t = 0, 1, \ldots, \bar{t} - 1)$.

For the finite games, there is a space of observables $O$. Also, there is a mapping $\tilde{o} \in M(P(S), O)$ that goes from environments to observables. Each player’s action at any time $t$ may be denoted by a function $x_{t} \in M(S \times O, X)$. Suppose player 1 is in state $s_{1, t-1}$ and the overall outside environment is $1_{s^{n}_{t-1,t-1}}$. Then, player 1’s action may be written as $x_{t}(s_{1, t-1}, \tilde{o}(1_{s^{n}_{1,t-1}}))$. There is a whole spectrum where $O$ and $\tilde{o}$ can be in. When $O = \{1\}$ and $\tilde{o}(\cdot) = 1$, each player is observation-blind; when $O = P(S)$ and $\tilde{o}$ is the identity map, each player is in full observation of the actual environment. When $S = L \times X$ as mentioned in Section 3.2, an intermediate case may be $O = P(X)$ and $\tilde{o}(\sigma) = \sigma \mid x$, the marginal distribution of $\sigma$ on $X$.

Theorem 3, however, nullifies the need to delve into the $(O, \tilde{o})$-related details about finite games. It says that an equilibrium of the NG counterpart, which is necessarily observation-blind, serves as a good asymptotic equilibrium for finite games when there are enough players;
and, the above asymptotic result is independent of the observation power of players in the finite games. The root cause of this is, as stated in Theorem 2, that the aggregate environment in large games evolves in a nearly deterministic fashion and hence the value of observation is diminished.

The value of Theorem 3 hinges on the relative ease with which the NG \( \Gamma(\sigma_0) \) can be analyzed over the finite games \( \Gamma_n(1_{\sigma_0}) \). For the former, we may name the transition from a given profile \( x = (x_t \mid t = 1, 2, \ldots, \bar{t}) \) to its resultant environment trajectory \( \sigma = (T^t(x_{[\bar{t}]}) \circ \sigma_0 \mid t = 0, 1, \ldots, \bar{t}) \) (around (7) and (8)) by the decision-to-environment operator \( Z^E_D : (M(S,X))^\bar{t} \to (P(S))^{\bar{t}+1} \); also, we may name the opposite transition of obtaining optimal profiles \( x \) under a given environment trajectory \( \sigma \) (around (10) to (14)) by the environment-to-decision operator \( Z^D_E : (P(S))^{\bar{t}+1} \to 2^{(M(S,X))^\bar{t}} \).

As evidenced by steps involved in (10) to (14), the second operator is by no means simple. Due to players’ abilities to influence their future gains through their current actions, some kind of dynamic programming rather than any myopic decision making is required. Still, if this step is overcome, we can identify equilibria \( x^* \) for \( \Gamma(\sigma_0) \) by studying fixed points of the operator \( Z^D_E \circ Z^E_D : (M(S,X))^\bar{t} \to 2^{(M(S,X))^\bar{t}} \). For a success story on this, the reader may check the nonatomic-game analysis of a dynamic pricing situation laid out in Yang and Xia (2008). For the latter finite games, however, even similar operators are hard to define.

4 Concluding Remarks

Leveraging convergence results on empirical distributions, we have illustrated links between finite-player games and infinite-player NGs. Under proper limiting regimes, we have shown that equilibria of NGs can be used in finite-player settings to produce asymptotically rational results. Through the filtering-out of the influence of each individual player’s action on the commonly-encountered environment, an NG is very often easier to handle than its finite-player counterpart. Our results therefore offer a justification for the study of NGs in situations where the number of involved parties, though huge, is finite.

Our message for normal-form models is sampling: an NG equilibrium can be applied to large finite games whose types are sampled from the NG’s type distribution. For extensive-
form models without individual states that bridge between past actions and future payoffs, the literature as well as our own unshown analysis gave out the signal of myopicity: the NG’s equilibrium which solves the current period’s problem only, can always be applied to large finite games to reap good rewards. For other extensive-form models without aggregate (global) shocks, our catch-word is observation-blindness: an equilibrium of NG, though not responsive to the current aggregate state, will nevertheless manage to help players in large games to obtain reasonably high payoffs.

Appendices

A. Proof of Proposition 2: Let \( A' \in \mathcal{B}(A) \) be chosen. If \( a \notin A' \), then
\[
(a, \pi)_n(A') \leq \pi(A') \leq (a, \pi)_n(A') + \frac{1}{n};
\]
if \( a \in A' \), then
\[
(a, \pi)_n(A') - \frac{1}{n} \leq \pi(A') \leq (a, \pi)_n(A').
\]
Hence, it is always true that
\[
| (a, \pi)_n(A') - \pi(A') | \leq \frac{1}{n}.
\]
In view of (1) and (2), we have
\[
\rho_A((a, \pi)_n, \pi) \leq \frac{1}{n}.
\]
We have thus completed the proof.

B. Proof of Theorem 1: Due to (S1), we may identify a \( \delta > 0 \), so that for any \( n \in \mathbb{N} \),
\[
\max_{m=1}^n | f(q_m, x_m, 1_{q_{-m}}(x^*)^{-1}) - f(q_m, x_m, p(x^*)^{-1}) | < \frac{\epsilon}{4},
\]
as long as \( q^n = (q_1, ..., q_n) \in Q^n \) satisfies
\[
\max_{m=1}^n \rho_X(1_{q_{-m}}(x^*)^{-1}, p(x^*)^{-1}) < \delta.
\]
If we let \( x^n = (x^*(q_1), x^*(q_2), ..., x^*(q_n)) \), we may see that \( 1_{q_{-m}}(x^*)^{-1} = 1_{x_{-m}} \) and \( 1_{q_{-m}}(x^*)^{-1} = 1_{x_{-m}} \) for any \( m = 1, 2, ..., n \). Therefore, from Proposition 2,
\[
\rho_X(1_{q_{-m}}(x^*)^{-1}, 1_{q^n}(x^*)^{-1}) = \rho_X(1_{x_{-m}}, 1_{x^n}) \leq \frac{1}{n}, \quad \forall m = 1, 2, ..., n.
\]
Combining Proposition 1 and (26), we know the existence of $Q' \in \mathcal{B}^\infty(Q)$ with $p^\infty(Q') = 1$, such that, for any $q = (q_1, q_2, \ldots) \in Q'$, there exists some integer $\bar{n}_0(q)$ for (25) to be true for any $n \geq \bar{n}_0(q)$. On the other hand, due to (S2), we may identify $\bar{n} \in N$, so that, for any $n \geq \bar{n}$ and $q^n = (q_1, \ldots, q_n) \in Q^n$,

$$\max_{m=1}^n \left| f_n(q_m, x_m, 1_{q_m}(x^*)^{-1}) - f(q_m, x_m, 1_{q_m}(x^*)^{-1}) \right| < \frac{\epsilon}{4}. \quad (27)$$

Also, because $x^*$ is an NG equilibrium for $\Phi(p)$, we know that, for any $q \in Q$ and $y \in X$,

$$f(q, x^*(q), p(x^*)^{-1}) \geq f(q, y, p(x^*)^{-1}). \quad (28)$$

Combining (24), (27), and (28), we see that, for any $q \in Q'$, $n \geq \bar{n}_0(q) \vee \bar{n}$, $x \in X^n$, and $m = 1, \ldots, n$,

$$f_n(q_m, x^*(q_m), 1_{q_m}(x^*)^{-1}) > f(q_m, x^*(q_m), 1_{q_m}(x^*)^{-1}) - \epsilon/4$$

$$> f(q_m, x^*(q_m), p(x^*)^{-1}) - \epsilon/2 \geq f(q_m, x_m, p(x^*)^{-1}) - \epsilon/2$$

$$> f(q_m, x_m, 1_{q_m}(x^*)^{-1}) - 3\epsilon/4 > f_n(q_m, x_m, 1_{q_m}(x^*)^{-1}) - \epsilon. \quad (29)$$

We have thus obtained the desired result. \hfill \blacksquare

**C. Necessity of Assumption (S1):** We consider an example where the type space $Q = [0, 1)$, the action space $X = \{+1, -1\}$, both $d_Q$ and $d_X$ are Euclidean, and payoff functions satisfy the following:

$$f_1(q, x, r) = f_2(q, x, r) = \cdots = f(q, x, r) = \frac{x}{(1 - q)^2} \cdot [r(\{+1\}) - r(\{-1\})]. \quad (30)$$

Note that $\mathcal{B}(X)$ coincides with the discrete $2^X$. We can check that (S2) is satisfied. On the other hand, we can verify that $f(q, x, \cdot)$ is continuous, but not uniformly in a $q$-equal fashion. That is, (S1) is slightly violated.

Consider the strategy profile $x = (x(q) \mid q \in Q) \in M(Q, X)$, where

$$x(q) =
\begin{cases} 
+1, & q \in \bigcup_{k=0}^\infty [1 - 1/2^k, 1 - 1/2^{k+1} - 1/2^{k+2}) = [0, 1/4) \cup [1/2, 5/8) \cup \cdots, \quad (31) \\
-1, & q \in \bigcup_{k=0}^\infty [1 - 1/2^{k+1} - 1/2^{k+2}, 1 - 1/2^{k+1}) = [1/4, 1/2) \cup [5/8, 3/4) \cup \cdots.
\end{cases}$$
In the NG involving the uniform type distribution $p$, this $x$ leads to $px^{-1}$ with $(px^{-1})(\{+1\}) = (px^{-1})(\{-1\}) = 1/2$. By (30), we know that all players under this action distribution will be indifferent between the two action choices. Hence, $x$ is an NG equilibrium.

For $a \in (0, 1)$, we use $\lfloor a \rfloor$ to represent the largest $1/2^k$, for $k = 1, 2, \ldots$, such that $1/2^k \leq a$. Let $Q' \in B^\infty(Q)$ be the set of sequences $q = (q_1, q_2, \ldots)$ satisfying the following: at an infinite number of $n$’s, the $(2n)$-th type $q_{2n}$ is in the interval $[1 - [1/\sqrt{2n}], 1)$ and $x(q_{2n})$ is of the opposite sign of $k_n \equiv (1_{q_{2n}} x^{-1})(\{+1\}) - (1_{q_{2n}} x^{-1})(\{-1\})$, which, as $q_{2n} = (q_1, \ldots, q_{2n-1})$ contains an odd number of components, satisfies $|k_n| \geq 1/(2n)$. When the latter occurs at a particular $n$, we may check through (30) that, in the game $\Phi_{2n}(1_{q_{2n}})$, the action $x(q_{2n})$ leaves player $2n$’s payoff at least 1 away from his best possible.

Let us find $p^\infty(Q')$. Based on the above description, we have

$$Q' = \bigcap_{l=1}^{+\infty} \bigcup_{n=l}^{+\infty} Q_n,$$

(32)

where, for $n = 1, 2, \ldots$,

$$Q_n = \{(q_1, \ldots, q_{2n}, \ldots) \in Q^\infty \mid q_{2n} \in [1 - [1/\sqrt{2n}], 1)\text{ and } x(q_{2n})\text{ is of the opposite sign of } (1_{q_{2n}} x^{-1})(\{+1\}) - (1_{q_{2n}} x^{-1})(\{-1\})\}.$$

(33)

From the above, especially (31) and (33), we can check that

$$p^\infty(Q_n) = \frac{1}{2} \cdot \frac{1}{\sqrt{2n}} \geq \frac{1}{4\sqrt{2n}},$$

(34)

and hence,

$$\sum_{n=1}^{+\infty} p^\infty(Q_n) = +\infty.$$

(35)

But all the $Q_n$’s are independent under $p^\infty$. So, by (32) and the second Borel-Cantelli Lemma, we can decide that

$$p^\infty(Q') = p^\infty\left( \bigcap_{l=1}^{+\infty} \bigcup_{n=l}^{+\infty} Q_n \right) = 1.$$

(36)

Therefore, the conclusion of Theorem 1 will not be true.

**D. Necessity of Assumption (S2):** We consider an example where the type space $Q = [0, 1)$, the action space $X = \{+1, -1\}$, and both $d_Q$ and $d_X$ are Euclidean. We let payoff
functions be $r$-independent. In particular, we let

$$f(q, +1, r) = 1, \quad f(q, -1, r) = 0, \quad \forall q \in [0, 1),$$

(37)

and for $n = 2, 3, \ldots$,

$$f_n(q, +1, r) = 1, \quad \forall q \in [0, 1),$$

(38)

$$f_n(q, -1, r) = 0, \quad \forall q \in [0, 1 - 1/n), \quad \text{and} \quad f_n(q, -1, r) = 2, \quad \forall q \in [1 - 1/n, 1).$$

We can check that (S1) is satisfied. On the other hand, we can verify that $f_n$ converges to $f$, but not in a $q$-uniform fashion. That is, (S2) is slightly violated.

Consider the strategy profile $x = (x(q) \mid q \in Q) \in M(Q, X)$ with $x(q) = +1$ for all $q \in Q$. In any NG, including the one involving the uniform type distribution $p$, this $x$ is clearly an NG equilibrium due to (37). Let $Q' \in \mathcal{B}^\infty(Q)$ be the set of sequences $q = (q_1, q_2, \ldots)$ satisfying the following: at an infinite number of $n$’s, the $n$-th type $q_n$ is in the interval $[1 - 1/n, 1)$. When the latter occurs at a particular $n$, we may check through (38) that, in the game $\Phi_n(1_{q_n})$, the action $x(q_n)$ leaves player $n$’s payoff 1 away from his best possible.

Let us find $p^\infty(Q')$. Based on the above description, we have

$$Q' = \bigcap_{l=1}^{+\infty} \bigcup_{n=l}^{+\infty} Q_n,$$

(39)

where, for $n = 1, 2, \ldots$,

$$Q_n = \{(q_1, \ldots, q_n, \ldots) \in Q^\infty \mid q_n \in [1 - \frac{1}{n}, 1)\}.$$  

(40)

We can check that

$$p^\infty(Q_n) = \frac{1}{n},$$

(41)

and hence,

$$\sum_{n=1}^{+\infty} p^\infty(Q_n) = +\infty.$$  

(42)

But all the $Q_n$’s are independent under $p^\infty$. So, by (39) and the second Borel-Cantelli Lemma, we have

$$p^\infty(Q') = p^\infty(\bigcap_{l=1}^{+\infty} \bigcup_{n=l}^{+\infty} Q_n) = 1.$$  

(43)

Therefore, the conclusion of Theorem 1 will not be true.
E. Proof of Proposition 3: Suppose sequence \( \{ \pi_{B1}', \pi_{B2}', \ldots \} \) weakly converges to the given probability measure \( \pi_B \), and sequence \( \{ \pi_{C1}', \pi_{C2}', \ldots \} \) weakly converges to the given probability measure \( \pi_C \). We are to show that the sequence \( \{ \pi_{B1}' \times \pi_{C1}', \pi_{B2}' \times \pi_{C2}', \ldots \} \) weakly converges to \( \pi_B \times \pi_C \).

Let \( F(B) \) denote the family of uniformly continuous real-valued functions on \( B \) with bounded support. Let \( F(C) \) be similarly defined for \( C \). We certainly have

\[
\begin{align*}
\lim_{k \to +\infty} \int_B f(b) \cdot \pi_{Bk}' (db) &= \int_B f(b) \cdot \pi_B (db), \\
\lim_{k \to +\infty} \int_C f(c) \cdot \pi_{Ck}' (dc) &= \int_C f(c) \cdot \pi_C (dc),
\end{align*}
\]

(44)

Define \( F \) so that

\( F = \{ f \mid f(b,c) = f_B(b) \cdot f_C(c) \text{ for any } (b,c) \in B \times C, \) where \( f_B \in F(B) \cup \{1\} \) and \( f_C \in F(C) \cup \{1\} \}, \) (45)

where 1 stands for the function whose value is 1 everywhere. (44) and (45) apparently lead to

\[
\lim_{k \to +\infty} \int_{B \times C} f(b,c) \cdot (\pi_{Bk}' \times \pi_{Ck}') (d(b,c)) = \int_{B \times C} f(b,c) \cdot (\pi_B \times \pi_C) (d(b,c)).
\]

(46)

According to Ethier and Kurtz (1986, Proposition III.4.4), \( F(B) \) and \( F(C) \) happen to be \( P(B) \) and \( P(C) \)'s convergence determining families, respectively. As \( B \) and \( C \) are complete, Ethier and Kurtz (1986, Proposition III.4.6, whose proof involves Prohorov’s Theorem, i.e., the equivalence between tightness and relative compactness of a collection of probability measures defined for complete separable metric spaces) further says that \( F \) as defined through (45) is convergence determining for \( P(B \times C) \). Therefore, we have the desired weak convergence by (46).

Let \( \epsilon > 0 \) be given. In view of the above product-measure convergence and the equivalence between the weak topology and that induced by the Prohorov metric, there must be \( \delta_B > 0 \) and \( \delta_C > 0 \), such that \( \rho_B(\pi_B', \pi_B) < \delta_B \) and \( \rho_C(\pi_C', \pi_C) < \delta_C \) will imply

\[
(\rho_B \times \rho_C)(\pi_B' \times \pi_C', \pi_B \times \pi_C) < \epsilon.
\]

(47)

By (1) and the given hypothesis, there is \( \bar{n}_1 \in N \), so that for \( n = \bar{n}_1, \bar{n}_2 + 1, \ldots \),

\[
(\pi_A)^n(\tilde{A}_n) > 1 - \frac{\epsilon}{2},
\]

(48)
where $\tilde{A}_n$ contains all $a \in A^n$ such that

$$\rho_B(1_{y_n(a)}, \pi_B) < \delta_B.$$  \hfill (49)

By (1) and Proposition 1, on the other hand, there is $\bar{n}^2 \in N$, so that for $n = \bar{n}^2, \bar{n}^2 + 1, \ldots$,

$$\left(\pi_C\right)^n(\tilde{C}_n) > 1 - \frac{\epsilon}{2},$$  \hfill (50)

where $\tilde{C}_n$ contains all $c \in C^n$ such that

$$\rho_C(1_c, \pi_C) < \delta_C.$$  \hfill (51)

For any $n = \bar{n}^1 \lor \bar{n}^2, \bar{n}^1 \lor \bar{n}^2 + 1, \ldots$, let $(a, c)$ be an arbitrary member of $\tilde{A}_n \times \tilde{C}_n$. We have from (47), (49), and (51) that,

$$\left(\rho_B \times \rho_C\right)(1_{y_n(a)}c, \pi_B \times \pi_C) < \epsilon.$$  \hfill (52)

Noting the facilitating $(a, c)$ is but an arbitrary member of $\tilde{A}_n \times \tilde{C}_n$, we see that

$$\left(\pi_A \times \pi_C\right)^n(\{(a, c) \in (A \times C)^n \mid \rho_B \times C(1_{y_n(a)}c, \pi_B \times \pi_C) < \epsilon\}) \geq \left(\pi_A\right)^n(\tilde{A}_n) \times \left(\pi_C\right)^n(\tilde{C}_n),$$  \hfill (53)

which by (48) and (50), is greater than $1 - \epsilon$. \hfill \blacksquare

**F. Proof of Proposition 4:** Let $\epsilon > 0$ be given. Note that $z$ is a continuous map from $B$ to $C$. By the continuous mapping theorem (Ethier and Kurtz 1986, Corollary III.1.9), there exists some $\delta \in (0, \epsilon]$, so that for any $\pi_B' \in P(B)$ satisfying $\rho_B(\pi_B', \pi_B) < \delta$,

$$\rho_C(\pi_B'z^{-1}, \pi_Bz^{-1}) < \epsilon.$$  \hfill (54)

On the other hand, by the hypothesis, we know for $n$ large enough,

$$\left(\pi_A\right)^n(\tilde{A}_n') > 1 - \delta,$$  \hfill (55)

where

$$\tilde{A}_n' = \{a \in A^n \mid \rho_B(1_{y_n(a)}, \pi_B) < \delta\} \in B^n(A).$$  \hfill (56)

So by (54) and (56), for $n$ large enough,

$$\rho_C(1_{y_n(a)}z^{-1}, \pi_Bz^{-1}) < \epsilon, \quad \forall a \in \tilde{A}_n'.$$  \hfill (57)
Combining this with (55), we obtain, for $n$ large enough,

$$
(\pi_A)^n(\{a \in A^n \mid \rho_C(1_{y_n(a)}z^{-1}, \pi_Bz^{-1}) < \epsilon\}) \geq (\pi_A)^n(A'_{n}) > 1 - \delta,
$$

(58)

which is further greater than $1 - \epsilon$. We have thus completed the proof. \hfill \Box

**G. Proof of Lemma 1:** Let $x \in M(S,X)$ and $\epsilon > 0$ be given. Define map $z \in M(S \times I, S)$, so that

$$
z(s, i) = \tilde{s}(s, \sigma, x(s), i), \quad \forall s \in S, i \in I,
$$

(59)

and for any $n \in N$ and $i \in I^n$. By (M1) and the continuity of $x$, we may see that $z \in K(S \times I, S)$. Also define operator $T'_n(i)$ on $P_n(S)$ so that $T'_n(i) \circ 1_s = 1_{s'}$, where for $m = 1, 2, \ldots, n$,

$$
s'_m = \tilde{s}(s_m, \sigma, x(s_m), i_m).
$$

(60)

In view of (8), (59), and (60), we may see that

$$
T(x) \circ \sigma = (\sigma \times \iota) \cdot z^{-1}, \quad \text{while} \quad T'_n(i) \circ 1_s = 1_{s'} \cdot z^{-1}.
$$

(61)

Combining the hypothesis on the convergence of $1_{s_n(a)}$ to $\sigma$ with Proposition 3, we obtain

$$
(\pi \times \iota)^n(\{(a, i) \in (A \times I)^n \mid \rho_{S \times I}(1_{s_n(a)i}, \sigma \times \iota) < \epsilon'\}) > 1 - \epsilon',
$$

(62)

for any $\epsilon' > 0$ and any $n$ large enough. By Proposition 4 and the fact that $z \in K(S \times I, S)$, this leads to

$$
(\pi \times \iota)^n(\{(a, i) \in (A \times I)^n \mid \rho_{S \times I}(1_{s_n(a)i} \cdot z^{-1}, (\sigma \times \iota) \cdot z^{-1}) < \epsilon'\}) > 1 - \epsilon',
$$

(63)

for any $\epsilon' > 0$ and any $n$ large enough. By (61), this is equivalent to the existence of $\tilde{n}_1 \in N$, so that for any $n = \tilde{n}_1, \tilde{n}_1 + 1, \ldots$,

$$
(\pi \times \iota)^n(\tilde{A}_n(\epsilon)) > 1 - \frac{\epsilon}{2},
$$

(64)

where

$$
\tilde{A}_n(\epsilon) = \{(a, i) \in (A \times I)^n \mid \rho_S(T(x) \circ \sigma, T'_n(i) \circ 1_{s_n(a)}) < \frac{\epsilon}{2}\} \in B^n(A \times I).
$$

(65)
On the other hand, note that the only difference between $T_n(x,i) \circ 1_s$ and $T'_n(i) \circ 1_s$ lies in that $1_{s_{m}}$ is used in (9) where $\sigma$ is used in (60). By (M1), there is $\delta \in (0, \epsilon/2]$, so that for any $(s,i) \in S \times I$ and any $\sigma' \in P(S)$ satisfying $\rho_S(\sigma, \sigma') < \delta$,

$$d_S(\bar{s}(s,\sigma,x(s),i),\bar{s}(s,\sigma',x(s),i)) < \frac{\epsilon}{2}.$$  \hspace{1cm} (66)

This leads to, for any $S' \in \mathcal{B}(S)$,

$$[T_n(x,i) \circ 1_s][(S')^{\epsilon/2}] \geq [T'_n(i) \circ 1_s](S'),$$  \hspace{1cm} (67)

whenever $\max_{m=1}^{n} \rho_S(\sigma, 1_{s_{m}}) < \delta$. By Proposition 2 and the hypothesis on the convergence of $1_{s_{n}(a)}$ to $\sigma$, there is $\bar{n} \in N$, so that for $n = \bar{n}^2, \bar{n}^2 + 1, \ldots$,

$$\pi^n(\tilde{B}_n(\delta)) > 1 - \delta \geq 1 - \frac{\epsilon}{2},$$  \hspace{1cm} (68)

where

$$\tilde{B}_n(\delta) = \{ a \in A^n \mid \max_{m=1}^{n} \rho_S(\sigma, (1_{s_{n}(a)})^n) < \delta \} \in \mathcal{B}^n(A).$$  \hspace{1cm} (69)

Consider arbitrary $n = \bar{n}^1 \lor \bar{n}^2, \bar{n}^1 \lor \bar{n}^2 + 1, \ldots$, $(a,i) \in \tilde{A}_n(\epsilon) \cap (\tilde{B}_n(\delta) \times I^n)$, and $S' \in \mathcal{B}(S)$. By (1) and (65), we see that

$$[T'_n(i) \circ 1_{s_{n}(a)}][(S')^{\epsilon/2}] + \frac{\epsilon}{2} \geq [T(x) \circ \sigma](S').$$  \hspace{1cm} (70)

Combining this with (67) and (69), we see that

$$[T_n(x,i) \circ 1_{s_{n}(a)}][(S')^{\epsilon/2}] + \epsilon \geq [T'_n(i) \circ 1_{s_{n}(a)}][(S')^{\epsilon/2}] + \epsilon > [T(x) \circ \sigma](S').$$  \hspace{1cm} (71)

In view of (1), this means

$$\rho_S(T_n(x,i) \circ 1_{s_{n}(a)}, T(x) \circ \sigma) < \epsilon.$$  \hspace{1cm} (72)

Therefore, for $n \geq \bar{n}^1 \lor \bar{n}^2$,

$$(\pi \times \iota)^n(\{(a,i) \in (A \times I)^n \mid \rho_S(T_n(x,i) \circ 1_{s_{n}(a)}, T(x) \circ \sigma) < \epsilon\}) \geq (\pi \times \iota)^n(\tilde{A}_n(\epsilon) \cap (\tilde{B}_n(\delta) \times I^n)),$$  \hspace{1cm} (73)

whereas the latter is, in view of (64) and (68), greater than $1 - \epsilon$. \hspace{1cm} \blacksquare
When $A = S$, $\pi = \sigma$, and $s_n(a) = a$, $1_{s_n(a)}$ is an empirical distribution of $\sigma$. We can obtain (64) in the proof much more easily: (61), in combination with the weak version of Proposition 1, will lead to the existence of $\bar{n} \in N$, so that (64) will be true for any $n = \bar{n}, \bar{n} + 1, \ldots$. Since Proposition 4 is not used, (M1) and the continuity of $x$ can be spared.

H. Proof of Theorem 2: We shall use induction to show that, for each $\tau = 0, 1, \ldots, \bar{t} - t$,

\[(\sigma_{t-1} \times \iota^\tau)^n(\tilde{A}_{n,\tau}(\epsilon')) > 1 - \frac{\epsilon'}{t - t + 1}, \quad (74)\]

for any $\epsilon > 0$ and $n$ large enough, where $\tilde{A}_{n,\tau}(\epsilon') \in B^\tau(S \times I^\tau)$ is such that, for any $(s_{t-1}, i_{[t,t+\tau-1]}) \in \tilde{A}_{n,\tau}(\epsilon')$,

\[\rho_S(T^\tau_n(x_{[t,t+\tau-1]}, i_{[t,t+\tau-1]}) \circ 1_{s_{t-1}}, T^\tau(x_{[t,t+\tau-1]}) \circ \sigma_{t-1}) < \epsilon'. \quad (75)\]

Once the above is achieved, we can then define $\tilde{A}_n(\epsilon)$ required in the theorem’s statement by

\[\tilde{A}_n(\epsilon) = \bigcap_{\tau=0}^{\bar{t}-t} [\tilde{A}_{n,\tau}(\epsilon) \times I^{n-(t-t-\tau)}]. \quad (76)\]

This and (74) will lead to

\[(\sigma_{t-1} \times \iota^{\bar{t}-t})^n(\tilde{A}_n(\epsilon)) > (1 - \frac{\epsilon}{t - t + 1})^{\bar{t}-t+1} > 1 - \epsilon, \quad (77)\]

for any $\epsilon > 0$ and $n$ large enough.

Now we proceed with the induction process. First, note that $T^0_n \circ 1_{s_{t-1}}$ is merely $1_{s_{t-1}}$ itself and $T^0 \circ \sigma_{t-1}$ is merely $\sigma_{t-1}$ itself. Hence, we will have (74) for $\tau = 0$ for any $\epsilon > 0$ and $n$ large enough just by Proposition 1. Then, for some $\tau = 1, 2, \ldots, \bar{t} - t$, suppose

\[(\sigma_{t-1} \times \iota^{-1})^n(\tilde{A}_{n,\tau-1}(\epsilon)) > 1 - \frac{\epsilon}{t - t + 1}, \quad (78)\]

for any $\epsilon > 0$ and $n$ large enough. We may apply Lemma 1 to the above, while at the same time identifying $S \times I^{\tau-1}$ with $A$, $\sigma_{t-1} \times \iota^{-1}$ with $\pi$, $T^{\tau-1}_n(x_{[t,t+\tau-2]}, i_{[t,t+\tau-2]}) \circ 1_{s_{t-1}}$ with $1_{s_n(a)}$, and $T^{\tau-1}(x_{[t,t+\tau-2]}) \circ \sigma_{t-1}$ with $\sigma$. This way, we will verify (74) for any $\epsilon > 0$ and $n$ large enough. Therefore, the induction process can be completed.
I. Proof of Lemma 2: We prove by induction on $t$. By (10), we know the result is true for $t = \bar{t} + 1$. Suppose for some $t = \bar{t}, \bar{t} - 1, ..., 2$, we have the continuity of $v_{t+1}(s_t, \sigma_t, x_{[t+1,\bar{t}]})$ in $s_t$. By this induction hypothesis, the continuity of $x_t$, (M1), and (M2), we may see the continuity of the right-hand side of (11) in $s_{t-1}$. Thus, $v_t(s_{t-1}, \sigma_{t-1}, x_{[t]}$) is continuous in $s_{t-1}$, and we have completed our induction process.

J. Proof of Lemma 3: We prove by induction on $t$. By (10) and (15), we know the result is true for $t = \bar{t} + 1$. Suppose for some $t = \bar{t}, \bar{t} - 1, ..., 2$, we have the convergence of $v_{n,t+1}(s_{t1}, (1_{sn})^n_{-1}, x_{[t+1,\bar{t}]})$ to $v_{t+1}(s_{t1}, \sigma_t, x_{[t+1,\bar{t}]}$) when $1_{sn}$ converges to $\sigma_t$. Now, suppose $1_{sn,t-1}$ converges to $\sigma_{t-1}$. Let $i = (i_1, i_2, ..., i_n)$ be generated through sampling on $(I, B(I), \iota)$.

Then by Lemma 1, we know that $T_n(x_t, i) \circ 1_{sn,t-1}$ converges to $T(x_t) \circ \sigma_{t-1}$ in probability. Due to Proposition 2, $(1_{sn,t-1})^n_{-1}$ will converge to $\sigma_{t-1}$, and $[T_n(x_t, i^n_{-1}) \circ 1_{sn,t-1}]^n_{-1}$ will converge to $T(x_t) \circ \sigma_{t-1}$. Thus,

1. $f(s_{t-1,1}, (1_{sn,t-1})^n_{-1}, x_t(s_{t-1,1}))$ will converge to $f(s_{t-1,1}, \sigma_{t-1}, x_t(s_{t-1,1}))$ in probability due to (M2); and,

2. the value $v_{n,t+1}(\tilde{s}(s_{t-1,1}, (1_{sn,t-1})^n_{-1}, x_t(s_{t-1,1}), i_1), T_n(x_t, i^n_{-1}) \circ 1_{sn,t-1}]^n_{-1}, x_{[t+1,\bar{t}]})$ will converge to $v_{t+1}(\tilde{s}(s_{t-1,1}, (1_{sn,t-1})^n_{-1}, x_t(s_{t-1,1}), i_1), T(x_t) \circ \sigma_{t-1}, x_{[t+1,\bar{t}]})$ in probability due to the induction hypothesis, which will in turn converge to $v_{t+1}(\tilde{s}(s_{t-1,1}, \sigma_{t-1}, x_t(s_{t-1,1}), i_1), T(x_t) \circ \sigma_{t-1}, x_{[t+1,\bar{t}]})$ in probability due to (M1) and Lemma 2.

As per-period payoffs are normalized, all value functions are uniformly bounded. The above convergences will then lead to the convergence of the right-hand side of (16) to the right-hand side of (11). That is, $v_{nt}(s_{t-1,1}, (1_{sn,t-1})^n_{-1}, x_{[t\bar{t}]}$) will converge to $v_t(s_{t-1,1}, \sigma_{t-1}, x_{[t\bar{t}]})$. We have thus completed the induction process.

K. Proof of Theorem 3: Let us consider subgames starting with some time $t = 1, 2, ..., \bar{t}$. For convenience, we let $\sigma_{t-1} = T^{t-1}(x^*_{[t,\bar{t}]}) \circ \sigma_0$. Now let $s_{t-1} = (s_{t-1,1}, s_{t-1,2}, ..., s_{t-1,n})$ be generated through sampling on $(S, B(S), \sigma_{t-1})$ and independently, $i = (i_1, i_2, ..., i_n)$ be generated through sampling on $(I, B(I), \iota)$.

By Proposition 1 and Lemma 1, we know that $1_{st-1}$ converges to $\sigma_{t-1}$ in probability, and also that $T_n(x^*_t, i) \circ 1_{st-1}$ converges to $T(x^*_t) \circ \sigma_{t-1}$ in probability. Due to Proposition 2,
(1_{s_{t-1}})_{n-1}^n$ and $[T_n(x_t^*, i_{t-1}^n) \circ 1_{s_{t-1}}]_{n-1}^n$ will have the same respective convergences. Then,

1. $f(s_{t-1,1}, (1_{s_{t-1}})_{n-1}^n, y(s_{t-1,1}))$ will converge to $f(s_{t-1,1}, \sigma_{t-1}, y(s_{t-1,1}))$ in probability at a $y$-independent rate due to (M2); and,

2. $v_{n,t+1}(\tilde{s}(s_{t-1,1}, (1_{s_{t-1}})_{n-1}^n, y(s_{t-1,1}), i_1), [T_n(x_t^*, i_{t-1}^n) \circ 1_{s_{t-1}}]_{n-1}^n, x_{[t+1,\Bar{t}]}^*)$ will converge to $v_{t+1}(\tilde{s}(s_{t-1,1}, (1_{s_{t-1}})_{n-1}^n, y(s_{t-1,1}), i_1), T(x_t^*) \circ \sigma_{t-1}, x_{[t+1,\Bar{t}]}^*)$ in probability at a $y$-independent rate due to Lemma 3, which will in turn converge to $v_{t+1}(\tilde{s}(s_{t-1,1}, \sigma_{t-1}, y(s_{t-1,1}), i_1), T(x_t^*) \circ \sigma_{t-1}, x_{[t+1,\Bar{t}]}^*)$ in probability at a $y$-independent rate due to (M1) and Lemma 2.

As per-period payoffs are normalized, all value functions are uniformly bounded. The above convergences will then lead to the convergence of the right-hand side of (19) to the right-hand side of (14). At the same time, the left-hand side of (19) minus $\epsilon$ will converge to the left-hand side of (14) due to the convergence of $(1_{s_{t-1}})_{n-1}^n$ to $\sigma_{t-1}$, Lemma 3, and the uniform boundedness of the value functions. From the satisfaction of (14), we know that (19) will be true for any $\epsilon > 0$ and $n$ large enough.

\section*{References}


