A Nonatomic-Game Model for Timing Clearance Sales Under Competition

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Abstract: We deal with dynamic revenue management (RM) under competition using the nonatomic-game approach. Here, a continuum of heterogeneous sellers try to sell the same product over a given time horizon. Each seller can lower his price once at the time of his own choosing, and faces Poisson demand arrival with a rate that is the product of a price-sensitive term and a market-dependent term. Different types of sellers interact, and their respective prices help shape the overall market in which they operate, thereby influencing the behavior of all sellers. Using the infinite-seller approximation, which deprives any individual seller of his influence over the entire market, we show the existence of a certain pattern of seller behaviors that collectively produce an environment to which the behavior pattern forms a best response. Such equilibrium behaviors point to the suitability of threshold-like pricing policies. Our computational study yields insights to RM under competition, such as profound ways in which consumer and competitor types influence seller behaviors and market conditions. © 2014 Wiley Periodicals, Inc. Naval Research Logistics 61: 365–385, 2014

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1. INTRODUCTION

Revenue management (RM) is about maximizing a seller’s profit by using pricing as well as other mechanisms such as overbooking, service customization, and discount allocation. Increasingly, it has become necessary for companies to bring the element of competition into consideration. The penetration of Internet access in virtually all homes and businesses has propelled price competition to an even more intense, global level. For example, customers can buy from many retailers listed at eBay.com for popular brands ranging from top-end luxury to modest clothes. In fact, the number of various retailers selling clothes, shoes, and accessories at eBay.com is increasing at double digit rates annually (Murray-West [21]). Besides eBay.com, many other online sites provide retailers with platforms to compete on. For instance, at least 50 retailers sell a particular Nike running shoe through Amazon.com. For the Samsung 17-inch SyncMaster LCD monitor, 83 sellers are listed at Google Shopping.

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It is, therefore, imperative for a good RM system to give heavy consideration to the outside competitive environment. Yet, current practices lag behind this imminent need. In the research community, the focus has been primarily on RM in a monopolistic setting. This is mostly because competitive RM needs to be modeled as a dynamic game, and hence is notoriously difficult to handle. This article aims to address the gap between the practical need and the state of research by taking a less frontal, nonatomic-game (NG) approach. Within the NG approach, we suppose that a continuum of heterogeneous sellers compete to sell the same product over a common time horizon. Each seller, with a given initial inventory, times his only price reduction action to maximize his revenue while anticipating a demand stream that is the product of all sellers’ collective decisions.

We categorize sellers into different types, so even under the same market condition, their abilities to attract demand might differ. This heterogeneity allows us to model seller differences in sales price, customer base, product quality, reputation, geographical characteristics, and various other factors. Our restriction to two price choices for each seller type is justifiable. Due to costs in paperwork and system
upgrading, advertisement and promotion, and manual relabeling, retailers do not take price changes lightly. They often conduct one significant markdown up to 50% during the sales season of a product, in the hope of enhancing sales before the product becomes obsolete. It is also noteworthy that airlines usually publish full and promotional fares in advance. Besides, insights derived in this article have the potential to be applied to more general competitive pricing problems in the future.

Under reasonable conditions, we show the optimality of a threshold-like pricing policy for any seller type under any given market condition: markdown is executed when the current time has passed a threshold level that decreases with the remaining inventory level. This part of the derivation, involving an optimal stopping-time problem solvable through variational inequalities, is dubbed the demand-to-decision direction. In the opposite decision-to-demand direction, we show that the evolution of the market condition is guided by a set of differential equations. The use of a continuity-based fixed point theorem then leads to the existence of an equilibrium threshold pricing policy.

The following features and results of our model are noteworthy as well:

I. We have carried out an experiment approximating a multiseller system by an ideal market unperturbable by any single seller, and with it have succeeded in finding a relatively simple solution to an otherwise unwieldy problem.

II. In our model, we have allowed the fraction of out-of-stock sellers to affect the market condition which is bound to be time-varying, and have considered multiple seller types which interact with each other through the common market.

III. We have gained the insight that, under intense competition, sellers should adopt threshold-like pricing policies, similar to the policy of a monopolist operating in a static environment. Of course, the actual policy is now a solution to a fixed-point rather than an optimization problem.

In addition, our simulation studies indicate that the presence of a reasonably large number of sellers would render predictions made by NG models accurate enough for practical purposes. Study of the NG model can, therefore, offer guidance to competitive RM when a reasonably large number of sellers are involved. Numerical analysis over a slew of model parameters produces useful insights into the competitive dynamic pricing problem. For instance, we verified that both consumer behavior and type compositions of competing sellers impact sellers’ responses and market conditions profoundly.

In the rest of the article, we first discuss relevant literature in Section 2 and give an overview of our model, approach, and results in Section 3. Detailed derivations in the demand-to-decision direction are conducted in Section 4 and those in the decision-to-demand direction are done in Section 5. We identify an equilibrium threshold-like policy in Section 6. Simulation and computational results are presented in Section 7. The article is concluded in Section 8, and most of its technical developments are left to appendices.

2. LITERATURE SURVEY

RM for a single seller with one-time inventory replenishment has been thoroughly understood under different scenarios. Gallego and van Ryzin [9], Bitran and Mondschein [4], and Zhao and Zheng [31] investigated cases where a seller can freely change his prices. Our research is more related to the optimal timing of irreversible price switches. Feng and Gallego [6] studied the optimal price-switching policy when there is a single opportunity for a price change in the entire sales horizon. They showed the optimality of a threshold-like policy, where a seller holding a certain number of remaining items will reduce (hike) his price if and only if the current time has passed (not passed) a certain time threshold that is decreasing in the aforementioned inventory level. Using more standardized procedures, Feng and Xiao [7] generalized the above result to the case where multiple price increases or decreases are permitted.

To study sellers under competitive pressure, we need game-theoretic price competition models. Perakis and Sood [22] examined a dynamic pricing game in which each seller adopts a price schedule that in the worst case is the best response to his competitors. Here, prices are not reactive to real-time sales data. Xu and Hopp [27] studied a multiseller dynamic pricing game where demand arrivals are governed by a geometric Brownian motion, the demand-price relation falls into a special family, and only the lowest-priced seller sells. Lin and Sibdari [16] showed that subgame-perfect equilibria exist for a discrete-time dynamic pricing game where opponents’ inventory levels are observable.

Levin et al. [15] and Liu and Zhang [18] all considered strategic consumers in competitive RM settings. Furthermore, Gallego and Hu [8] took a limiting approach to dynamic price competition as well, letting the demand arrival rate instead of the number of competing sellers go to infinity. In particular, they obtained structural insights to a differential game dealing with deterministic demand; then, for corresponding stochastic versions, they used earlier solutions to generate heuristics that turned out to be asymptotically optimal in the heavy-traffic limit.

We use NG models to study the dynamic pricing problem. They possess the following advantages over finite-game models:

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A. There is less interdependence among seller actions, as each individual’s decision has no discernible impact on the “environment” (public information), even though the latter is driven by the aggregate of all individuals’ decisions.

B. It is no longer necessary to assume that sellers are able to make real-time observations about the environment and base decisions on them; rather, the less demanding requirement is imposed in which sellers’ own collective actions fulfill their expectations about the environment.

Due to the above, NG models tend to offer more tractability than their finite-game counterparts. Notable works on NG include Schmeidler [23], Mas-Colell [20], Green [11], Housman [13], and Khan et al. [14]. The NG approach has found its use in economics, for instance, in the search literature; see Shimer and Smith [25] for randomized models and Eckhout and Kircher [5] for directed models. Incidentally, directed search studies are related to this article. They center around the match between sellers and buyers in the face of diverse preferences and contested trading opportunities.

Yang and Xia [29] applied the NG concept to a reversible dynamic pricing problem in which sellers can choose between two prices freely. The current article’s emphasis on irreversible pricing fills a different void in practice. For many businesses, especially retailers, reverting back to higher prices that have been visited in earlier times might be a virtually impossible option. Often times, customers, especially fashion-good shoppers, expect prices to decline over time; see, for example, Mantrala and Rao [19]. Thus, facing unknown and potentially prohibitively expensive consequences, sellers shun away from irksome customers.

The current article’s new feature of seller heterogeneity enables us to model complex situations, where sellers that react differently to external environments interact with each other through a common market. In its demand-to-decision direction, each seller is faced with a stopping time problem, which is somewhat more difficult than its counterpart in the earlier article by Yang and Xia [29]. Even when given the same threshold levels, sellers behave differently between the free-pricing and markdown cases. Hence, the current article’s decision-to-demand direction is also substantially different from that of the earlier article.

Hopenhayn [12] proposed the concept of stationary equilibrium that is based on time-invariant aggregate market prices and individual firm decisions. To further cope with situations involving nonignorable market shares of leading firms, Weintraub et al. [26] proposed the concept of oblivious equilibrium. In that approach, firms are only aware of the long-run average state of the industry instead of the latter’s exact present state. Adlakha and Johari’s [1] study of mean field equilibria of games exhibiting complementarity traits used the same steady-state approach. The current problem is prominently transient in nature, where each seller’s inventory level decreases over time; yet, these levels drive sellers’ behaviors. This renders the long-run average criterion inapplicable.

3. AN OVERVIEW

We suppose there is a continuum of sellers trying to sell the same product in time horizon \([0, T]\). Sellers are heterogeneous and their types form a finite set \(\Theta\). For each \(\theta \in \Theta\), let \(S(\theta) \in [0, 1]\) stand for the proportion of type-\(\theta\) sellers among the entire seller population. We certainly have

\[
\sum_{\theta \in \Theta} S(\theta) = 1. \tag{1}
\]

Sellers can lower their prices once based on their own inventory and the market condition. While different types of sellers can differ in prices, sellers of the same type \(\theta \in \Theta\) have the same price choices \(\bar{p}^0(\theta)\) and \(\bar{p}^1(\theta)\), with \(\bar{p}^1(\theta) > \bar{p}^0(\theta) > 0\). No restriction, however, is placed on prices adoptable by sellers of different types. For instance, \(\bar{p}^1(\theta) < \bar{p}^0(\theta)\) is allowed for \(\theta \neq \theta'\). To each seller, demand would come in as a Poisson process independent of demands faced by other sellers, with arrival rate dependent on the current time, the seller’s type, the price he charges, and the current market condition.

We also assume that the maximum number of initial items is \(N\). Now, at time \(0\), every seller possesses some \(n\) number of items with \(n = 0, 1, \ldots, N\). If we use \(f_n(0|\theta)\) to denote the relative fraction among all type-\(\theta\) sellers of those possessing \(n\) items at time \(0\), then the vector \(f(0|\theta) = (f_n(0|\theta)|n = 1, 2, \ldots, N)\) ought to mean the initial stock distribution of all type-\(\theta\) sellers and \(f(0) = (f(0|\theta)|\theta \in \Theta)\) that of all sellers. Note that \(1 - \sum_{n=1}^{N} f_n(0|\theta)\) is the portion among type-\(\theta\) sellers that are without stock in the beginning.

For convenience, we suppose out-of-stock sellers are effectively charging the \(+\infty\) price. In our model, market condition is reflected in the various fractions of sellers that are charging the particular prices \(\bar{p}^0(\theta), \bar{p}^1(\theta), \) and \(+\infty\) over all types \(\theta \in \Theta\). For any time \(t \in [0, T]\) and type \(\theta \in \Theta\), we use \(m^0(t|\theta)\) to denote the fraction, among the entire population, of sellers charging price \(\bar{p}^0(\theta)\) and \(m^1(t|\theta)\) the fraction of sellers charging price \(\bar{p}^1(\theta)\). Then, the fraction of type-\(\theta\) sellers that run out of stock is

\[
m^\infty(t|\theta) = S(\theta) - m^0(t|\theta) - m^1(t|\theta), \tag{2}
\]

and due to (1),

\[
\sum_{\theta \in \Theta} (m^0(t|\theta) + m^1(t|\theta) + m^\infty(t|\theta)) = 1. \tag{3}
\]
We use the $2 \cdot |\Theta|$-long vector $m(t) = (m^n(t)|\theta), m^0(t)|\theta \in \Theta)$ to represent the time-$t$ instantaneous market condition. Given $m(t)$, the fraction of sellers charging the high price $\bar{p}^1(\theta)$ for any $\theta$ can be automatically deduced from (2) as $S(\theta) - m^n(t)|\theta) - m^0(t)|\theta).$ The stream $m = (m(t)|t \in [0, T]) = (m^n(t)|\theta), m^0(t)|\theta) \in \Theta, t \in [0, T])$ describes the evolution of the market condition.

We use $\lambda^k(m, t|\theta)$ to denote the time-$t$ demand arrival rate under seller type $\theta \in \Theta$, price index $k \in [0, 1]$, and instantaneous market condition $m = (m^n(\theta), m^0(\theta)|\theta \in \Theta).$ A market-condition process $m = (m(t)|t \in [0, T])$ would lead to a common arrival-rate stream $\lambda = (\lambda(\theta)|\theta \in \Theta) = (\lambda^k(t)|\theta \in \Theta, k \in [0, 1], t \in [0, T]) = (\lambda^k(m(t)|t\theta) \in \Theta, k \in [0, 1], t \in [0, T]).$ Facing the common market process $m$, every type-$\theta$ seller should brace himself for arrival rate $\lambda^k(t)|\theta) = \lambda^k(m(t), t|\theta)$ if he is to charge price $\bar{p}^k(\theta)$ at time $t$. Each individual type-$\theta$ seller, knowing the negligibility of his contribution to the overall market condition, will respond optimally to the anticipated arrival-rate stream $\lambda(\theta)$.

We will study this response for the special arrival-function form $\lambda^k(m, t|\theta) = \alpha_k(\theta) \cdot \beta(m, t|\theta).$

The market condition $m = (m^n(\theta), m^0(\theta)|\theta \in \Theta)$ at time $t$ will affect a type-$\theta$ seller’s demand at that moment through the second multiplicative term $\beta(m, t|\theta).$ Note that $m$ contains information on the proportions of all prices on the market at the present time—$\sum_{\theta \in \Theta} m^n(\theta)$ is the proportion of out-of-stock sellers, each $m^0(\theta)$ is the proportion of sellers charging the type-$\theta$ low price $\bar{p}^0(\theta),$ and each $S(\theta) - m^n(\theta) - m^0(\theta)$ is the proportion of those charging the type-$\theta$ high price $\bar{p}^1(\theta).$ Hence, the $\beta$-term facilitates interaction of different sellers on the market.

Taken together, the two terms $\alpha_k(\theta)$ and $\beta(m, t|\theta)$ can reflect a variety of finite-seller price-demand relationships. Consider, for instance, the following demand form for seller 1 in a $V$-seller situation:

$$D_{V1}(p_1, p_2, ..., p_V, t) = \frac{a(t) \cdot \exp(-b p_1)}{c(t) + \exp(-B_{V-1}(p_2, ..., p_V))} \cdot 1(p_1 \leq \bar{p}),$$

(4)

where $\bar{p}$ is a positive constant, $1(\cdot)$ stands for the indicator function, and

$$B_{V-1}(p_2, ..., p_V) = b' / V \times \text{the sum of those}$$

$$p_2, p_3, ..., p_V \text{ below the bound } \bar{p}$$

$$+ b' \times \text{the proportion of prices above the bound } \bar{p}.$$

(5)

This form leaves much flexibility to the demand’s scaling with time $t$; at the same time, it allows demand to decrease with seller 1’s own price and increase with other sellers’ prices. The ratio and exponential functions are reminiscent of the multinomial logit relationship widely adopted in the competitive pricing literature; see, for example, Anderson et al. [3] and Aksoy-Pierson et al. [2]. However, we caution that in our case, seller 1’s demand is supposed to converge as $V$ approaches $+\infty$, whereas in the latter case, the $V$ sellers are supposed to virtually divide some total demand.

To imitate (4) and (5) in the current nonatomic setting, we can let

$$\alpha_k(\theta) = \exp(-b \cdot \bar{p}^k(\theta)),$$

$$\beta(m, t|\theta) = a(t) \cdot [c(t) + \exp(-b' \cdot \sum_{\theta'' \in \Theta} m^0(\theta'') \cdot \bar{p}^0(\theta'')) + (S(\theta) - m^n(\theta) - m^0(\theta')) \cdot \bar{p}^1(\theta'))] - b'' \cdot \sum_{\theta'' \in \Theta} m^n(\theta'')].$$

(6)

The multiplicative form certainly has its limitation in the rigid way in which it forces demand to scale with market changes across all sellers of the same type, independent of the prices they charge. But in view of the tractability it is to afford us, especially in the demand-to-decision direction, the mild sacrifice on model generality is probably worth taking.

If the market condition affects an individual’s demand in a more aggregate form, say through a function of $m$, then it certainly does not hurt to still model the demand as a function of $m$. Conversely, we believe it unnecessary to let demand be dependent on something less aggregate like the $f^k_n(\theta'')$’s, namely, the proportions of type-$\theta''$ sellers charging prices $\bar{p}^k(\theta'')$ and with $n$ items left—buyers usually do not have information about and will not base their purchase decision on the stocking levels of sellers.

With the multiplicative demand form, we can show that a threshold policy $\tau = (\tau_n(\theta)|\theta \in \Theta) = (\tau_n(\theta)|\theta \in \Theta, n = 1, 2, ..., N)$, satisfying $0 \leq \tau_n(\theta) \leq \tau_{n-1}(\theta) \leq \cdots \leq \tau_1(\theta) \leq T$ for any $\theta \in \Theta$, is an optimal response to a given market process. Under this policy, a type-$\theta$ seller should lower his price at the time when the threshold time $\tau_n(\theta)$ for his current inventory level $n$ is to be passed. We shall relate the resultant optimal threshold policy $\tau$ with the given arrival-rate stream $\lambda$ by the operation $\tau = Z^D_A(\lambda)$. Here, "$D$" resonates with decision and "$A$" resonates with arrival.

Once sellers have adopted the threshold policy $\tau$, their collective behavior would determine the course of the market-condition evolution. To describe this evolution, we introduce environment (shorthand for price-stock distribution process) $f = (f(\theta)|\theta \in \Theta) = (f(t)|\theta)|\theta \in \Theta, t \in [0, T]) = (f^k_n(t)|\theta)|\theta \in \Theta, k \in [0, 1], t \in [0, T], n = 1, 2, ..., N, t \in [0, T]).$ Inside $f$, each $f(t|\theta) = (f^k_n(t)|\theta)|k \in [0, 1], n = 1, 2, ..., N)$ is the time-$t$ price-stock distribution for type-$\theta$ sellers, with each $f^k_n(t|\theta)$ representing the portion of population among
type-θ sellers that are with stock level n and charging the price \( \bar{p}(θ) \) at time \( t \). An environment describes the all-time evolution of the price-stock distribution of the sellers.

Given threshold policy \( τ \) and arrival-rate stream \( λ \), a group of differential equations would govern the dynamics of the resulting environment \( f \). These equations are variants of a differential equation that is also instrumental to the definition of the earlier operator \( Z^D_A \). We have presented in detail this equation and its solution in Appendix A. Note that we denote the map from a given \( (τ, λ) \) pair to its corresponding \( f \) by \( f = Z^E_{DA}(τ, λ) \) wherein “\( E \)” stands for environment.

An environment \( f \) contains information about type, price, and inventory; yet, a market-condition process \( m \) that ultimately affects each seller’s payoff contains type and pricing information. Thus, we need to work with the transition from \( f \) to \( m \) and then from \( m \) to an arrival-rate stream \( \bar{λ} \). We use the map \( Z^A_A \) to denote the transition from \( f \) to \( \bar{λ} \). Finally, we suppose that buyers’ responses are at varied speeds; hence the actual arrival-rate stream \( \bar{λ} \) experienced by sellers is a lagged and time-averaged version of \( \bar{λ} \). We denote this transition by \( Z^A_A \).

Our main objective is to identify an equilibrium arrival-rate stream, one that induces a pricing policy as a best response, whose adoption by all sellers would in turn lead to the arrival-rate stream itself. The induced pricing policy is certainly an equilibrium pricing policy, since its adoption by all sellers would result in an arrival-rate stream that solicits the pricing policy as a best response. We achieve the equilibrium arrival-rate stream by focusing on the arrival-to-arrival operator \( Z^A_A(\cdot) = Z^A_A(Z^E_{DA}(Z^D_A(\cdot, \cdot))) \). To picture how the various operators form the composite operator \( Z^A_A \), the reader may refer to Fig. 1.

Suppose we have identified a fixed point \( λ^* \) to the composite operator \( Z^A_A \). We can then use \( τ^* = Z^D_A(λ^*) \) to identify the threshold policy used by all sellers in response to the arrival-rate stream \( λ^* \), use \( f^* = Z^E_{DA}(τ^*, λ^*) \) to denote the environment resulting from \( τ^* \) and \( λ^* \), and use \( \bar{λ}^* = Z^E_A(f^*) \) to denote the instantaneous arrival-rate stream corresponding to the environment \( f^* \). Since \( λ^* = Z^A_A(\lambda^*) = Z^A_A(Z^E_Z(Z^D_A(\lambda^*, λ^*))) = Z^A_A(Z^E_A(Z^D_A(\tau^*, λ^*))) = Z^A_A(\bar{λ}^*), \) the dynamics of the actual arrival-rate stream \( \bar{λ}^* \) is indeed governed by the policy \( τ^* \), environment \( f^* \), and instantaneous arrival-rate stream \( \bar{λ}^* \) that issue from it. Thus, \( \bar{λ}^* \) would qualify as an equilibrium arrival-rate stream.

4. FROM ARRIVAL-RATE STREAM TO DECISION

We take a product form for the arrival-rate function, so that \( \bar{λ}(m, t|θ) = \bar{p}(θ) \cdot \bar{P}(m, t|θ) \). Here, \( \bar{p}(θ) \) indicates that the demand arrival is dependent on the seller’s type and his charging price, and \( \bar{P}(m, t|θ) \) reflects the market condition’s influence on a type-θ seller’s demand arrival at time \( t \).

It is through the \( \bar{P} \) term that different seller types interact. Note the time- \( t \) market condition \( m(t) = (m^∞(t|θ), m^0(t|θ))|θ ∈ Θ \) contains information on aggregate proportions of sellers that are charging the various prices \( \bar{p}^1(θ), \bar{p}^0(θ), \) and \( +∞ \) across all types \( θ ∈ Θ \). The pricing policies of all seller types help form \( m(t) \), and the latter impacts all the arrival-rate streams seen by all seller types.

Let \( Δ^M \) be a subset of \( \mathbb{R}^{2|θ|} \), so that

\[
Δ^M = \left\{ (m^∞(θ), m^0(θ))|θ ∈ Θ \right\} \in [0, 1]^{2|θ|} | m^∞(θ') + m^0(θ') ≤ S(θ') \text{ for any } θ' ∈ Θ \right\}. \tag{7}
\]

All instantaneous market conditions reside in \( Δ^M \). We assume that:

- **S1.** in terms of revenue generation rate, \( \bar{p}^0(θ) \cdot \bar{p}^0(θ) > \bar{p}^1(θ) \cdot \bar{p}^1(θ) > 0 \) for every \( θ ∈ Θ \);
- **S2.** the function \( \bar{P}(θ, t|θ) \) is continuous on \( Δ^M \times [0, T] \) for every \( θ ∈ Θ \); and
- **S3.** the constant \( \bar{β} = \min_{θ ∈ Θ} \inf_{m ∈ Δ^M} \bar{P}(m, t|θ) \) is strictly positive.

Assumption (S1) means that a lower price will always lead to a higher revenue generation rate. If it were not true, there would be no reason to consider the lower price. Since \( \bar{p}^0(θ) < \bar{p}^1(θ) \), (S1) implies (S1’): \( \bar{p}^0(θ) > \bar{p}^1(θ) > 0 \) for every \( θ ∈ Θ \). That is, a lower price can attract more demand than a higher price. Assumption (S2) is merely a regularity condition. It says that arrival rates will not change dramatically with respect to market-condition and time variations. Because \( Δ^M \times [0, 1] \) is a compact set, by assumptions (S1’) and (S2), we can define the arrival-rate upper bound \( \bar{λ} \) through

\[
\bar{λ} = \max_{θ ∈ Θ} \left[ \sum_{θ ∈ Θ} \bar{p}^0(θ) \cdot \sup_{m ∈ Δ^M, t|θ} \bar{P}(m, t|θ) \right]. \tag{8}
\]

Assumption (S3) is essential in showing the continuity of a seller’s pricing decision with respect to the market process he
experiences. It means that, regardless of his external market environment, a seller will be able to attract some demand to himself by charging either the low or high price.

Now, suppose a time-continuous stream \( \beta = (\beta(t|\theta)\in \Theta, t \in [0, T]) \) is given, so that the arrival-rate stream \( \lambda(\theta) = (\lambda^k(t|\theta)\in [0, 1], t \in [0, T]) \) faced by a type-\( \theta \) seller is such that \( \lambda^k(t|\theta) = \tilde{\tau}^k(t) \cdot \beta(t|\theta) \) for \( k \in [0, 1] \) and every \( t \in [0, T] \). This typical type-\( \theta \) seller faces a stopping time problem, of deciding the optimal time to switch from his high to his low price using information about his own past inventory levels.

One optimal policy for this type-\( \theta \) seller comes in the threshold form \( \tau(\theta) = (\tau_n(\theta))_{n = 1, 2, ..., N} \), satisfying \( 0 \leq \tau_N(\theta) \leq \tau_{N-1}(\theta) \leq \cdots \leq \tau_1(\theta) \leq T \). To understand why, define \( a_n(t|\theta) \) so that

\[
a_n(t|\theta) = \exp(-\tilde{\lambda}^0(t,T|\theta)) \cdot \sum_{k=0}^{n-1} \frac{\left(\tilde{\lambda}^k(t,T|\theta)\right)^k}{k!}, \tag{9}
\]

where we have used \( \tilde{\lambda}^0(s,t|\theta) \) to denote the integral \( \int_s^t \lambda^0(u|\theta) \cdot du \). Taking derivative over \( t \), we have

\[
d_t a_n(t|\theta) = \lambda^0(t|\theta) \cdot \exp(-\tilde{\lambda}^0(t,T|\theta)) \cdot \frac{\left(\tilde{\lambda}^0(t,T|\theta)\right)^{n-1}}{(n-1)!}. \tag{10}
\]

Note the following:

a. By (S3) and (10), we know that \( d_t a_n(t|\theta) \) is strictly positive.

b. We have \( a_n(T|\theta) = 1 \), which, by (S1), is strictly greater than \( \tilde{\tau}^1(\theta) \cdot (\tilde{\rho}^1(\theta) - \tilde{\rho}^0(\theta))/\left(\tilde{\rho}^0(\theta) - (\tilde{\tau}^1(\theta) - \tilde{\tau}^0(\theta))\right) \).

c. It is easy to see from (9) that \( a_n(t|\theta) \) is increasing in \( n \).

By (a) and (b), we can let \( \tau_n(\theta) \) be the earliest time \( t \) that satisfies

\[
a_n(t|\theta) \geq \frac{\tilde{\tau}^1(\theta) \cdot (\tilde{\rho}^1(\theta) - \tilde{\rho}^0(\theta))}{\tilde{\rho}^0(\theta) - (\tilde{\tau}^1(\theta) - \tilde{\tau}^0(\theta))}, \tag{11}
\]

when this time exists in \([0, T]\), and \( \tau_n(\theta) = 0 \) otherwise. By (c), we know that \( \tau_n(\theta) \) is decreasing in \( n \). With the help of Feng and Xiao [7], we can show that \( \tau(\theta) \) provides an optimal pricing policy for the type-\( \theta \) seller.

**PROPOSITION 1:** When faced with arrival-rate stream \( \lambda(\theta) = (\lambda^k(t|\theta)\in [0, 1], t \in [0, T]) \), a type-\( \theta \) seller has the threshold policy \( \tau(\theta) \) defined through (9) and (11). Under this policy, the seller should switch from his high to low price when time \( t \) hits \( \tau_n(\theta) \) while its inventory level is at \( n \).

We shall use the demand-to-decision operator \( Z^D_A \) to denote the process from a given arrival-rate stream \( \lambda = (\lambda(\theta)\in \Theta) \) to its best-responding threshold pricing policy \( \tau = (\tau(\theta)\in \Theta) \) as identified in Proposition 1.

### 5. FROM DECISION TO ARRIVAL-RATE STREAM

To describe the environment, or the evolution of the sellers’ stock-price distribution, we use \( f^k_0(t|\theta) \) to denote the relative fraction among type-\( \theta \) sellers, of those with state \( (k, n) \) at time \( t \), that is, the fraction of population among type-\( \theta \) sellers who have \( n \) items left and are charging the price \( \tilde{p}^k(\theta) \) at time \( t \). Given initial stock distribution \( f(0|\theta) = (f_n(0|\theta))_{n = 1, 2, ..., N} \), suppose that all type-\( \theta \) sellers follow the same threshold policy \( \tau(\theta) = (\tau_n(\theta))_{n = 1, 2, ..., N} \) and the arrival-rate stream is known to be the continuous \( \lambda(\theta) = (\lambda^k(t|\theta)\in [0, 1], t \in [0, T]) \). Then for \( n = 1, 2, ..., N \), the portion of environment that is made up of type-\( \theta \) sellers would evolve as follows:

\[
\begin{align*}
f_n^0(0|\theta) &= 0, \\
d_t f_n^0(t|\theta) &= \lambda^0(t|\theta) \cdot (f_{n+1}^0(t|\theta) - f_n^0(t|\theta)), \\
&\quad \forall t \in (0, \tau_n(\theta)) \cup (\tau_n(\theta), T), \\
f_n^0(\tau_n(\theta)|\theta) &= f_n^0(\tau_n(-\theta)|\theta) + f_n^1(\tau_n(-\theta)|\theta), \\
f_n^1(0|\theta) &= f_n(0|\theta), \\
d_t f_n^1(t|\theta) &= \lambda^1(t|\theta) \cdot (f_{n+1}^1(t|\theta) - f_n^1(t|\theta)), \\
&\quad \forall t \in (0, \tau_n(\theta)), \\
f_n^1(\tau_n(\theta)|\theta) &= 0, \quad \forall t \in [\tau_n(\theta), T],
\end{align*}
\tag{12}
\]

where we have let \( f_{N+1}^0(t|\theta) = f_{N+1}^1(t|\theta) = 0 \) for \( t \in [0, T] \). In (12), the first and fourth equalities reflect that all type \( \theta \) sellers start with their high price \( \tilde{p}^1(\theta) \) in the beginning; the second and fifth equalities are due to the demand arrival rate of \( \lambda^k(t|\theta) \) at the price \( \tilde{p}^k(\theta) \); also, the third and sixth equalities come from the transition of the high price \( \tilde{p}^k(\theta) \) to the low price \( \tilde{p}^0(\theta) \) taking place at time \( \tau_n(\theta) \). By understanding the interval \((0, 0)\) as \( \emptyset \), the first equality as \( f_n^0(0^-|\theta) = 0 \), and the fourth equality as \( f_n^1(0^-|\theta) = f_n(0|\theta) \), we can allow the possibility of \( \tau_n(\theta) = 0 \) in (12).

In Appendix C, we show an iterative way to achieve the solution to (12). For all player types, the transition from a given pair of policy \( \tau = (\tau(\theta)|\theta \in \Theta) \) and arrival-rate stream \( \lambda = (\lambda(\theta)|\theta \in \Theta) \) to an environment \( f = (f(\theta)|\theta \in \Theta) \) that satisfies (12) fulfills our definition of the operator \( Z^D_A \).

When \( \lambda \) is continuous in time, \( f = Z^D_A f(\tau, \lambda) \) is piecewise continuous in \( t \) with potentially \( N \cdot |\Theta| \) discontinuities due to the presence of that many threshold points contained in \( \tau \).

Conversely, the arrival-rate stream \( \lambda \) may be thought of as a consequence of its underlying \( f \). Given an environment \( f \),
we let, for each $\theta \in \Theta$,
\[
\begin{align*}
m^k(t|\theta) &= \sum_{n=1}^{N} f^k_n(t|\theta) \cdot S(\theta), \forall k \in [0,1], \\
m^\infty(t|\theta) &= S(\theta) - m^0(t|\theta) - m^1(t|\theta).
\end{align*}
\] (13)

The thus obtained $m^0(t|\theta), m^1(t|\theta),$ and $m^\infty(t|\theta)$ would be the fractions of type-$\theta$ sellers, among the entire population, that are charging price $\bar{p}^0(\theta)$, charging price $\bar{p}^1(\theta)$, and out of stock, respectively. To the market-condition process for any $\theta$, we can learn later from Appendix D that both the fractions of type-$\theta$ that are charging price $\bar{p}^0(\theta)$, charging price $\bar{p}^1(\theta)$, and out of stock are discontinuous points from $f$.

That is, when all buyers see $m$ simultaneously, their collective response would lead to $\lambda$. This process from a given environment $f$ to its corresponding instantaneous arrival-rate stream $\lambda$ using (13) and (14) is denoted as operator $Z^A_{E}$. Note both the $f$ resulted $m$ and $\lambda = Z^A_{E}(f)$ can potentially inherit $N \cdot |\Theta|$ discontinuous points from $f$.

We suppose that the actual arrival-rate stream experienced by sellers is a lagged and time-averaged version $\lambda = \{\lambda(\theta)|\theta \in \Theta\} = \{\lambda^k(t|\theta)|\theta \in \Theta, k \in [0,1], t \in [0,T]\}$ of $\lambda$, such that
\[
\lambda^k(t|\theta) = \frac{1}{\xi} \cdot \int_{t-\xi}^{t} \lambda^k(s|\theta) \cdot ds,
\forall \theta \in \Theta, k \in [0,1], t \in [0,T],
\] (15)

where we have adopted the convention that $\lambda^k(t|\theta) = \lambda^k(0|\theta)$ for any $\theta \in \Theta, k \in [0,1], t \in [-\xi, 0]$. This assumption reflects that some time is needed for buyers to react to changes which have occurred in the market, and that different buyers might react at different speeds. Note that even information travels at the finite speed of light and relative to the seller, buyers are geographically dispersed. Our implicit assumption of the uniformly distributed buyer response time is merely an innocuous simplification. Now, at any moment in time, the current buyer behavior, being a function of the environment in the past, would nevertheless impact the environment instantaneously.

Even though $f, m,$ and $\lambda$ are only guaranteed to be piecewise continuous in time, we can learn later from Appendix D that $\lambda$ would be continuous. So, methodologically, (15) allows us to deal with the space of continuous functions rather than that of piecewise continuous functions. For the former, it is much easier to identify topologies conducive to the establishment of fixed points. Let us denote the instantaneous-to-actual transition through (15) by operator $Z^A_{D}$.

6. EQUILIBRIUM AND A GAME-NG CONNECTION

We use the Tychonoff fixed point theorem (see, e.g., Granas and Dugundji [10], p. 147) to establish the existence of a threshold pricing policy in equilibrium. To this end, we first define proper topological spaces that house actual arrival-rate streams $\lambda$, threshold policies $\tau$, environments $f$, and instantaneous arrival-rate streams $\lambda$. We then define the composite operator $Z^A_{E}$ through
\[
Z^A_{E}(\lambda) = Z^A_{E}(Z^E_{DA}(Z^D_{A}(\lambda), \lambda)).
\] (16)

The bulk of our work involves showing continuity properties of the four operators introduced in Sections 4 and 5, which lead to the continuity of the $Z^A_{E}$ operator. Finally, combining the last continuity result and properties of the $\lambda$-space, we can use the aforementioned fixed point theorem to establish the following main equilibrium existence result.

THEOREM 1: There exist actual arrival-rate stream $\lambda^*$, threshold policy $\tau^*$, environment $f^*$, and instantaneous arrival-rate stream $\lambda^*$, such that $\tau^* = Z^E_{DA}(\lambda^*)$, $f^* = Z^D_{A}(\tau^*, \lambda^*)$, and $\lambda^* = Z^A_{E}(\lambda^*)$.

The reader can find detailed derivations in Appendix D. As discussed at the end of Section 3, the fixed point $\lambda^*$ identified by Theorem 1 constitutes an equilibrium arrival-rate stream. It would induce all sellers to respond with pricing policies prescribed in $\tau^*$, which then leads back to the stream $\lambda^*$.

Besides existence, we have also identified the shape of an equilibrium pricing policy $\tau^*$ to be adopted by all sellers. Under $\tau^*$, a type-$\theta$ seller has only to react to his own inventory level and base his mark-down decision on a comparison between the threshold level $\tau^*_\ell(\theta)$ and the present time. This policy is easy to implement as it does not impose on sellers the burden of either remembering past plays or responding in real time to opponents’ evolving inventory levels. That sellers form a continuum has helped to mute the issue of whether inventory levels of other sellers are observable—regardless, the joint price-inventory distribution would follow a predetermined pathway $f^*$; whereas, they are often not observable in real practice but sometimes assumed to be so for theoretical tractability.

It would be pointless to study the equilibrium policy $\tau^*$ for the NG setting unless the policy could be used in the actual finite-seller setting to produce meaningful results. The applicability of NG equilibria in finite-player situations have found positive answers in the discrete-time setting; see Yang [28]. For the current continuous-time setting, however, we have so far only been able to show numerically that a given pricing policy would induce very similar environments in large finite-seller systems as well as in the NG setting. This nevertheless serves as empirical evidence for the practical value of NG-equilibrium policies.

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Conversely, we can formulate the problem of measuring the closeness of environments that result from applying one given pricing policy to the NG setting and a V-seller setting, respectively, while the number V of sellers in the latter setting keeps growing. The interested reader may consult the conjecture presented in our Appendix F. One major observable difficulty is that, even all sellers follow a given policy, such that to achieve the resulting environment, one must have already looked for a fixed point of a complex operator.

7. COMPUTATIONAL EXPERIMENTS

We have two goals for our computational experiments. First, we want to show the feasibility of using an NG-generated policy in a real finite-seller environment. Second, we are to use NG approximation to generate managerial insights for sellers engaged in RM competition.

7.1. Implementation Details

We discretize the time axis, so that for some integer I, each i = 0, 1, ..., I is a time index. With δT = T/I, index i now represents time t = i · δT in the original interval [0, T]. We always suppose Θ = {1, ..., |Θ|}. Now it is possible to use a 2N · |Θ| · (I + 1)-long vector f = (f^k(i|θ))k ∈ {0, 1}, θ ∈ Θ, n = 1, 2, ..., N, i = 0, 1, ..., I) to represent the environment, and 2 · |Θ| · (I + 1)-long vectors λ = (λ^k(i|θ))k ∈ {0, 1}, θ ∈ Θ, 0, 1, ..., I) and ̂λ = (̂λ^k(i|θ))k ∈ {0, 1}, θ ∈ Θ, 0, 1, ..., I) to represent the arrival-rate streams. Also, we can use an N · |Θ|-long vector τ = (τ^k(i|θ))k ∈ {0, 1}, θ ∈ Θ, n = 1, 2, ..., N) to represent a threshold policy, in which every τ^k(i|θ) is an integer in [0, 1, ..., I]. We approximate any integral \( \int_T^T h(u) \cdot du \) over the time interval \([t, s]\) by the summation \( \sum_{i=t}^{s} h(i \cdot \delta T) \cdot \delta T \).

Let there be positive constants p, Λ, and η, constants μ, ν, and δ in (0, 1), as well as constants γ, ν, and φ in [1]. For prices, we let

\[
\hat{\varphi}^0(θ) = μ p \cdot (1 + γ \cdot (θ - 1) / |Θ|),
\]

\[
\hat{\varphi}^1(θ) = p \cdot (1 + γ \cdot (θ - 1) / |Θ|).
\]

For arrival rates, we let the t-independent \( \lambda^k(m|θ) \) be of the product form with price-sensitive factors

\[
\varphi^0(θ) = φ^0 \cdot Λ, \quad \varphi^1(θ) = μ ν φ^20 \cdot Λ,
\]

and (t, θ)-independent market-dependent factor

\[
\bar{β}(m^∞(θ), m^0(θ))|θ ∈ Θ = 1 + \sum_{θ ∈ Θ} (γ \cdot m^∞(θ) - δ \cdot θ^{-n} \cdot m^0(θ)).
\]

Note that

\[
\hat{p}^1(θ) \cdot \hat{\varphi}^1(m|θ) = ν φ^θ < 1.
\]

Hence, assumptions (S1) to (S3) are all satisfied. Also, \( θ \) measures the sensitivity of customers to prices imposed by a type-θ seller, with a larger θ indicating a higher sensitivity. By (19), we see that \( \hat{β}(m) \) is increasing in \( m^∞(θ) \) and decreasing in \( m^0(θ) \) for every \( θ ∈ Θ \). The following are also clear from (17) and (19): When ξ > 0, the prices of different seller types are different; also, when η > 0, different seller types have different influences on arrival-rate streams.

Unless otherwise specified, we let the type distribution \( S(θ) |θ ∈ Θ \) be uniform, so that \( S(θ) = 1 / |Θ| \) for every \( θ ∈ Θ \). At each \( θ \), we let the initial stock distribution \( f(0) = f_n(0)|θ ∈ Θ, n = 1, 2, ..., N \) be uniform at strictly positive levels, so that \( f_n(0) = 1 / N \) for \( n = 1, 2, ..., N \). To implement (15), we let

\[
\lambda^k(i|θ) = \frac{\hat{λ}^k(θ - 1|θ) + ̂λ^k(i|θ)}{2},
\]

where we have adopted the convention that \( \hat{λ}^k(-1|θ) = \hat{λ}^k(0|θ) \).

To locate a fixed point \( λ^* \) for the composite operator \( Z^A_\Lambda \) defined in (16), we go through an iterative procedure, namely, a Tatônnement scheme as it is often called in the economics literature. We shall obtain arrival-rate stream \( λ^* \) as a limit of the sequence \( (λ)^i|θ ∈ Θ, i = 0, 1, ..., I \), where \( λ^0 \) is arbitrarily chosen and for every \( i \), \( λ^i = Z^A_\Lambda(λ^{i-1}) \). Our stopping criterion is \( ||λ^i - λ^{i-1}|| < ε \) for some error bound \( ε > 0 \), where

\[
||λ^i - λ^i|| = \max_{θ ∈ Θ} \max_{k = 0, 1} |λ^k(i|θ) - λ^k(i|θ)|.
\]

For this study, we have let \( ε = 0.001 \).

Theorem 1 is reached through the use of Tychonoff’s fixed point theorem, a generalization of Brouwer’s fixed point theorem. It is not generally known that a Tatônnement scheme could converge to a fixed point under the latter theorem’s conditions. Due partially to the involvement of the demand-to-decision mapping, or strict concavity. Therefore, a theoretical guarantee to our Tatônnement scheme is hard to come by.

Conversely, our extensive numerical tests have demonstrated that convergence through the scheme could take place quickly and always to unique equilibria. In all our tests, we fix \( T = 1, Λ = N/T = N, \) and \( p = 1 \). Also, we let \( |Θ| \) take
values 1, 2, and 5 and the number $N$ take values 5, 10, 30, and 50. At each $|\Theta|$ and $N$, we randomly generate 100 problem instances by sampling over the parameters $\mu, v, \delta, \eta, \xi, \gamma$, and $\phi$ from the uniform distribution on $(0, 1)$. At each problem instance, we initiate the corresponding Tat\’onnement scheme from 10 different starting points $\lambda^0$.

We found that convergence depends on the $I$ value we assume on the division of the time horizon. Most cases would converge within 20 iterations at $I = 1000$. For the few cases that failed to do so, we let $I = 2000$ and allowed them another 20 iterations. We could then try $I = 3000$, and so on. For all the cases we tested, we did not find one that would need to go beyond $I = 3000$. We present in Table 1 the average numbers of iterations needed for convergence under various $|\Theta|$ and $N$ values. Here, the number of iterations includes those wasted at smaller nonfinal $I$ values.

### 7.2. Convergence to Finite-Seller Situations

We use simulations involving finite numbers of sellers to illustrate that not many sellers are needed for the NG-equilibrium policy to become attractive in the real finite-seller setting. Once an equilibrium arrival-rate stream $\lambda^*$ is identified for an NG instance, we can use $\tau^* = Z_0^D(\lambda^*)$ and $f^* = Z_0^E(\tau^*, \lambda^*)$ to obtain the corresponding equilibrium pricing policy $\tau^*$ and environment $f^*$, respectively. We then conduct a series of simulations. We let there be some $Q$ independent runs in each simulation and let $V$ sellers be simulated in each run. All random variables within each run are also independent of each other. We identify sellers as $1, 2, ..., V$. In every run, we let seller $v$’s type $C_v$ be a sample from the distribution $(S(\theta) | \theta \in \Theta)$ and his initial inventory level $N_v(0)$ be a sample from the distribution $f(0|C_v)$. All the $V$ sellers are required to follow the NG policy $\tau^* = (i^*_n(\theta) \cdot \delta T | \theta \in \Theta, n = 1, 2, ..., N)$ to sell their products.

To compute the actual run- $q$ environment $f^q = (f^{q,k}(i|\theta))_{k \in [0, 1], \theta \in \Theta, n = 1, 2, ..., N, i = 0, 1, ..., I - 1}$ as a starting point, we let all sellers charge their respective high prices, and correspondingly, let $f^{q,0}(0|\theta) = f_n(0|\theta)$ and $f^{q,0}(0|\theta) = 0$ for $n = 1, 2, ..., N$. In each step $i$, we change the price index $k$ associated with a type-$\theta$ inventory- $n$ seller from 1 to 0 if $i \geq i^*_n(\theta)$ occurs for the time index $i$. Then, we let the random drop in this seller’s stock level in time interval $[i \cdot \delta T, (i + 1) \cdot \delta T]$ be the minimum between $n$ and the Poisson random variable with parameter $\lambda^{\delta T}$, where $\lambda^{\delta T}$ is computed from the time- $i$ price-stock distribution vector $f^q(i) = (f^{q,k}(i|\theta))_{k \in [0, 1], \theta \in \Theta, n = 1, 2, ..., N}$ using (13). Finally in this step, we compute $f^q(i + 1)$ from the current price-stock levels of all $V$ sellers.

We use the following formula to compute the distance $||f^q - f^*||$ between the actual run- $q$ environment $f^q$ and the NG-equilibrium environment $f^*$:

$$||f^q - f^*|| = \frac{1}{I + 1} \sum_{i=0}^{I} \sum_{k=0}^{1} \left| \sum_{\theta \in \Theta} f^{q,k}_n(i|\theta) - f^*_n(i|\theta) \right| \cdot S(\theta) .$$

(23)

As $\sum_{k=0}^{1} \sum_{n=1}^{N} \sum_{\theta \in \Theta} f^{q,k}_n(i|\theta) \cdot S(\theta)$ is the fraction of sellers with stocks left at time index $i$, this definition of distance “scales well” with $N$. We now use $\mu_{sf}$ and $\sigma_{sf}$ to denote, respectively, the sample mean and standard deviation of the environmental differences within a simulation. That is,

$$\mu_{sf} = \frac{1}{Q} \sum_{q=1}^{Q} ||f^q - f^*||,$$

(24)

and

$$\sigma_{sf} = \sqrt{\frac{1}{Q - 1} \cdot \sum_{q=1}^{Q} (||f^q - f^*|| - \mu_{sf})^2} .$$

(25)

In all simulations, we fix $\mu = 0.65$, $\nu = 0.7$, $\delta = 0.5$, $\eta = 0.5$, $\xi = 0.1$, $\gamma = 0.5$, and $\phi = 0.9$. In Figs. 2, 3, and 4, we depict $\mu_{sf} \pm \sigma_{sf}$ values for simulations with $|\Theta| = 1, 2, 5$, and 5, respectively. In each figure, we take $N = 30$ and 50, and let the $V$ values vary.

From the three figures, we can unequivocally discern the converging trend of the V-seller system’s aggregate environment towards the NG-equilibrium environment when all $V$ sellers adopt the NG-equilibrium pricing policy. The convergence seems to be reasonably fast and not much dependent on the heterogeneity of sellers and initial inventory levels. At $N = 10$, we notice that the environmental distance between a 50-seller system and the NG is already below 10%. This reflects that NG is a good proxy for reality when competition is sufficiently intense.

### 7.3. Managerial Insights

We can use the NG as a proxy to derive managerial insights for real, finite-seller situations when the number of competitors is large enough. We first present sellers’ threshold...
policies under different parameters. We shall plot each threshold policy in the $t-n$ plane, with time $t \in [0, T]$ marked on the horizontal axis and inventory level $n \in \{0, 1, ..., N\}$ marked on the vertical axis. This way, given an inventory-level trajectory, we can easily tell the price sample path under the given policy. For illustration purposes, we have plotted an example with $N = 4$ in Fig. 5.

In the figure, the arrows mark the trajectory of a seller’s inventory level. The seller starts at $t = 0$ with inventory level $n = 4$ and the high price $\bar{p}^1$. He then switches to the low price $p^1$. 

Figure 2. Environmental distances at varying $V$ values—when $|\theta| = 1$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Figure 3. Environmental distances at varying $V$ values—when $|\theta| = 2$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Figure 4. Environmental distances at varying $V$ values—when $|\theta| = 5$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]
\[ \bar{p}^0 \] when he has an inventory level of \( n = 3 \) and the time \( t \) is about to cross the threshold point \( r_3 \). In the end the seller is out of stock.

Let us now fix \( N = 30, T = 1, \mu = 0.65, \xi = 0.1, \phi = 0.9, \) and \( \eta = 0.5 \), while letting each one of the three parameter \( \nu, \gamma, \) and \( \delta \) vary around its default value. As before, the default values are \( \nu = 0.7, \gamma = 0.5, \) and \( \delta = 0.5 \).

First, we let \( \nu \) vary within \((0, 1)\), while \( \Theta = \{1\} \) and all other parameters are kept at their default values. Note that \( \nu \) measures buyers’ acceptance of the high price. In Fig. 6, we depict the policy \( \tau(1) \) at different \( \nu \) values.

From Fig. 6, we see that \( \tau_n(1) \) increases in \( \nu \) at every \( n \). That is, as buyers become more receptive of the high price, sellers are more reluctant to lower their prices. This is intuitive, as sellers would naturally want to exploit the buyers’ higher tolerance of the high price.

Then, we let \( \gamma \) vary within \([1]\), while \( \Theta = \{1\} \) and all other parameters are kept at their default values. Note that \( \gamma \) reflects how the proportion of out-of-stock sellers affect the sales of the current seller. In Fig. 7, we depict the policy \( \tau(1) \) at different \( \gamma \) values.

From Fig. 7, we see that \( \tau_n(1) \) increases in \( \gamma \) at every \( n \). When out-of-stock sellers affect demands at with-stock sellers more decisively, sellers tend to be more reluctant to lower their prices. With demand edging higher, sellers can be more relaxed in their attempts to lure demand with the low price.

Next, we let \( \delta \) vary within \((0, 1)\), while \( \Theta = \{1, 2, 3, 4, 5\} \) and all other parameters are kept at their default values. Note that \( \delta \) reflects how adversely other sellers’ charging the low price would affect the current seller’s sales. In Fig. 8, we depict the policy \( \tau(1) \) at different \( \delta \) values.

From Fig. 8, we see that \( \tau_n(1) \) decreases in \( \delta \) at each \( n \). Note a larger \( \delta \) is associated with the increased power of other sellers to affect the sales of the current seller. When other sellers are more capable of subduing the current seller’s demand by charging the low price, all sellers would compete more eagerly to undercut their competitors’ prices.
To understand the impact of seller heterogeneity on seller behaviors, we illustrate in Fig. 9 threshold policies of sellers of different types. We let $\Theta = \{1, 2, 3, 4, 5\}$ and all other parameters be at default values.

Figure 9 clearly indicates a decreasing trend of $\tau(\theta)$ in $\theta$. This has much to do with (20), which dictates that a bigger increase in revenue rate from price reduction would be associated with a higher seller type $\theta$. Such a seller would be in more favor of price changes.

Finally, to understand the impact of seller-type composition on market evolution, we deviate from the uniformity of the type distribution $(S(\theta)|\theta \in \Theta)$. We take $\Theta = \{1, 2\}$, $\phi = 0.5$, and keep almost all other parameters at their default values. However, we let $S(1)$, the proportion of type-1 sellers, go through 0.1 and 0.9, and effectively let $S(2)$ go through 0.9 and 1.1. In Fig. 10, we depict the relative market process $m = (m^0(t)/S(\theta), m^\infty(t)/S(\theta))_{t \in [0, T], \theta \in \Theta}$.

In Fig. 10, the left portion reflects the market evolution of type-1 sellers and the right portion reflects that of type-2 sellers. Because the general patterns are similar, we can concentrate on the left portion. In it, we may fix any time $t$ and focus on one particular $S(1)$ value, say 0.1, that is represented by the two solid curves. If we draw a vertical line segment at the horizontal coordinate $t$ that goes from 0 to 1, it will intersect the solid curves at two vertical coordinates, $m^0(t)/S(1)$ and $1 - m^\infty(t)/S(1)$. This results in three intervals: $[0, m^0(t)/S(1)]$, $[m^0(t)/S(1), 1 - m^\infty(t)/S(1)]$, and $[1 - m^\infty(t)/S(1), 1]$. The lengths of these intervals, $m^0(t)/S(1)$, $m^\infty(t)/S(1) = 1 - m^0(t)/S(1) - m^\infty(t)/S(1)$, and $m^\infty(t)/S(1)$, exactly correspond to the fractions of sellers charging the low price, those charging the high price, and those that have run out of stock, respectively.

By comparing the solid and dashed curves, we see that, when $S(1)$ increases, sellers will be slower to reduce their prices and quicker to run out of stock. Note type-1 sellers are innately low-price-charging and type-2 sellers are innately high-price-charging. When $S(1)$ increases at the expense of $S(2)$, the market will have more sellers charging low prices. When all else is the same, there will be more demand coming to both types of sellers. This way, both types of sellers will feel less urgency to reduce their prices. However, this relaxed attitude will not be enough to avoid the inevitability of items going out of stock more quickly.

From all these numerical tests, the following observations have become obvious: (a) buyers’ behaviors profoundly affect the equilibrium behaviors of sellers and (b) the composition of seller types impact the evolution of market conditions. A seller in an intensely competitive market should take stock of the types of customers and competing sellers he will face, and then adjust his pricing policies accordingly.

8. CONCLUDING REMARKS

We explored the possibility of using an NG scheme, as opposed to a finite-game one, to examine dynamic RM under competition. The outcome of our study suggested that the
tractability afforded by the NG approach might have far outweighed the loss of precision caused by its infinite-seller approximation. Our computational experiments verified that NG models could serve as proxies of real competitive situations and be used to generate managerial insights for the latter.

It is desirable to relax the two-price restriction for each seller type. However, we caution that the generalization might be difficult to carry out. Liu and Yang [17] have shown that threshold levels for this case are not necessarily monotone in price indices, rendering the decision-to-demand portion of the derivation much harder than that reported in Section 5. Finally, it could prove worthwhile to study the more realistic scenario where buyers with different preferences optimize their own purchasing behavior. For this endeavor, extant researches on directed search, for example, Shi [24] and Eeckhout and Kircher [5], should be of great help.

APPENDIX A: A DIFFERENTIAL EQUATION AND ITS SOLUTION

Given real numbers \(t, s\), and \(f_0\), continuous functions \(a\) and \(b\) defined on interval \([t, s]\), we use \(F[t, s, f_0, a, b]\) to denote function \(f\) defined on \([t, s]\) that satisfies the following equation:

\[
\begin{align*}
  f(t) &= f_0, \\
  d_uf(u) &= a(u) \cdot (b(u) - f(u)) \quad \forall u \in (t, s). \\
\end{align*}
\]

This differential equation has a unique solution \(F[t, s, f_0, a, b]\) as follows. For any \(u \in [t, s]\),

\[
F[t, s, f_0, a, b](u) = f_0 \cdot \exp \left( - \int_t^u a(v)dv \right) + \int_u^s a(v)b(v) \\
\cdot \exp \left( - \int_v^u a(w)dw \right) \cdot dv.
\]

The above will be useful for both operators \(Z_D^A\) and \(Z_B^D\).

APPENDIX B

PROOF OF PROPOSITION 1: As this result concerns one particular seller type, we opt to exclude \(\theta\) from our notation. That is, we use \(\tilde{\beta}\) instead of \(\beta(\theta), \tilde{\sigma}^0\) instead of \(\sigma^0(\theta), \beta(t)\) instead of \(\beta(t|\theta)\), \(\tau_n\) instead of \(\tau_n(\theta)\), and so on and so forth.

Use \(N^0(s, t)\) for the random number of arrivals under price \(\tilde{\beta}\) and the given stream \(\beta\) during period \([s, t)\). Let \(v^0_n(t)\) be the maximum expected revenue the seller can make in time interval \([t, T]\) when he starts time \(t\) with inventory level \(n\) and price \(\tilde{\beta}\). Since \(\tilde{\beta}\) is the last price the seller can charge before going out of stock, we have

\[
v^0_n(t) = \tilde{\beta}^0 \cdot E[N^0(t, T) \wedge n].
\]

When the seller is currently charging \(\tilde{\beta}^1\), he gets to decide when to switch to \(\tilde{\beta}^0\). Hence,

\[
v^1_n(t) = \sup_{t \in \mathcal{T}(t)} \left[ \tilde{\beta}^1 \cdot \left( N^0(t, T) \wedge n \right) \right. \\
\left. + \tilde{\beta}^0 \cdot (N^0(t, T) \wedge (n - N^0(t, t))) \right].
\]

where we have let \(T(t)\) stand for the set of stopping times in the time interval \([t, T]\). Define infinitesimal generators \(G_n^0(t)\) corresponding to the high price \(\tilde{\beta}^1\) and various \(n\) and \(t\) levels, so that

\[
G_n^0(t)v^0 + \tilde{\beta}^1 \cdot \beta(t) \leq 0, \quad \text{when } t \in [\tau_n, T], \\

\[
G_n^0(t)v^0 + \tilde{\beta}^1 \cdot \beta(t) > 0, \quad \text{when } t \in [0, \tau_n).
\]

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Now let us construct a function $u_n(t)$. First, for $t \in [0, T]$, define
$$u_0(t) = v_0^0(t) = 0. \quad (38)$$
Next, for $n = 1, 2, ..., N$, we iteratively define $u_n(t)$ for $t \in [0, \tau_n)$ as the unique solution for the differential equation
$$\frac{d}{dt} v_n^0(t) + \beta^1 \cdot \partial_1 \cdot \beta(t) = 0, \quad \forall t \in (0, \tau_n),$$
$$u_n(\tau_n) = v_0^0(\tau_n). \quad (39)$$
Indeed, following (26) and (27), we can solve (39) to obtain, for any $t \in [0, \tau_n)$,
$$u_n(t) = v_0^0(\tau_n) \cdot \exp(-\tau_n \cdot \beta(t, \tau_n))$$
$$+ \tau_n^1 \cdot \int_0^{\tau_n} \beta(s) \cdot (1 + \exp(-\tau_n \cdot \beta(t, \tau_n))) \cdot ds. \quad (40)$$
Conversely, from the second half of (37), we have, for $t \in [0, \tau_n)$,
$$v_0^0(t) < v_n^0(\tau_n) \cdot \exp(-\tau_n \cdot \beta(t, \tau_n))$$
$$+ \tau_n^1 \cdot \int_0^{\tau_n} \beta(s) \cdot (1 + \exp(-\tau_n \cdot \beta(t, \tau_n))) \cdot ds. \quad (41)$$
Using (38), (40), and (41), we can inductively prove that
$$u_n(t) > v_n^0(t), \quad \forall n = 1, 2, ..., N \text{ and } t \in [0, \tau_n). \quad (42)$$
The constructed $u_n(t)$ is uniformly bounded and absolutely continuous in $t$ for every $n$. By (37), (38), (39), and (42), as well as the fact that $v_n^0(T) = 0$ for every $n$, we can see that $u_n(t)$ also satisfies (i) to (iv) stipulated earlier. Hence, we have $u_n(t) = u_n(t)$. By (37), $\tau_n$ is also the earliest time $t$ for $v_n^0(t) = 0$, and hence defines an optimal switching time. □

**APPENDIX C: DETAILS ON THE $f = Z^k_{\text{DA}}(\tau, \lambda)$ OPERATION**

For convenience, we again omit the $\theta$-dependence; also, we let $\tau_{N+1} = 0$ and $\tau_0 = T$. For $n = 1, 2, ..., N$, we use symbol $f_{k,N}^{n+1}(\tau_{n+1})$ to represent $f_{k}(0)$. For $m = 0, 1, 2, ..., N$, we let $f_{k,m+1}$ be a function defined on $[\tau_{m+1}, \tau_{m}]$ that is always 0. Now, we iteratively define $N \cdot (N + 1)$ functions $f_{k,m}^{n}$ for $n = 1, 2, ..., N$ and $m = 0, 1, ..., N$, such that each $f_{k,m}^{n}$ is defined on $[\tau_{m+1}, \tau_{m}]$. Specifically, for $n = N, N - 1, ..., 1$, we let
$$f_{k,m}^{n} = \begin{cases} F[\tau_{m+1}, \tau_{m}, f_{k,m+1}^{n+1}(\tau_{m+1}), \lambda, f_{1,m}^{1}] & \text{when } m = N, N - 1, ..., n, \\ 0 & \text{when } m = n - 1, n - 2, ..., 0, \end{cases} \quad (43)$$
where the $F[\cdot]$ notation is defined through (27) in Appendix A.

Similarly, for $n = 1, 2, ..., N$, we let $f_{k,N}^{n}$ be a function defined on $[\tau_{N+1}, \tau_N]$ that is always 0, while for $m = 0, 1, ..., N$, we let $f_{k,m}^{N+1}$ be a function defined on $[\tau_{m+1}, \tau_{m}]$ that is always 0. We can iteratively define $N^2$ functions $f_{k,m}^{n}$ for $n = 1, 2, ..., N$ and $m = 0, 1, ..., N - 1$, such that each $f_{k,m}^{n}$ is defined on $[\tau_{m+1}, \tau_{m}]$. Specifically, for $n = N, N - 1, ..., 1$, we let
$$f_{k,m}^{n} = \begin{cases} F[\tau_{m+1}, \tau_{m}, f_{k,m+1}^{n+1}(\tau_{m+1}), \lambda, f_{1,m}^{1}] & \text{when } m = N, N - 1, ..., n, \\ 0 & \text{when } m = n - 1, n - 2, ..., 0, \end{cases} \quad (44)$$
where again $F[\cdot]$ is defined through (27) in Appendix A.

By iteratively checking on initial conditions, we can match the problem (12) about the environment $f = f_k(t) \in [0, 1], n = 1, 2, ..., N, t \in [0, T]$ of a particular seller type with the problem described by (26) in Appendix A, and then verify the following.

**PROPOSITION 2:** For $k \in [0, 1]$ and $n = 1, 2, ..., N$,
$$f_{k,m}^{n}(t) = f^k_m(t), \text{ when } t \in [\tau_{m+1}, \tau_m] \text{ for some } m = 0, 1, ..., N.$$ The above basically says that $f_{k,m}^{n}$ as defined on $[0, T]$ is patched up from $N + 1$ pieces: $f_{k,m}^{n}$ on $[0, \tau_m)$, $f_{k,m+1}$ on $[\tau_m, \tau_{m+1}]$, $\cdots$, $f_{k,N+1}$ on $[\tau_N, T]$. It fulfills the portion of the $Z_{\text{DA}}^k$ operator for a given seller type.

**APPENDIX D: DERIVATIONS THAT LEAD TO THEOREM 1**

The Tychonoff fixed point theorem says that, if $S$ is a nonempty compact, convex subset of a locally convex linear topological space and $f : S \to S$ is continuous, then $f$ has a fixed point: there exists an $x^* \in S$ such that $x^* = f(x^*)$. To use this theorem, we need to specify our topologies. To space $\Delta^k$ for instantaneous market conditions, we prescribe metric $|| \cdot ||$, so that for $m, m' \in \Delta^k$, their distance is understood to
$$||m - m'|| = \max_{\theta \in \Theta} (|m(\infty) - m'(\infty)| \vee |m(0) - m'(0)|). \quad (45)$$
For any strictly positive integer $j$, we introduce norm $|| \cdot ||^j$ for the linear space $\mathbb{R}^j$, so that for any $x = (x_j'j = 1, 2, ..., j) \in \mathbb{R}^j$,
$$||x||^j = \max_{j' = 1, 2, ..., j} |x_j'|. \quad (46)$$
Of course, $|| \cdot ||^1$ is simply $\cdot$.

We introduce the $L^\infty$-norm $|| \cdot ||_{\infty}$ to $\mathbb{R}^j$-valued functions $g$ defined on any interval $[t, s]$, so that
$$||g||_{\infty} = \sup_{u \in [t, s]} ||g(u)||^j. \quad (47)$$

For convenience, we denote $|| \cdot ||_{\infty}^j$ by $|| \cdot ||_{\infty}$. We can express a few useful inequalities delineating the continuity of the solution $F[\cdot]$ of (26) with respect to various parameters. The continuity of $a$ and $b$ guarantees that $|a|_{\infty} < +\infty$ and $|b|_{\infty} < +\infty$. Now suppose $a$ is positive valued. Using the expression (27) of $F[\cdot]$, we can check that
$$|F[t, s, f_0, a, b](u) - F[t', s, f_0, a, b](u)| \leq |a|_{\infty} \cdot |(a + |b|_{\infty})| \cdot |t - t'|, \quad (48)$$
$$|F[t, s, f_0, a, b](u) - F[t, s, f_0', a, b](u)| \leq |f_0 - f_0'|, \quad (49)$$
and
$$|F[t, s, f_0, a, b](u) - F[t, s, f_0, a, b']|(u) | \leq |(a)(s - t) \cdot |b - b'|_{\infty}|. \quad (50)$$

Furthermore, we have
$$|F[t, s, f_0, a, b](u) - F[t, s, f_0, a, b'](u)| \leq |(a)(s - t) \cdot |b - b'|_{\infty}| \cdot |u - u'|. \quad (51)$$

Inequality (51) says that $F[t, s, f_0, a, b]$ is a Lipschitz continuous function on $[t, s]$. We have this inequality because, for $u \in [t, s]$ and $\Delta u \in [0, s - u]$,
$$f_0 \cdot \exp\left( -\int_t^u a(v)dv \right) - f_0 \cdot \exp\left( -\int_t^{u+\Delta u} a(v)dv \right) \leq |f_0| \cdot |\Delta u|_{\infty} \cdot |\Delta u|, \quad (52)$$
while
\[
\begin{align*}
&\left| \int_{f}^{a}(v)b(v)\cdot\exp\left(-\int_{a}^{u}a(u)du\right) \cdot dv \right|
- \int_{f}^{a+\Delta u}a(v)b(v)\cdot\exp\left(-\int_{a}^{u}a(u)du\right) \cdot dv \\
\leq &\left| \int_{f}^{a}(v)b(v)\cdot\exp\left(-\int_{a}^{u}a(u)du\right) \cdot dv \right|
\left(1-\exp\left(-\int_{a}^{a+\Delta u}a(u)du\right)\right) \cdot dv \\
&+ \int_{a}^{a+\Delta u}a(v)b(v)\cdot\exp\left(-\int_{a}^{u}a(u)du\right) \cdot dv \\
\leq &\left(\|a\|_{\infty}\right)^{2} \cdot \left(\|b\|_{\infty} \cdot \|s-t\| \cdot \Delta u + \|a\|_{\infty} \cdot \|b\|_{\infty} \cdot \Delta u. \right)
\end{align*}
\]

(53)

There is a strictly positive constant $x'$, so that for any $x \in [0,x')$,
\[
\exp(x) \leq 1 + 2x .
\]

(54)

Now, we have
\[
\begin{align*}
|F_{\lambda}(s) - F_{\lambda}(s_{0})| &\leq |a - a'| \cdot |s - t| \cdot \left(2\|f\|_{\infty} + 2\|a'\|_{\infty} \cdot |s - t| \cdot \|a - a'\|_{\infty} \right)
\end{align*}
\]

(55)

when $|a - a'| \leq x' / |s - t|$. This is because

Recall that $\xi$ first appeared in (15) for the definition of the $Z_{\lambda}$ operator. For $j = 1, 2, \ldots$, let $X_{j}$ be the space of $\mathbb{R}^{j}$-valued functions defined on $[-\xi, T]$, $Y_{j}$ the space of continuous $\mathbb{R}^{j}$-valued functions on $[0, T]$, and let $\mathcal{L}$ be the space of continuous bounded functions on $[0, T]$. For $f \in X_{j}$, we can identify any $f \in Y_{j}$, as a member of $X_{j}$ by agreeing on the convention that $f(t) = f(0)$ for any $t \in [-\xi, 0)$. It is possible to show that each member of $X_{j}$ has a finite $L^{\infty}$-norm. For $j = 1, 2, \ldots$, let $X_{j}$ be the space of $\mathbb{R}^{j}$-valued functions defined on $[-\xi, T]$ that are right continuous with left limits, and let $Y_{j}$ be the space of continuous $\mathbb{R}^{j}$-valued functions on $[0, T]$. We can identify any $f \in Y_{j}$ as a member of $X_{j}$ by agreeing on the convention that $f(t) = f(0)$ for any $t \in [-\xi, 0)$. It is possible to show that each member of $X_{j}$ has a finite $L^{\infty}$-norm.

**LEMMA 1:** For any $f \in X_{j}$, it is true that $\|f\|_{\infty} < +\infty$.

We also have the occasion to use the Skorohod topology as presented next. Let $K$ stand for the class of strictly increasing, continuous mappings of $[-\xi, T]$ onto itself, and let $I$ be the identity map on $[-\xi, T]$. For $f, g \in X_{j}$, we define
\[
\rho^{j}(f, g) = \inf_{\theta \in K} \{ \|f - g\|_{\infty} \}
\]

(57)

Here, $g^{t}$ stands for the function with value $g(t)$ at any $t \in [-\xi, T]$. We can think of $I$ as a time-rescaling function to be used to closely match the values $f(t)$ and $g(t)$. The metric $\rho^{j}$ for $X_{j}$ in turn induces the Skorohod topology.

To explicate the topology adopted for a given space when reporting continuity results, we use the notation $(A, \Delta)$ to indicate that space $A$ is endowed with topology $\Delta$. For instance, we shall use $(X^{j}, \rho^{j})$ for space $X^{j}$ endowed with the Skorohod topology $\rho^{j}$, and use $(Y^{j}, \|\cdot\|_{\infty})$ for space $Y^{j}$ endowed with the topology that results from $\|\cdot\|_{\infty}$ norm.

Recall that $\Theta$ is the set of seller types. While $\mathbb{P}^{\Theta}$ is a natural topology for instantaneous arrival-rate streams ... it does not make $(X^{j}, \rho^{j})$, a linear topological space—open sets induced by $\rho^{j}$ are not preserved under linear translation. Also, it is difficult to construct a nonempty, convex, and compact subset in $(X^{j}, \rho^{j})$ that contains all potential instantaneous arrival-rate streams. Thus, we are compelled to introduce actual arrival-rate streams $\lambda = (\lambda^{j}(t)\theta| \theta \in \Theta, k \in [0,1], t \in [0,T])$, which are continuous and make up a linear topological space under the $\|\cdot\|_{\infty}$-induced topology in which the simultaneous convexity and compactness of subsets are relatively easy to come by. Therefore, (15) and $Z_{\lambda}$ are also necessitated from technical considerations.

Our threshold polices all come from the multiple of closed simplices $\Delta^{q} = \{ (\tau_{1}(\theta)\theta| \theta \in \Theta, n = 1, \ldots, N) | 0 \leq \tau_{1}(\theta) \leq \cdots \leq \tau_{q}(\theta) \leq T \}$. For any $\tau_{q} \in \Theta^{q}$ and $\mathbb{P}^{\Theta}$, the simple topological space is generated by $\|\cdot\|_{\infty}$.

Let $X_{\lambda}^{2N}[\theta]$ be the subset of $X_{\lambda}^{2N}$, whose every member $f = (f(t)\theta| t \in [-\xi, T]) = (f(t)\theta| t \in \Theta, t \in [0,1], n = 1, 2, 3, \ldots, N)$. Therefore, (15) satisfies the following: (E-I) each $f(t)$ is in $\Delta^{q}$, (E-II) $m(t) = m(0)$ for $t \in [-\xi, 0)$, (E-III) $m(t)$ has at most $N\cdot q$ discontinuities in $t$, and (E-IV) $f$ Lipschitz continuous in $t$ with coefficient $\mathcal{L} \cdot ((\Delta T) + 2)$ in each of its continuous segments, where $\mathcal{L}$ is defined in (8). By Proposition 2, (S2), and the bound (51), we can treat environments as coming from $X_{\lambda}^{2N}$.

Let $X_{\lambda}^{2N}$ be the subset of $X_{\lambda}^{2N}$, whose every member $m = (m(t)\theta| t \in [-\xi, T]) = (m(t)\theta| t \in \Theta, t \in [0,1], n = 1, 2, 3, \ldots, N)$. Therefore, (15) satisfies the following: (M-I) each $m(t)$ is in $\Delta^{q}$, (M-II) $m(t) = m(0)$ for $t \in [-\xi, 0)$, (M-III) $m(t)$ has at most $N\cdot q$ discontinuities in $t$, and (M-IV) $f$ is Lipschitz continuous in $t$ with coefficient $\mathcal{L} \cdot ((\Delta T) + 2)$ in each of its continuous segments. In view of the nature of $X_{\lambda}^{2N}$ and (13), we can treat all market processes as coming from $X_{\lambda}^{2N}$.

**THEOREM 1:** For any $f \in X_{2N}$, whose every member $\lambda = (\lambda(t)\theta| t \in [0, T]) = (\lambda(t)\theta| t \in \Theta, t \in [0, T], \lambda(t)\theta| t \in [-\xi, 0)$, the following holds: (A-I) each $\lambda(t)$ is in $(0, X_{\lambda}^{2N})$, (A-II) $\lambda(t) = \lambda(0)$ for $t \in [-\xi, 0)$, (A-III) $\lambda(t)$ has at most $N\cdot q$ discontinuities in $t$. In view of the nature of $X_{\lambda}^{2N}$, (14), and (S2), we can treat all instantaneous arrival-rate streams as coming from $X_{\lambda}^{2N}$.

Let $Y_{\lambda}^{2N}$ be the subset of $Y_{\lambda}^{2N}$, whose every member $\lambda = (\lambda(t)\theta| t \in [0, T]) = (\lambda(t)\theta| t \in \Theta, t \in [0, T], \lambda(t)\theta| t \in [-\xi, 0)$. Therefore, (15) satisfies the following: (A-I) each $\lambda(t)$ is in $(0, X_{\lambda}^{2N})$, (A-II) $\lambda(t)$ is Lipschitz continuous with coefficient $\mathcal{L} \cdot \Delta T$ and (A-III) each $\lambda(t)\theta$ is greater than $\mathcal{L} \cdot \theta$, where $\mathcal{L}$ is defined in (S3). We can treat all actual arrival-rate streams as coming from $Y_{\lambda}^{2N}$, as by (15), we have, for any $t \in [0, T]$ and $\Delta t \in (0, (T-t) \wedge \xi)$,
\[
\begin{align*}
&\int_{t-\xi}^{t+\Delta t} \lambda^{0}(s) \cdot ds \leq \int_{t-\xi}^{t+\Delta t} \lambda^{0}(s) \cdot ds + \int_{t+\Delta t}^{t+\Delta t} \lambda^{0}(s) \cdot ds \\
&\leq 2\|\lambda^{0}\|_{\infty} \cdot \Delta t,
\end{align*}
\]

(58)

which converges to 0 at a rate faster than $\Delta T$, as $\Delta T$ approaches 0.

Recall we have used $\tau = Z_{\lambda}^{j}(\lambda)$ to represent the mapping from an arrival-rate stream $\lambda \in Y_{\lambda}^{2N}$ to a threshold policy $\tau \in \Delta^{q}$ as prescribed by Proposition 1. The transition $Z_{\lambda}^{j}$ can be shown to be continuous.

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LEMMA 2: The map $Z^D_\lambda$ is continuous from $(Y^{(2)\theta}_A, ||\cdot||_{\infty})$ to $(\Delta^D, ||\cdot||_{\infty})$.

Under the given initial stock distribution $f(0) = (f_0(0)|\theta \in \Theta, n = 1, 2, \ldots, N)$, we recall we have used $f = Z_{DA}(\tau, \lambda)$ to represent the solution $f \in X^{(2)\theta}_E$ of (12) when given threshold vector $\tau \in \Delta^D$ and actual arrival-rate stream $\lambda \in Y^{(2)\theta}_A$. The arrangement here again is $f(t) = f(0)$ for $t \in [-\xi, 0]$. We can verify the continuity of $Z^D_{DA}$ in $\lambda$ and $\tau$ individually.

LEMMA 3: At a fixed actual arrival-rate stream $\lambda \in Y^{(2)\theta}_A$, the map $Z^D_{DA}(\cdot, \lambda)$ from $(\Delta^D, ||\cdot||_{\infty})$ to $(X^{(2)\theta}_E, ||\cdot||_{\infty})$ is Lipschitz continuous with a coefficient independent of $\lambda$.

LEMMA 4: At a fixed threshold vector $\tau \in \Delta^D$, the map $Z^D_{DA}(\tau, \cdot)$ from $(Y^{(2)\theta}_A, ||\cdot||_{\infty})$ to $(X^{(2)\theta}_E, ||\cdot||_{\infty})$ is Lipschitz continuous in a small-enough range with a coefficient independent of $\tau$.

Recall we have used $\hat{\lambda} = Z^D_{\lambda}(f)$ to denote the process of turning an environment $f \in X^{(2)\theta}_E$ into a market-condition process $m \in X^{(2)\theta}_M$ through (13), and then turning the latter into an instantaneous arrival-rate stream $\tilde{\lambda} \in X^{(2)\theta}_E$ through (14). This map possesses a nice continuity property.

LEMMA 5: The map $Z^D_{\hat{\lambda}}$ from $(X^{(2)\theta}_E, ||\cdot||_{\infty})$ to $(X^{(2)\theta}_E, ||\cdot||_{\infty})$ is uniformly continuous.

Recall $\lambda = Z^D_{\hat{\lambda}}(\hat{\lambda})$ has been used to denote the transfer from an instantaneous arrival-rate stream $\tilde{\lambda} \in X^{(2)\theta}_E$ into an actual stream $\lambda \in Y^{(2)\theta}_A$ through (15). We have the following important result.

LEMMA 6: The map $Z^D_{\lambda}$ from $(X^{(2)\theta}_E, ||\cdot||_{\infty})$ to $(Y^{(2)\theta}_A, ||\cdot||_{\infty})$ is continuous.

Consider the composite map $Z^D_{\lambda}$ defined by (16) from $Y^{(2)\theta}_A$ to itself. By combining Lemmas 2 to 6, we can verify its continuity.

PROPOSITION 3: The map $Z^D_{\lambda}$ from $(Y^{(2)\theta}_A, ||\cdot||_{\infty})$ to $(Y^{(2)\theta}_A, ||\cdot||_{\infty})$ is continuous.

Moreover, we can show that $Y^{(2)\theta}_A$ is a special subset of $Y^{(2)\theta}_A$.

PROPOSITION 4: The set $Y^{(2)\theta}_A$ is nonempty, convex, and compact under the metric generated by $||\cdot||_{\infty}$.

As $(Y^{(2)\theta}_A, ||\cdot||_{\infty})$ is a normed linear space, we certainly have the following.

PROOF OF LEMMA 1: Suppose, on the contrary, that $||f||_{\infty} = +\infty$ for some $f \in X^\lambda$. Then, there must exist a sequence $\{t_i\}_{i=1}^\infty$ in $[\xi, T]$ such that $||f(t_i)||_{\infty} \geq i$ for each $i$. As $[\xi, T]$ is a compact subset of the real line $\mathbb{R}$, there exists a subsequence of $\{t_i\}_{i=1}^\infty$ that converges to some point $t \in [-\xi, \xi]$. Then, either $||f(t)||_{\infty}$ or $||f'(t)||_{\infty}$ would be $+\infty$. But this contradicts the fact that $f \in X^\lambda$.

PROOF OF LEMMA 2: As each type-$\theta$ seller responds to the type-specific actual arrival-rate stream he faces, the map $Z^D_\lambda$ can be decomposed into type-specific components. Since $|\Theta|$ is finite, we can concentrate on the continuity of $Z^D_\lambda$’s components $Z^D_{\lambda}(\theta)$ at particular types $\theta$. The following derivation is good for any $\theta \in \Theta$, which we omit mentioning.

To emphasize its $\lambda$-dependence, though, we use $a_\theta(\lambda,t)$ to denote the left-hand side of (9). For $\lambda_1, \lambda_2 \in Y^{(2)\theta}_A$ and $t \in [0, T]$, we apparently have

$$\left| a_\theta(\lambda_1,t) - a_\theta(\lambda_2,t) \right| \leq T \cdot ||\lambda_2 - \lambda_1||_{\infty}. \quad (59)$$

It can be checked that

$$\left| a_\theta(\lambda_1,t) - a_\theta(\lambda_1,t) \right| = \left| \exp(-\lambda_1^0(t)) - \sum_{k=0}^{n-1} \frac{(\lambda_2^0(t))^k}{k!} \right|$$

$$\leq \left| \exp(-\lambda_2^0(t)) - \exp(-\lambda_1^0(t)) \right| \cdot \sum_{k=0}^{n-1} \frac{(\lambda_2^0(t))^k}{k!}$$

$$+ \exp(-\lambda_1^0(t)) \cdot \left| \sum_{k=0}^{n-1} \frac{(\lambda_2^0(t))^k}{k!} \right| \leq 2 \cdot \left| \exp(-\lambda_1^0(t)) - (1 - e^{-\lambda_1^0(t)}) \right|. \quad (60)$$

which is below

$$\exp(-\lambda_1^0(t)) - (1 - e^{-\lambda_1^0(t)}) \leq 1 - e^{-\lambda_1^0(t)} - e^{-\lambda_2^0(t)}$$

$$+ \exp(-\lambda_1^0(t)) \cdot \left| \exp(-\lambda_1^0(t)) - \exp(-\lambda_2^0(t)) \right|$$

$$\leq 2 \cdot \left| \exp(-\lambda_1^0(t)) - (1 - e^{-\lambda_1^0(t)}) \right|. \quad (61)$$

The above would be further smaller than $4 \cdot \left| \lambda_2^0(t) - \lambda_1^0(t) \right|$ when $\left| \lambda_2^0(t) - \lambda_1^0(t) \right| \leq 1$, whereas the latter is defined around (54).

Conversely, due to (A-1), we can learn from (10) that $d_\theta(\lambda_1,t) \leq \mathbb{T}$ when $t \geq (T - T^\prime) \vee 0$, where $\mathbb{T}$ is defined in (8). Define strictly positive constant $\tilde{t}$ so that

$$\tilde{t} = \frac{p^0_{\theta} \sigma^0 - p^0_{\theta} \sigma^1}{p^0_{\theta} X^\prime - p^0_{\theta} X^\prime(\mathbb{T} - \tilde{t})} \wedge T < \frac{1}{\tilde{t}}. \quad (62)$$

From the above upper bound on $d_\theta(\lambda_1,t)$, the fact that $a_\theta(\lambda_1,t) = 1$, and the definition in Proposition 1 of $\tau_\theta$, i.e., $(Z^D_\lambda(\lambda_1))$, we can see that

$$(Z^D_\lambda(\lambda_1)) \leq T - \tilde{t}. \quad (63)$$

for any $\lambda \in Y^{(2)\theta}_A$ and any $n = 1, 2, \ldots, N$. By (10) again and (A-III), we see that $d_\theta(\lambda_1,t) \geq \tilde{t}$ for $t \in [0, T - \tilde{t}]$, where

$$\tilde{t} = \frac{p^0_{\theta} \sigma^0 \cdot \exp(-\mathbb{T}) - (\mathbb{P}^0_{\theta} \tilde{t} \wedge 1)^{N-1}}{(N-1)!} > 0, \quad (64)$$

where $\tilde{t}$ is defined in (S3). So, for $\lambda \in Y^{(2)\theta}_A$ and $s, t \in [0, T - \tilde{t}]$,

$$|a_\theta(\lambda_1,t) - a_\theta(\lambda_1,s)| \geq \tilde{t} \cdot |t - s|. \quad (65)$$

In view of the definition in Proposition 1 of $(Z^D_\lambda(\lambda_1))$, which has just been shown to be in $[0, T - \tilde{t}]$, we can see from (59), (60), (61), and (64) that

$$\left| Z^D_\lambda(\lambda_2) - Z^D_\lambda(\lambda_1) \right| \leq \max_{\theta = 1, 2, \ldots, N} \left| (Z^D_\lambda(\lambda_2))_{\theta} - (Z^D_\lambda(\lambda_1))_{\theta} \right|$$

$$\leq 4T \cdot ||\lambda_2 - \lambda_1||_{\infty}. \quad (66)$$

whenever $||\lambda_2 - \lambda_1||_{\infty} \leq \mathbb{x}^\prime / T$.

PROOF OF LEMMA 3: As $\Theta$ is finite, we can prove for each individual $\theta$. For convenience, we avoid making the $\theta$-dependence explicit. Now, given $\tau = (\tau_n | n = 1, 2, \ldots, N)$ and $\tau^\prime = (\tau_n^\prime | n = 1, 2, \ldots, N)$, we let

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\[ \tau_{n+1} = \tau'_{n+1} = 0 \quad \text{and} \quad \tau_0 = \tau'_0 = T = T \text{ for convenience.} \]

We introduce increasing mapping \( l \), so that \( l(t) = t \) for \( t \in [-\varepsilon, 0) \), \( l(t_n) = t_n' \) for \( n = 0, 1, \ldots, N, N + 1 \), while for \( n = 0, 1, \ldots, N \) and \( t \in (t_{n+1}, t_n) \),

\[ l(t) = t_n' + (t_n' - t_n') \cdot (t - t_n')/(t_n - t_{n+1}). \]  

(67)

Obviously, \( l \) is a time-rescaling function, and its distance from the identity map \( I \) is given by \( \|l - I\|_{\infty} = \max_{n=1,2,\ldots,N} \| t_n' - t_n' \| = \|t - t\| \). Let environments \( f = (f^n_1(t) \in [0,1], n = 1, 2, \ldots, N, t \in [0,T]) \) and \( f' = (f^n_1'(t) \in [0,1], n = 1, 2, \ldots, N, t \in [0,T]) \) be solutions to (12) when threshold vectors \( \tau \) and \( \tau' \) are adopted, respectively. That is, \( f = Z_{DA}^E(\tau, \lambda) \) and \( f' = Z_{DA}^E(\tau', \lambda) \). Our next goal is to show that, for some positive constant \( C \),

\[ |f^n_k(l(t)) - f^n_k(t)| \leq C \cdot \|t - t\|^N \quad \forall k \in [0,1], n = 1, 2, \ldots, N, t \in [0,T]. \]  

(68)

Once (68) has been proven, then, according to the definition of the metric \( \rho \), we know that

\[ \rho_\infty^N(Z_{DA}^E(\tau, \lambda), Z_{DA}^E(\tau', \lambda)) \leq (C + 1) \cdot \|t - t\|^N. \]  

(69)

That is, \( Z_{DA}^E(\cdot, \cdot) \) is Lipschitz continuous with a coefficient independent of \( \lambda \).

Now we turn to (68). We shall use relations (43) and (44) extensively, and obtain bounds using (48) to (51). In our development, we will repeatedly use the relation that

\[ |F[t, s, f_0, \lambda_k, f(t)] - g(t)| \leq |F[t, s, f_0, \lambda_k, f(t)] - F[t, s, f_0, \lambda_k, f(l(t))]| + |F[t, s, f_0, \lambda_k, f(l(t))]| - g(t)| \leq |F[t, s, f_0, \lambda_k, f(l(t))]| - g(t)| + \|\nabla F(l(t))\|_{\infty} \cdot \|t - t\| \cdot \|t - t\| \]  

(70)

where \( t, s \in [0,T] \), \( f_0 \in [0,1] \), and \( f \) is a function defined on \([t, s]\) with \( \lambda \) assuming the role of \( \tau \). Now, for \( n = N + 1, N, \ldots, 1 \) and \( m = N + 1 \), we have

\[ |f^n_{m+1}(t_n') - f^n_{m+1}(t_n)| = |0 - 0| = 0. \]  

(72)

For \( n = N + 1, N, \ldots, 1, m = n - 1, n - 2, \ldots, 1 \), and \( t \in [t_{m+1}, t_m] \), we have

\[ |f^n_{m+1}(l(t)) - f^n_{m+1}(t)| = |0 - 0| = 0. \]  

(73)

Conversely, for \( n = N, N - 1, \ldots, 1, m = N, N - 1, \ldots, n \), we have

\[ |f^n_{m+1}(l(t)) - f^n_{m+1}(t)| \leq T_1 + T_2 + T_3 + T_4 \leq |f^n_{m+1}(l(t)) - f^n_{m+1}(t)| + \|T\| \cdot \|f^n_{m+1}(l(t)) - f^n_{m+1}(t)| \]  

(74)

where

\[ T_1 = |l(t_{m+1}) - l(t_{m+1})| \]  

(75)

\[ T_2 = |F[t_{m+1}, t_{m+1}, f^n_{m+1}(l(t_{m+1})), \lambda_1, f^n_{m+1}(l(t_{m+1}))]| \]  

(76)

\[ T_3 = |F[t_{m+1}, t_{m+1}, f^n_{m+1}(l(t_{m+1})), \lambda_1, f^n_{m+1}(l(t_{m+1}))]| \]  

(77)

\[ T_4 = |F[t_{m+1}, t_{m+1}, f^n_{m+1}(l(t_{m+1})), \lambda_1, f^n_{m+1}(l(t_{m+1}))]| \]  

(78)

If for positive constants \( b_{m+1} \) and \( b_{m+1} \), we know that

\[ |f^n_{m+1}(t_{m+1}) - f^n_{m+1}(t_{m+1})| \leq b_{m+1} \cdot \|t - t\|^N \quad \text{and} \quad |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| \leq b_{m+1} \cdot \|t - t\|^N \]  

for every \( t \in [t_{m+1}, t_m] \), then we clearly have

\[ |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| \leq |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| + \|T\| \cdot \|f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| \]  

(79)

Now, for \( n, m = N + 1, N, \ldots, 0 \), we define \( b_{m+1} \) as follows. For \( n = N + 1, N, \ldots, 1 \) and \( m = N + 1 \) as well as \( n = 1, n - 2, \ldots, 1 \), we let \( b_{m+1} = 0 \); for \( n = N, N - 1, \ldots, m = N, N - 1, \ldots, 1 \), we define \( b_{m+1} \)’s iteratively through the relationship

\[ b_{m+1} = b_{m+1} + \|T\| \cdot b_{m+1} + \|T\|^2 + 2\|T\|^2 \]  

(80)

It may be difficult to obtain closed-form expressions for the \( b_{m+1} \). But it suffices to know that all of them are well defined and positive. More specifically, we know that each \( b_{m+1} \) is \( \lambda \) being multiplied by a positive \((N+1-n)\)-th order polynomial of \( T \). Combining the above, we obtain

\[ |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| \leq b_{m+1} \cdot \|t - t\|^N \]  

(82)

\[ \forall m = N, N, \ldots, 1, m = N, N, \ldots, n, n = N, N, \ldots, 1, \text{ and } t \in [t_{m+1}, t_m]. \]  

Again, for \( n = N + 1 \) and \( m = N, N, \ldots, 0, \) or \( m = N, N, \ldots, 1 \), we have

\[ |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| = |0 - 0| = 0. \]  

(82)

Conversely, for \( n = N, N, \ldots, 1, m = N, N, \ldots, n, n = 2, \ldots, 0, \) and \( t \in [t_{m+1}, t_m] \), we have, similar to (74),

\[ |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| \leq |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| + \|T\| \cdot \|f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| \]  

(83)

while for \( n = N, N, \ldots, 1, \) and \( t \in [t_{m+1}, t_m] \), we have

\[ |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| \]

\[ = |F[t_{m+1}, t_{m+1}, f^n_{m+1}(l(t)), \lambda, f^n_{m+1}(l(t))]| \]

\[ - F[t_{m+1}, t_{m+1}, f^n_{m+1}(l(t)), \lambda, f^n_{m+1}(l(t))]| \]

\[ \leq T_5 + T_6 + T_7 + T_8 \]

\[ = |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| + |f^n_{m+1}(l(t)) - f^n_{m+1}(l(t))| \]

(84)

\[ \text{Navy Research Logistics DOI 10.1002/nav} \]
\[ T_S = |F(t'_n, t_n, \ldots, t_1, f^0_{n}(t'_n) + f^{1}_{n}(t'_n), \lambda, f_{n-1,1,\ldots,1}(t'_n)); f^0_{n}(t'_n) + f^{1}_{n}(t'_n), \lambda, f_{n-1,1,\ldots,1}(t'_n)); f^0_{n}(t'_n) + f^{1}_{n}(t'_n), \lambda, f_{n-1,1,\ldots,1}(t'_n))| \]
\[ T_b = |F(t'_n, t_n, \ldots, t_1, f^0_{n}(t'_n) + f^{1}_{n}(t'_n), \lambda, f_{n-1,1,\ldots,1}(t'_n)); f^0_{n}(t'_n) + f^{1}_{n}(t'_n), \lambda, f_{n-1,1,\ldots,1}(t'_n)); f^0_{n}(t'_n) + f^{1}_{n}(t'_n), \lambda, f_{n-1,1,\ldots,1}(t'_n))| \]

Now, for \( n = N + 1, N, \ldots, 1 \) and \( m = N, N - 1, \ldots, 0 \), we define \( b^0_{n,m} \) as follows. For \( n = N + 1 \) and \( m = N - 1, \ldots, 0 \), or \( m = N \) and \( n = N, N - 1, \ldots, 1 \), we let \( b^0_{n,m} = 0 \); for \( n = N, N - 1, \ldots, 1 \) and \( m = N, N - 1, \ldots, 0 \), we define \( b^0_{n,m} \) iteratively through the relationship

\[ b^0_{n,m} = \sum_{k=1}^{N} b^0_{n,k} T \]

Again, each \( b^0_{n,m} \) is \( \lambda \) being multiplied by a positive \((N + 1)\)-th order polynomial of \( \lambda T \), and we have

\[ \| f^0_{n,m}(t') - f^0_{n,m}(t) \| < b^0_{n,m} \| \tau - \tau' \| \]

\[ \forall n = N, N - 1, \ldots, 0, m = N - 1, N - 2, \ldots, 0, t \in [t_m, t_n]. \]

Define \( C \) so that

\[ C = \max_{n=0}^{N} \max_{m=0}^{N-n} b^0_{n,m} \]

\[ \| f^0_{n,m} \| \leq b^0_{n,m} \]

\[ \forall n = 0, 1, \ldots, N, m = 0, 1, \ldots, n \]

We can use this \( C \) as the constant for (88).

PROOF OF LEMMA 4: As \( \Theta \) is finite, we can prove for each individual \( \tau \). For convenience, we avoid making the \( \Theta \)-dependence explicit. Now, consider \( \lambda, \lambda' \in \mathbb{Z}^2 \) satisfying \( ||\lambda' - \lambda||_2^2 < \epsilon \) for some \( \epsilon > 0 \) around \((n, \tau)\). By \( f^0_{n,m} \), we mean \( f^0_{n,m} \) with \( \lambda' \) assuming the role of \( \lambda \) in \( f^0_{n,m} \) around \((n, \tau)\).

\[ \| f^0_{n,m}(t') - f^0_{n,m}(t) \| < C_{n,m} \| \tau - \tau' \| \]

\[ \forall t, t' \in [-\tau, \tau], \theta \in \mathbb{Z}, \lambda \in \mathbb{Z}^2, \]

\[ t - t' < \delta(\epsilon, \lambda). \]

Given \( \epsilon > 0 \), we define \( \delta \) so that

\[ \delta = \epsilon \cdot \frac{\delta(\epsilon/3, \lambda)}{2} \]

\[ \| f^0_{n,m}(t') - f^0_{n,m}(t) \| < C_{n,m} \| \tau - \tau' \| \]

\[ \forall t, t' \in [-\tau, \tau], \theta \in \mathbb{Z}, \lambda \in \mathbb{Z}^2, \]

\[ t - t' < \delta(\epsilon, \lambda). \]

For any instantaneously arriving rate stream \( \lambda \), satisfying \( \| \lambda' - \lambda \|_2^2 < \delta \), we can identify time-rescaling function \( t \in [\lambda(t), \lambda(t')] \)

\[ \| f^0_{n,m}(t') - f^0_{n,m}(t) \| < C_{n,m} \| \tau - \tau' \| \]

\[ \forall t, t' \in [-\tau, \tau], \theta \in \mathbb{Z}, \lambda \in \mathbb{Z}^2, \]

\[ t - t' < \delta(\epsilon, \lambda). \]
By the definition (15), we have, for any particular \( t \in [0, T] \),
\[
|| (Z^A_\lambda(t) - Z^\lambda(t)) ||^2 = \int_{t - \varepsilon}^t || \lambda(s) - \tilde{\lambda}(s) ||^2 \, \text{d}s/\xi
\]
\[
\leq \left( \int_{t - \varepsilon}^t || \lambda(s) - \tilde{\lambda}(s) ||^2 \, \text{d}s + \int_{t - \varepsilon}^t || \tilde{\lambda}(s) - \tilde{\lambda}(s) ||^2 \right) / \xi
\]
\[
= T_1 + T_2 + T_3,
\]
where
\[
T_1 = \int_{t - \varepsilon}^t || \lambda(s) - \tilde{\lambda}(s) ||^2 \, \text{d}s/\xi,
\]
\[
T_2 = \int_{t - \varepsilon}^t || \tilde{\lambda}(s) - \tilde{\lambda}(s) ||^2 / \xi,
\]
\[
T_3 = \int_{t - \varepsilon}^t || \tilde{\lambda}(s) - \tilde{\lambda}(s) ||^2 / \xi,
\]
while \( W \) is a subset of \([t - \varepsilon, t]\) defined as follows:
\[
W = \{ s \in [t - \varepsilon, t] | s \text{ is more than 23 away from any discontinuity point of } \tilde{\lambda} \}.
\]

By (101), we know that \( T_1 < 2 \delta \), which is smaller than \( \varepsilon/3 \) by (99).

By the definition (104) of \( W \) and (100), we know that any \( s \in W \) satisfies that \( s \) and \( \tilde{\lambda}(s) \) belong to the same continuity segment of \( \tilde{\lambda} \). Then, according to (100), the definition of \( \delta \) in (99), and the definition of \( \delta(\cdot, \cdot) \) revolving around (98), we know that \( ||\tilde{\lambda}(s) - \tilde{\lambda}(s) ||^2 < \varepsilon/3 \) for any \( s \in W \). Since \( W \) is the union of a finite number of intervals, it is Lebesgue measurable with measure \( |W| \leq \varepsilon \). Thus, we have \( T_2 < \varepsilon/3 \).

By \((A' - III)\) again and (104), we know that the Lebesgue measure \( |(t - \xi, t)] \cap W| \leq 4N \delta \cdot |\xi|/\theta \), so that \( T_3 \leq 4N \delta \cdot |\xi| / \theta \), which is smaller than \( \varepsilon / 3 \) by (99) again.

Combining results from the above, we see that \( ||Z^A_\lambda(t) - Z^\lambda(t)||^2 < \varepsilon \) whenever \( \rho^2(\tilde{\lambda}, \tilde{\lambda}) \) is strictly below the \( \delta \) defined in (99).

PROOF OF PROPOSITION 2: By Lemma 2, we know that, for any \( \varepsilon > 0 \) and \( \lambda \in Y^A_\lambda \), there is some \( \delta^A_\lambda(\varepsilon, \lambda) > 0 \), so that, for any \( \lambda' \in Y^A_\lambda \) satisfying \( ||\lambda - \lambda'||^2 < \delta^A_\lambda(\varepsilon, \lambda) \),
\[
||Z^A_\lambda(t) - Z^\lambda(t)||^2 < \varepsilon.
\]

From Lemmas 3 and 4, we know that, there are strictly positive constants \( c^A_\varepsilon, \tilde{c}^A_\varepsilon \), and \( d^A_\varepsilon \), such that, for any \( \tau, \tau' \in \Delta^A \) and \( \lambda, \lambda' \in Y^A_\lambda \) with \( ||\lambda - \lambda'||^2 \leq d^A_\varepsilon \),
\[
\rho^{2N}(Z^A_\varepsilon(\tau, \lambda), Z^A_\varepsilon(\tau', \lambda'))
\]
\[
\leq \rho^{2N}(Z^A_\varepsilon(\tau, \lambda), Z^A_\varepsilon(\tau', \lambda')) + \rho^{2N}(Z^A_\varepsilon(\tau', \lambda), Z^A_\varepsilon(\tau', \lambda'))
\]
\[
< c^A_\varepsilon ||\tau - \tau'||^2 + \tilde{c}^A_\varepsilon ||\lambda - \lambda'||^2 < \frac{1}{\delta^A_\varepsilon}.
\]

From Lemma 5, we know that, for any \( \varepsilon > 0 \), there is some \( \delta^A_\lambda(\varepsilon) > 0 \), so that, for any \( f, f' \in X^A_\lambda \) satisfying \( \rho^{2N}(f, f') < \delta^A_\lambda(\varepsilon) \),
\[
\rho^{2N}(Z^A_\varepsilon(f), Z^A_\varepsilon(f')) < \varepsilon.
\]

By Lemma 6, for any \( \varepsilon > 0 \) and \( \lambda \in X^A_\lambda \), there is some \( \delta^A_\lambda(\varepsilon, \lambda) > 0 \), so that, for any \( \lambda' \in X^A_\lambda \) satisfying \( \rho^{2N}(\lambda, \lambda') < \delta^A_\lambda(\varepsilon, \lambda) \),
\[
||Z^A_\lambda(\lambda) - Z^A_\lambda(\lambda')||^2 < \varepsilon.
\]

Suppose we are given \( \varepsilon > 0 \) and \( \lambda, \lambda' \in Y^A_\lambda \) satisfying
\[
||\lambda - \lambda'||^2 < \delta^A_\lambda(\varepsilon, \lambda) \wedge \frac{\delta^A_\lambda(\varepsilon, \lambda)}{2c^A_\varepsilon} \wedge d^E_\varepsilon,
\]
where we have let
\[
\lambda = Z^A_\varepsilon(Z^A_\varepsilon(\lambda, \lambda')).
\]

Then, from the definition of \( \delta^A_\lambda(\cdot, \cdot) \) revolving around (105), we have
\[
||Z^A_\varepsilon(\lambda) - Z^A_\varepsilon(\lambda')||^2 < \delta^A_\lambda(\varepsilon, \lambda)/2c^A_\varepsilon.
\]

Combining (109) and (111) with definitions of \( c^E_\varepsilon, c^A_\varepsilon \), and \( d^E_\varepsilon \) revolving around (106), we have
\[
\rho^{2N}(Z^A_\varepsilon(Z^A_\varepsilon(\lambda, \lambda')), Z^A_\varepsilon(Z^A_\varepsilon(\lambda', \lambda')) < \delta^A_\lambda(\varepsilon, \lambda).
\]

From the definition of \( \delta^A_\lambda(\cdot, \cdot) \) revolving around (108) and the definition of \( \lambda \) in (110), we have
\[
||Z^A_\varepsilon(Z^A_\varepsilon(\lambda, \lambda')) - Z^A_\varepsilon(Z^A_\varepsilon(\lambda', \lambda'))||^2 < \varepsilon.
\]

According to the definition (16), we have achieved the desired result. □

PROOF OF PROPOSITION 3: \( Y^{2N}_\lambda \) is clearly nonempty and convex.
Since all members \( \lambda \) of \( Y^{2N}_\lambda \) satisfy (A-I), we must have
\[
sup_{\lambda \in Y^{2N}_\lambda} ||\lambda||^2 \leq \tilde{\lambda} < +\infty.
\]

Since all members \( \lambda \) of \( Y^{2N}_\lambda \) satisfy (A-II), we must have
\[
sup_{\lambda \in Y^{2N}_\lambda} ||\lambda(t) - \lambda(s)||^2 \leq \frac{2 \varepsilon \delta}{\xi}.
\]

and hence
\[
\lim_{\delta \to 0} \sup_{\lambda \in Y^{2N}_\lambda} ||\lambda(t) - \lambda(s)||^2 = 0.
\]

By the Ascoli-Arzela Theorem (Yosida [30], p. 85), we know that \( Y^{2N}_\lambda \) is relatively compact.

Suppose a sequence \( \{\lambda'\}' \in Y^{2N}_\lambda \) converges to some \( \lambda \in Y^{2N}_\lambda \), it is easy to check that \( \lambda \) satisfies (A-I) to (A-III), and hence \( \lambda \in Y^{2N}_\lambda \) still.

Therefore, \( Y^{2N}_\lambda \) is closed and hence compact. □

APPENDIX E

PROOF OF THEOREM 1: Due to Propositions 3, 4, and 5, we can use the Tychonoff fixed point theorem to establish the existence of an actual arrival-rate stream \( \lambda^* \in Y^{2N}_\lambda \) satisfying \( \lambda^* = Z^A_\lambda(\lambda^*) \). Now define threshold policy

Naval Research Logistics DOI 10.1002/nav
\( \tau^* = Z_{DA}^A(\lambda^*), \) environment \( f^* = Z_{DA}^E(\tau^*, \lambda^*), \) and instantaneous arrival-rate stream \( \lambda^* = Z_{DA}^E(f^*). \) These and the definition of \( Z_{DA}^A \) through (16) would lead to \( \lambda^* = Z_{DA}^A(\lambda^*). \)

**APPENDIX F: A CONJECTURE ON THE SUITABILITY OF THE NG APPROACH**

Let an initial inventory-level distribution \( f(0) = (f_0(0)|\theta = 1, 2, ..., N) \) be given. When there are \( V \) sellers, let \( \Omega_1, ..., \Omega_V \) be \( V \) independent unit-rate Poisson processes on the time horizon \([0, T]\), where \( T \) is defined in (8). That is, each \( \Omega_v = (\Omega_v(t)|t \in [0, T]) \) is an increasing, integer-valued process that is right-continuous with left limits and satisfies \( \Omega_v(0) = 0 \) as well as \( \Omega_v(t) - \Omega_v(t^-) = 0 \) or 1; also, for any \( n_1, ..., n_V \in [0, T] \) and positive integers \( a_1, ..., a_V \),

\[
\Pr[\Omega_1(t_1) = a_1, ..., \Omega_V(t_V) = a_V] = e^{-(t_1+...+t_V)} \frac{(t_1)^{a_1} \cdots (t_V)^{a_V}}{a_1! \cdots a_V!}.
\]

Suppose seller \( v \) is of type \( C_v \) and starts time 0 with inventory level \( N_v(0) = 0, 1, ..., N \). We may let \( C_v = (C_v|v = 1, 2, ..., V) \) be randomly sampled from \((S(\theta)|\theta \in \Theta)\) and the initial inventory-level vector \( N^V(0) = (N_v(0)|v = 1, 2, ..., V) \) be randomly sampled from \( f(0) \). That is,

\[
\Pr[C_1 = \theta_1, ..., C_V = \theta_V] = S(\theta_1) \times \cdots \times S(\theta_V), \quad \forall \theta_1, ..., \theta_V \in \Theta,
\]

and for \( n_1, ..., n_V = 0, 1, ..., N \),

\[
\Pr[N(0) = n_1, ..., N(0) = n_V] = f_{n_1}(0|C_1) \times \cdots \times f_{n_V}(0|C_V).
\]

When adopting pricing policy \( \tau = (\tau(\theta)|\theta \in \Theta) = (\tau_k(\theta)|\theta \in \Theta, k \in [0, 1], t \in [0, T]), \) while facing Poisson process \( \Omega_v \) and starting with inventory level \( N_v(0) \), seller \( v \)'s inventory process \( N_v(0) = (N_v(t)|t \in [0, T]) \) and price-switching epoch \( t^\tau_v \in [0, T] \) satisfy the following:

\[
\begin{align*}
N_v(t) &= N_v(0) - \Omega_v(t) \int_0^t \lambda^A(s|C_v(s) - ds), \quad t \in [0, t^\tau_v], \\
N_v(t) &= [N_v(0) - \Omega_v(t) \int_0^t \lambda^A(s|C_v(s) - ds + \int_0^t \lambda^E(s|C_v(s) - ds)]^+, \\
&\quad \text{when } t \in (t^\tau_v, T], \\
t^\tau_v &= \sup\{t|t \in [0, T] \text{ and } t < \tau_v(\Omega_v(t))\}.
\end{align*}
\]

From the inventory-process vector \( N^V = (N_v|v = 1, 2, ..., V) \) and switching-epoch vector \( t^\tau^V = (t^\tau_v|v = 1, 2, ..., V) \), we may obtain the price-inventory empirical distribution process \( f = f(\tau(\theta)\theta \in \Theta) = f(\tau^V(\theta)|\theta \in \Theta, k \in [0, 1], t \in [0, T]) \) for \( \theta \in \Theta \) and \( \theta = 1, 2, ..., N \),

\[
\begin{align*}
f^\tau_k(t|\theta) &= \sum_{v=1}^V 1(C_v = \theta, N_v(t) = n) \text{ and } t \geq t^\tau_v, \\
f^\tau_k(t|\theta) &= \sum_{v=1}^V 1(C_v = \theta, N_v(t) = n) \text{ and } t < t^\tau_v.
\end{align*}
\]

where 0/0 may be treated as 1. Under any \( \lambda, \Omega^V = (\Omega_v|v = 1, 2, ..., V), C^V, \) and \( N^V(0) \), we may denote the \( f \) resulting from (121) and (122) by \( Z_{DA}^E(\tau, \lambda, v, C^V, N^V(0)) \). Note that the corresponding NG operator under the same initial distribution \( f(0) \) is \( Z_{DA}^E \).

Conversely, the actual arrival-rate stream \( \lambda^* \) is dependent on buyer-behavior operators \( Z_{DA}^E \) and \( Z_{DA}^A \). So the process from a given policy to its resulting environment must involve some fixed point problem. For the NG setting, define operator \( Z_{DA}^E \) by

\[
Z_{DA}^E(\tau, f) = Z_{DA}^E(\tau, Z_{DA}^A(\lambda^*(f))),
\]

while for the V-player setting, define operator \( Z_{DA}^E \) by

\[
Z_{DA}^E(\tau, f, \Omega^V, C^V, N^V(0)) = Z_{DA}^E(\tau, Z_{DA}^A(\lambda^*(f)), \Omega^V, C^V, N^V(0))
\]

Under a given policy \( \tau \), we may denote the fixed point of \( Z_{DA}^E(\tau, \cdot) \), if it is in existence and unique. Under a given policy \( \tau \), \( \Omega^V \)-realization \( C^V \)-realization \( v^V \), and \( N^V(0) \)-realization \( v^0(0) \), we may denote the fixed point of \( Z_{DA}^E(\tau, v^V, C^V, N^V(0)) \) by \( Z_{DA}^E(\tau, v^V, C^V, N^V(0)) \), if it is in existence and unique. The following is our conjecture.

**CONJECTURE 1:** Given a policy \( \tau \), the operator \( Z_{DA}^E(\tau, f, v^V, C^V, N^V(0)) \) is well defined, and the operator \( Z_{DA}^E(\tau, v^V, C^V, N^V(0)) \) is well defined for every \( v^V, C^V \), and \( N^V(0) \). Also,

\[
Z_{DA}^E(\tau, v^V, C^V, N^V(0)) \xrightarrow{v \rightarrow \infty} Z_{DA}^E(\tau, v^V, C^V, N^V(0))
\]

In the above, we have supposed that the convergence of two environments is measured in the \( \rho^{(N)}(\cdot, \cdot) \)-metric defined in Appendix D. Conjecture 1 says that the randomness in the environment experienced by sellers, caused by the randomness in the \( \Omega \), \( C \), \( v^V \), and \( N^V(0) \)'s, will eventually vanish as the number \( V \) of sellers tends to \( +\infty \).

When the given policy in the conjecture is an NG-equilibrium pricing policy \( \tau^* \), the resulting \( f^* = Z_{DA}^E(\tau^*) \) in the NG will be an equilibrium environment as defined in Theorem 1. To see this, note from the definition of \( Z_{DA}^E \) around (123), that

\[
f^* = Z_{DA}^E(\tau^*, v^V(Z_{DA}^E(\lambda^*(f^*)))) = Z_{DA}^E(\tau^*, Z_{DA}^A(v^V(\lambda^*))) = Z_{DA}^E(\tau^*, \lambda^*)
\]

Thus, Conjecture 1 implies the following. When an NG-equilibrium pricing policy \( \tau^* \) is adopted by all sellers in a V-member system, the resultant environment, though random, will converge as \( V \) tends to \( +\infty \) to the deterministic NG-equilibrium environment \( f^* \) corresponding to the policy \( \tau^* \). Due to the system’s various continuities, very likely sellers in the V-member system will become ever closer to viewing \( \tau^* \) as an equilibrium when \( V \) goes to \( +\infty \).

Three main steps probably need to be taken to prove the conjecture. First, we need to get a sense in which \( Z_{DA}^E(\tau, \lambda, v^V, C^V, N^V(0)) \) converges to \( Z_{DA}^E(\tau, \lambda) \). Then, we need to move on to a certain convergence of \( Z_{DA}^E(\tau, f, v^V, C^V, N^V(0)) \) to \( Z_{DA}^E(\tau, f) \). Finally, we need to use this convergence to obtain the concept described in the conjecture. The first two convergences must be in some strong sense, because the last convergence involves fixed points of those operators involved in the earlier convergences.

Due to the complex dynamics of the environment in both the NG setting as exemplified in (12) and the finite-seller setting as exemplified in (121) to (122), as well as difficulties associated with (13), (14), and (15), a proof of the conjecture remains elusive to us. However, our computational study says that the conjecture is very plausible.
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