A Nonatomic-Game Approach to Dynamic Pricing under Competition

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We study a revenue management problem involving competing firms. We assume the presence of a continuum of infinitesimal firms where no individual firm has any discernable influence over the evolution of the overall market condition. Under this nonatomic-game approach, the unanimous adoption of an equilibrium pricing policy by all firms will yield a market-condition process that in turn will elicit the said policy as one of the best individual responses. For both deterministic- and stochastic-demand cases, we show the existence of equilibrium pricing policies that exhibit well-behaving monotone trends. Our computational study reveals many useful insights, including the fact that only a reasonable number of firms are needed for our approach to produce near-rational pricing policies.

Key words: revenue management; nonatomic games; fixed point; competitive firms; dynamic pricing

1. Introduction

Revenue management (RM) attempts to maximize a firm’s profit through the use of pricing as well as other tools such as overbooking, service customization, and discount allocation. Examples of successful RM executions abound. According to a Harvard Business Review article (Friend and Walker 2001), retailers normally achieve gross-margin increases from 5% to 15% by optimizing pricing and assortment. Areas in which RM already boasts a proven track record include hotels (Choi and Mattila 2006), restaurants (Kimes 2005), cruise and ferry lines (Lieberman and Dieck 2002), rental cars and other heavy equipment (Savin et al. 2005), and so on.

In today’s marketplace, companies are often obliged to take the competition element into their RM considerations. The proliferation of Internet access among virtually all homes and businesses brings price competition to an even more intense, global level. Sites that pool dynamically changing prices from tens or even hundreds of providers and let customers choose the most suitable ones for themselves include Expedia, Hotwire, Orbitz, and Travelocity. The competition is intense in several industries and is most manifest in the hospitality industry and the secondary market for sport and entertainment event tickets. In the hospitality industry, a search on Hotel.com resulted in around 40 small brand-less hotels with a 2.5 star rating within 15 miles of downtown Atlanta. At the same time, FanSnap.com featured more than 50 vendors selling tickets for the NBA game between the Miami Heat and the Boston Celtics on October 26, 2010.

Yet, the understanding of RM in competitive settings remains less than desirable. It is RM in a monopolistic setting that has attracted the most attention and is currently the most mature. In part, this is because competitive RM is dynamic game in nature and hence is notoriously difficult. The current research aims to narrow the gap between the practical need and the methodological state-of-the-art by taking a less frontal assault. More particularly, we take the nonatomic-game (NG) approach, effectively assuming that there is a continuum of competing firms. Concentrating on a two-price dynamic pricing problem, we shall first conduct a theoretical study under the NG setting and then a computational test to determine the accuracy of the approximation for real scenarios with finite firms.

We consider both a deterministic and a stochastic model in this study. In both models, firms compete to earn higher revenues by selling the same product over a fixed time horizon. They all first anticipate and subsequently experience a common, external market pro-
cess. The instantaneous demand arrival to a firm is
governed by both the price it currently charges and the
present market condition. In turn, the instantaneous
market condition is reflected in the distribution of
prices currently charged by all firms. In the stochastic
case, the actual demand arrival to a firm is Poisson.

For both cases, a pricing policy prescribes to a firm
the prices it should charge at various inventory levels
and time points. Such a policy is said to have reached
equilibrium when its universal adoption by all firms
results in a market process for which the current policy
serves as a best individual response. Using order-
and continuity-based fixed point theorems, we
succeed in showing for both cases, that one of the
equilibrium pricing policies is well behaved. Compu-
tationally, NG-equilibrium pricing policies can be
shown to work well in real, finite-firm settings. For
instance, when as few as 30 competitors follow an NG
equilibrium policy, the revenues and market process
that they generate are already reasonably close to
those anticipated for the NG situation.

1.1. Literature Survey
The RM for a single seller has been studied in depth.
For instance, Gallego and van Ryzin (1994) examined
the problem of dynamically pricing inventories with
stochastic demands. They showed the optimality of a
monotone pricing policy as well as the asymptotic opti-
mality of two fixed-price heuristics. Dealing with a sim-
ilar model containing a finite number of price choices,
Feng and Xiao (2000a) derived the optimality of a com-
parable monotone pricing policy and provided a proce-
dure for computing this policy. When demand is time-
varying on top of being price-dependent, Zhao and
Zheng (2000) identified the decline over time of cus-
tomers’ willingness to pay premiums as a sufficient
condition for the monotonicity of the optimal pricing
policy. Gallego and van Ryzin (1997) gave asymptoti-
cally optimal heuristics that are based on the solution
to the problem’s deterministic-control counterpart.

Significant literature has been published on RM
settings where price changes are irreversible. Feng
and Gallego (1995) studied the optimal price-switch-
ing policy when there is one chance for a price change
in the entire sales horizon. For this problem, they
showed the optimality of a threshold-like policy. Feng
and Xiao (2000b) generalized the above result to a
case with multiple price switches in the same direc-
tion. Comprehensive reviews on single-seller RM
research have been compiled by McGill and van
Ryzin (1999), Bitran and Caldentey (2003), and

Besides the above monopolistic RM literature, a few
works have addressed finite-game RM models. Perakis
and Sood (2006) studied a dynamic pricing game in
which each firm reacts most conservatively to the price
schedules of its competitors. In their study, prices are
not reactive to real-time sales data. Xu and Hopp
(2006) studied a multi-seller dynamic pricing game,
where demand arrival is governed by a geometric
Brownian motion and only the lowest-priced seller
sells. Lin and Sibdari (2009) showed that subgame-per-
fect equilibria exist for a discrete-time dynamic pricing
game in which the opponents’ inventory levels are
observable. In another dynamic pricing game with
mutually observable inventory levels, Levin et al.
(2009) allowed consumers to be strategic as well. Nei-
ther work revealed structural properties of their equi-
libria. Relying on results from its deterministic version,
Gallego and Hu (2007) found approximate fixed-price
equilibria for a dynamic pricing game.

We simplify the analysis at hand by considering a
continuum of nonatomic firms, effectively removing
influences by individual firms on the overall market
condition. We caution that the market condition is still
affected by the firms’ pricing decisions in aggregation.
Under our NG setting, we are able not only to establish
the existence of equilibrium pricing policies but also to
reveal their structural properties. In particular, we can
show the inventory- and time-monotonicity of our
equilibrium pricing policies, much like we can for an
optimal policy in a monopolistic setting. Here, we
deem a policy both inventory- and time-monotone
when its prescribed price is decreasing in both the
inventory level of the current firm and the present time.

A considerable body of literature focuses on NG.
Aumann (1964) studied a cooperative game involving
a continuum of participants. Schmeidler (1973) treated
a nonatomic game in its general normal form, and
Mas-Colell (1984) took an alternative approach to the
same problem. Housman (1988) introduced a frame-
work that could deal with both nonatomic and atomic
players, where each of the latter type of players has a
non-negligible impact on other players. Recent theoreti-
cal works on games played by a continuum of players
include Khan et al. (1997), Balder (1999), and so on.

A very pertinent point is the suitability of an NG’s
equilibrium to finite-player situations. There have
been a few positive results. Carmona (2004) illustrated
that an NG’s strategy profile is a pure equilibrium if
and only if its restricted versions on a certain sequence
of finite games provide approximate pure equilibria
for the games. Al-Najjar (2008) studied discrete large
games on the basis of finitely additive probabilities.
Among other things, he established that restrictions
of an equilibrium of the discrete large game offer
approximate equilibria for a predetermined sequence
of finite games. Yang (2011) established that an NG’s
mixed strategy is its equilibrium if and only if it helps
to achieve asymptotic equilibria for increasingly large
finite games whose types are randomly sampled from
the NG’s defining distribution.
However, all these studies were confined to normal-form game settings. A theory meeting our needs has to cover dynamic stochastic games involving private information. Green (1980), Sabourian (1990), and Al-Najjar and Smorodinsky (2001) did work on large dynamic games involving randomness. However, they all aimed at establishing the near-myopic property of the dynamic games’ equilibria. Thus, our computational study will play an important role in empirically affirming the connection between the current NG and its related finite games.

In our approach, each individual firm bases its decision on the anticipated outside market process, which is itself a consequence of all firms’ collective actions and hence reveals information beyond the current firm’s local demand patterns. This framework also is related to the rational expectations (RE) approach initiation which is generally attributed to Muth (1961). However, our models are not really RE in the strict sense defined by the economics literature. Here, firms deal with outside customers instead of trading among themselves, and their decisions are local prices rather than bid–sell quantities that help to achieve prevailing market prices.

In the operations management literature, Wu and Chen (2010) combined the NG and RE approaches in their study of a competitive equilibrium production-inventory model with applications to the petroleum refining industry. Recently, Yang et al. (2007) applied the NG concept to a dynamic pricing problem. In their study, every firm involved is allowed one chance to lower its price. The key, therefore, is the rational timing of price reductions for all firms. In this study, we let firms freely choose prices from a given set, imposing no limit on repeated visits to the same price level.

Finally, we note that Weintraub et al. (2008) recently proposed the concept of oblivious equilibrium (OE) for large dynamic games exhibiting stationary traits. An OE is a profile of firm strategies that allows each firm to achieve the best outcome at every juncture of the game when, besides its own state, the firm is only aware of the long-run average state of the industry instead of the latter’s exact present state. When market power is dispersed among more and more firms afforded by an ever-increasing market size, the authors showed the asymptotic rationality of OE even when a concerned firm could be re-equipped with real-time industry information. OE’s efficiency and accuracy in computation was confirmed in Weintraub et al. (2010a), and its connections with other solution concepts for both finite- and infinite-player games were probed in Weintraub et al. (2010b). In addition, Xu (2008) successfully applied OE to an empirical study involving the research and development activities of the Korean electric motor industry.

Nevertheless, we caution that OE is not suitable for the stochastic case of our problem, which is prominently transient in nature. Aside from its continuous-time setup, our problem has the feature that each firm’s individual state, that is, its inventory level, is not observable to other firms. This problem’s observable state of the industry is the empirical distribution of prices charged by firms, and the hidden driver behind it is the empirical distribution of firms’ inventory levels. However, these levels decrease over time, rendering the long-run average concept inapplicable. Even in the NG limit, every firm has to optimally respond to a central time-varying market path.

1.2. Research Outline

For the deterministic model, we are able to identify reasonable conditions under which all firms will adopt increasing equilibrium price schedules. Our basic analytical tool is Tarski’s order-based fixed point theorem. In the stochastic model, we assume Poisson demand arrival for each firm, with a rate that is dependent on both the firm’s own price and the outside market condition.

Our analysis for the latter model begins with the identification of its Hamilton–Jacobi–Bellman (HJB) equation. We succeed in establishing the existence of NG equilibria for the case where the demand rate can be decomposed into two multiplicative factors, one price dependent and another market dependent. We rely on the point-map version of the Kakutani–Glicksberg–Fan fixed point theorem to derive the existence of an equilibrium pricing policy that is both inventory- and time-monotone. Along the way, we show many continuity results that may prove to be useful in other settings.

As noted above, a prominent question is always how useful our knowledge about NG equilibria can be in predicting/prescribing actions for firms in real settings where, after all, only a finite number of firms are present. Our computational study shows that a firm blindly following one of the NG equilibria will fare well in a market consisting of a reasonably large number of players. For instance, in a 30-firm environment where every firm behaves according to an NG equilibrium, firms will generate actual market conditions that are not too much different from those predicted by the NG approximation. This in turn results in firms’ actual revenue gains that are not much different from those predicted by the NG approximation either.

In addition, our computational study confirms that knowledge gained about the deterministic case can be very useful in the stochastic setting even if only a finite number of firms are in competition. This is especially evident in the heavy volume limit. The deterministic pricing policy, when properly interpreted,
also exhibits inventory- and time-monotone trends. So, in terms of managerial insights, a major contribution of this study is showing that firms under intense competition will adopt pricing policies that share characteristics with policies in monopolistic settings. Therefore, the results of this study can serve as guidelines to industries where competition is sufficiently intense, such as the hospitality industry, the second market of ticket selling, and, to a lesser degree, airlines and rental car industries.

Without a doubt, a model that considers multiple price options will be more practical, even though the trends demonstrated by our two-price model can serve as guidelines. Though this kind of a model is outside the current study’s scope, solving it will likely be facilitated by the tools already identified by us, such as the partial ordering of markets, the two fixed point theorems, the closed-form solution to a particular differential equation, and the various monotonicity and continuity results.

In the remainder of this article, we treat the deterministic model in section 2 and the stochastic model in section 3, present our computational results in section 4, and make concluding remarks in section 5.

2. The Deterministic-Demand Case

2.1. The Setting

A continuum of identical firms compete to sell the same type of product in the time interval \([0, \bar{t}]\). Each firm’s price is restricted to some set \(P\). Note that different prices can exist for the same product type due to differences in consumer locations and the presence of transportation costs. At a given time, all firms are influenced by the same market condition \(m\), which is essentially the cumulative distribution function (cdf) of prices charged by all firms:

\[
m = (m(p') | p' \in P)
\]

\[

\equiv (\text{Pr}[\text{The randomly sampled price } \Pi \leq p'] | p' \in P).
\]

We use \(M\) to denote the set of all instantaneous market conditions.

For two market conditions \(m^1, m^2 \in M\), we rank them by the stochastic dominance order: we say that condition \(m^2\) is milder than condition \(m^1\), or \(m^1 \leq m^2\), when

\[
m^1(p') \geq m^2(p'), \quad \forall p' \in P.
\]

Basically, we deem a market condition to be milder (or harsher) when more firms charge higher (or lower) prices, thus giving any firm in the market a less (or more) difficult time. Apparently, the above ranking constitutes a partial order for set \(M\). To describe the evolution of the market condition over a period of time, we introduce \(\mathcal{M} \equiv M^{[0, \bar{t}]}\), the set of all market processes. We can rank market processes \(m^1 = (m^1(t) | t \in [0, \bar{t}]) \in \mathcal{M}\) and \(m^2 = (m^2(t) | t \in [0, \bar{t}]) \in \mathcal{M}\) by \(m^1 \leq m^2\) when

\[
m^1(t) \leq m^2(t), \quad \forall t \in [0, \bar{t}].
\]

This extended ranking provides a partial order for \(\mathcal{M}\).

Let \(\text{cdf } F(x)\) denote the initial stock distribution of all firms. We assume that \(F(x)\) induces a piecewise continuous probability density function (pdf) \(d_x F(x)\) with a finite upper bound \(f = \sup_{x \in [0, \infty)} d_x F(x)\). For convenience, we call a firm with the initial inventory level \(x\) an \(x\)-firm.

Now, let the price set \(P\) be \(\{\bar{p}^L, \bar{p}^H, \bar{p}^{\infty}\}\), with \(\bar{p}^{\infty} > \bar{p}^H > \bar{p}^L > 0\). Here, \(\bar{p}^{\infty}\) is a demand-stopping price, which is also the default price charged by a firm when it is out of stock. For the two demand-allowing prices, we may think of \(\bar{p}^L\) as the manufacturer’s recommended price and \(\bar{p}^H\) as the price after an industry-standard discount. For any instantaneous market condition \(m\), we write \(m = (m^L, m^H)\), where \(m^L\) is the fraction of firms charging the low price \(\bar{p}^L\) and \(m^H\) is the fraction of firms charging the high price \(\bar{p}^H\). Automatically, \(m^{\infty} = 1 - m^L - m^H\) is the fraction of firms charging the demand-stopping price \(\bar{p}^{\infty}\). Now, the previous definition for \(m^1 \leq m^2\) translates into the following:

\[
m^{1L} \geq m^{2L}, \quad \text{and } m^{1L} + m^{1H} \geq m^{2L} + m^{2H}.
\]

For \(i = L,H\), demand is assumed to arrive to a \(\bar{p}^i\)-charging firm in a market with condition \(m\) at the rate \(\lambda^i(m)\).

Naturally, a firm should be able to draw a larger crowd with a lower price, as customers see prices charged by the firm as their costs and they are rational profit seekers themselves. Also, the low price needs to produce a higher revenue per item sold for it to be a viable option for the firm. In addition, the arrival pattern to the firm should not change abruptly when the market condition has only been mildly perturbed. These requirements lead to the following conditions:

\[(D1) \lambda^L(m) > \lambda^H(m) > 0, \quad \text{so that the low price entices more demand.}\]
\[(D2) \text{More than (D1), we have also } \bar{p}^L \cdot \lambda^L(m) > \bar{p}^H \cdot \lambda^H(m); \text{this way, the low price leads to a higher revenue generation rate.}\]
\[(D3) \text{Both } \lambda^L(m) \text{ and } \lambda^H(m) \text{ are continuous in the set } \Delta^M \equiv \{(m^L, m^H) \in \mathbb{R}^2 | m^L + m^H \leq 1, m^L, m^H \geq 0\}. \text{ On top of these, we introduce two more assumptions:}\]
Suppose that every firm is expecting a time-based market condition: for \( m^1 \) and \( m^2 \) with \( m^1 \leq m^2 \),
\[
\frac{\lambda^H(m^1)}{\lambda^H(m^2)} \geq \frac{\lambda^L(m^1)}{\lambda^L(m^2)},
\]
or, equivalently,
\[
\frac{\lambda^H(m^1)}{\lambda^L(m^2)} \leq \frac{\lambda^L(m^1)}{\lambda^H(m^2)}.
\]

(D5) Both \( \lambda^L(m) \) and \( \lambda^H(m) \) are increasing in the market condition \( m \), where the order for \( m \) is defined through Equation (4); hence, a milder market promotes demand to a firm.

Condition (D4) essentially says that the harsher the outside market condition becomes the more prominent the crowd-drawing effect of the low price. In other words, when entangled in a race to the bottom, a firm will lose a huge market share when it attempts to raise its price; when the competition becomes milder, however, the same firm will not lose as much when it tries to do the same. This condition plays a critical role in the identification of an NG equilibrium pair: a progressively increasing price schedule and a progressively improving market. (D4) will be automatically true when the demand rate function is of the product form, with \( \lambda^H(m) = \beta^H(m) \) for \( i = L,H \).

On the other hand, (D5) reflects that demand is more easily drawn to a firm when its competitors are more prone to charge high prices. Suppose market condition \( m^1 \) depicts the situation with 50% of the firms charging the low price, 40% of the firms charging the high price, and the remaining 10% charging the prohibitive price; also, market condition \( m^2 \) depicts the situation with, respectively, the 20%, 50%, and 30% ratios. Then, the satisfaction of Equation (4) tells us that \( m^1 \leq m^2 \). A firm immersed in the \( m^1 \) condition should expect to face more difficulty in attracting customers than one in the \( m^2 \) condition, simply because there are bigger portions of low-price-charging competitors in the former case. (D5) merely sums up this observation.

2.2. Preliminary Analysis
Suppose that every firm is expecting a time-continuous market process \( m = (m(t)) \, t \in [0, \bar{t}] \).

Due to (D3), the resultant demand-rate stream \( \lambda = (\lambda^L(m(t)), \lambda^H(m(t)) \, t \in [0, \bar{t}]) \) is continuous in time. We confine our consideration to the case where the price path of every firm is right continuous with left limits. Every firm thus charges \( \bar{p}, \overline{p}_H \), and \( \bar{p}_e \) intermittently in left-closed-right-open intervals until running out of stock, at which time the firm effectively starts to charge the demand-stopping price \( \bar{p}_e \) until the end. As rational numbers in \([0, \bar{t}]\) form one of its dense subsets, every aforementioned interval must contain a distinct rational number. Thus, we have countable intervals. Moreover, we can show that only the low and high prices need to be considered by a firm before it runs out of stock.

**Proposition 1.** Before running out of stock, a firm will not charge the demand-stopping price \( \bar{p}_e \).

For any time points \( t^1, t^2 \in [0, \bar{t}] \) with \( t^1 < t^2 \), we define demand levels \( \chi(t^1, t^2) \) and \( \bar{x}(t^1, t^2) \), so that
\[
\chi(t^1, t^2) = \int_{t^1}^{t^2} \lambda^H(m(t)) \, dt, \quad \bar{x}(t^1, t^2) = \int_{t^1}^{t^2} \lambda^L(m(t)) \, dt.
\]

Note that \( \chi(t^1, t^2)(\bar{x}(t^1, t^2)) \) is the cumulative demand level to a firm in time interval \([t^1, t^2] \) when it charges the high(low) price during the interval. By (D1), we know that \( \chi(t^1, t^2) < \bar{x}(t^1, t^2) \). We find that, prior to running out of stock or time, every firm with a below-\( \chi(0, \bar{t}) \) initial stock should continue charging the high price, and every firm with an above-\( \bar{x}(0, \bar{t}) \) initial stock should continue charging the low price.

**Proposition 2.** An x-firm operating under the given market process \( m \) with \( x \leq \chi(0, \bar{t}) \) will charge the high price \( \bar{p}_e \) until running out of stock, and an x-firm with \( x \geq \chi(0, \bar{t}) \) will charge the low price \( p_L \) until running out of time.

For any Lebesgue-measurable subset \( T \) of \([0, \bar{t}]\), we use \( |T| \) to denote its Lebesgue measure. For any pricing policy \( p \equiv \{ p(t) = \bar{p}, \overline{p}_H, p^e \, t \in [0, \bar{t}] \}, \) let \( T^L(p) \), \( T^H(p) \), and \( T^\infty(p) \) be sets of time points \( t \) at which \( p(t) = \bar{p}_e \), \( \overline{p}_H \), and \( p^e \), respectively. Note that these three sets exactly partition the interval \([0, \bar{t}]\), both \( T^L(p) \) and \( T^H(p) \) are countable unions of left-closed-right-open intervals; also, by Proposition 1, \( T^\infty(p) \) is a closed interval containing \( \bar{t} \).

Under policy \( p \), a firm makes sales in \( s(p) = |T^L(p)| + |T^H(p)| \) amount of time, and, under market process \( m = (m(t)) \, t \in [0, \bar{t}] \), the firm can sell a \( y(p) \) quantity of products and earn a revenue of \( r(p) \), with
\[
\begin{align*}
\{ y(p) &= \int_{T^L(p)} \lambda^H(m(t)) \, dt + \int_{T^H(p)} \lambda^L(m(t)) \, dt, \\
r(p) &= \bar{p}_e \cdot \int_{T^L(p)} \lambda^L(m(t)) \, dt + \overline{p}_H \cdot \int_{T^H(p)} \lambda^H(m(t)) \, dt. 
\end{align*}
\]

For the limited-capacity case in which the firm’s inventory level \( x \) is within \( (\chi(0, \bar{t}), \bar{x}(0, \bar{t})) \), we can verify that it always sells all items just in time. Note that the same perfect time-item matching result appeared
as Proposition 4 in Gallego and van Ryzin (1994), where the focus was dynamic pricing involving multiple price choices and stationary but price-dependent demand patterns.

**Lemma 1.** For \( x ∈ (x(0, \bar{t}), \bar{x}(0, \bar{t})) \), an \( x \)-firm operating under the given market process \( m \) always adopts a policy \( p \) with \( s(p) = 1 \) and \( y(p) = x \).

2.3. When the Market Improves over Time

We now suppose that the market process \( m = (m(t) \mid t ∈ [0, \bar{t}] ) \) is improving over time. That is, for any \( t^1, t^2 ∈ [0, \bar{t}] \) with \( t^1 ≤ t^2 \),

\[
m^L(t^1) ≥ m^L(t^2), \quad \text{and} \quad m^L(t^1) + m^H(t^1) ≥ m^L(t^2) + m^H(t^2).
\]

(7)

While Lemma 1 states that a firm should charge the low price during the time interval \([t^1, t^2]\) when it charges the low(high) price during \([t^1, t^{HL}(t^{HL})]\) and the high(low) price during \([t^{HL}(t^{HL}), t^2]\). We have

\[
\begin{align*}
\bar{p}^L &= \bar{p}^L \cdot x^H_r + \bar{p}^H \cdot x^H_l = \bar{p}^L \cdot x^L_r + \bar{p}^H \cdot x^H_l + \bar{p}^H \cdot (x^H_l - x^H_r), \\
\bar{p}^H &= \bar{p}^H \cdot x^H_r + \bar{p}^H \cdot x^H_l = \bar{p}^L \cdot x^L_r + \bar{p}^H \cdot x^H_l + \bar{p}^H \cdot (x^H_l - x^H_r).
\end{align*}
\]

(9)

The direct consequence of Lemma 2 is that

\[
\bar{p}^H - \bar{p}^L = (\bar{p}^H - \bar{p}^L) \cdot (x^H_l - x^H_r) \geq 0.
\]

(10)

In the following, we summarize the above findings.

**Lemma 3.** Let time points \( t^1, t^2 ∈ [0, \bar{t}] \) with \( t^1 < t^2 \) and inventory level \( x ∈ (x(t^1, t^2), \bar{x}(t^1, t^2)) \) be given. Suppose a firm is required to sell an \( x \)-quantity of products in exactly the time interval \([t^1, t^2]\) and is offered the two options of \( \bar{p}^H \)-first-\( \bar{p}^L \)-next and \( \bar{p}^L \)-first-\( \bar{p}^H \)-next. Then, under the time-improving market process \( m \), the firm will be better off by choosing the former option.

Using Lemmas 1 and 3, as well as an exchange-based argument, we can achieve the following important result.

**Proposition 3.** Under the time-improving market process \( m \), an \( x \)-firm with \( x ∈ (x(0, \bar{t}), \bar{x}(0, \bar{t})) \) will charge the low price \( \bar{p}^L \) first, the high price \( \bar{p}^H \) next, and then run out of stock at exactly the time \( \bar{t} \).

We can represent the results in Propositions 2 and 3 in a uniform fashion. For \( x ∈ (x(0, \bar{t}), \bar{x}(0, \bar{t})) \), we define \( \bar{p}^{HL}(x) \) as the solution for

\[
x = \int_{0}^{\bar{p}^{HL}(x)} \bar{x}^H(m(t)) \cdot dt + \int_{\bar{p}^{HL}(x)}^{\bar{p}^L(x)} \bar{x}^L(m(t)) \cdot dt.
\]

(11)

For \( x ≤ x(0, \bar{t}) \), we define \( \bar{p}^{HL}(x) = 0 \), and, for \( x ≥ \bar{x}(0, \bar{t}) \), we define \( \bar{p}^{HL}(x) = \bar{t} \). Note that the thus-defined \( \bar{p}^{HL}(x) \) is increasing in \( x \). We see from the two aforementioned propositions how an \( x \)-firm should behave when the market condition improves over time—it should charge the low price before time \( \bar{p}^{HL}(x) \) and then charge the high price afterwards until the depletion of either time or stock.

For \( x ≥ x(0, \bar{t}) \), we can let \( \bar{p}^{HL}(x) \) be the time at which an \( x \)-firm adopting the above policy runs out of stock, and it is the solution to the following:

\[
x = \int_{0}^{\bar{p}^{HL}(x)} \bar{x}^H(m(t)) \cdot dt.
\]

(12)
We may let $t^{H}\phi(x) = \bar{t}$ when $x \geq \bar{z}(0, \bar{t})$. In this way, the function $t^{H}\phi(x)$ is increasing in $x$ too.

2.4. The Existence of an Equilibrium
When every firm follows the aforementioned policy, the fraction of low-price-charging firms can only decrease over time. Meanwhile, it is always true that the fraction of out-of-stock firms is on the rise over time. In view of Equation (4), this means that the resultant market process is indeed improving over time. Therefore, the time-improvement property of a market process is self-sustaining.

Before proceeding further, we caution that a condition opposite to (D4) does not seem to be able to generate self-sustaining market processes: Only under a time-degrading market will we be able to obtain the same result as Lemma 2, which will make it profitable for firms to charge the high price first and the low price next. However, the fraction of out-of-stock firms is still on the rise over time. So, the resultant market process is neither improving nor degrading.

For a given policy, we let the inverse function of the corresponding $t^{H}\phi(x)$ be $x^{H}(t)$ and the inverse function of the corresponding $t^{H}\phi(x)$ be $x^{H}(t)$. A firm with an initial inventory level $x^{L}(t)$ switches from the low price to the high price at time $t$, and a firm with an initial inventory level $x^{H}(t)$ runs out of stock at time $t$. We can understand $(x^{H}(t), x^{H}(t)  \mid t \in [0, \bar{t}])$ as the policy $p$ because the latter says no more than the following: For an $x$-firm at time $t$, it will charge the low price when $x > x^{H}(t)$, the high price when $x^{H}(t) < x \leq x^{H}(t)$, and be out of stock when $x \leq x^{H}(t)$.

From Equations (11) and (12), we have the following market-to-decision relation:

$$
\begin{align*}
\begin{cases}
  x^{H}(t) = \int_{0}^{t} \lambda^{L}(m(s)) \cdot ds + \int_{t}^{\bar{t}} \lambda^{H}(m(s)) \cdot ds, \\
  x^{H}(t) = \int_{0}^{t} \lambda^{H}(m(s)) \cdot ds.
\end{cases}
\end{align*}
$$

Suppose all firms adopt the same pricing policy $p = (x^{H}(t), x^{H}(t)  \mid t \in [0, \bar{t}])$. Then, based on the interpretation of this policy, we must have the following decision-to-market relation:

$$
m^{L}(t) = 1 - F(x^{H}(t)), m^{H}(t) = F(x^{H}(t)) - F(x^{H}(t)).
$$

Recall that $F(x)$ is the fraction of $x'$-firms with $x' \in [0, x]$. Therefore, Equation (14) reflects the meaning of $p$ stated at the end of the preceding paragraph.

Upon combining Equations (13) and (14), we see that $m$ will be a self-sustaining market process if and only if it is a solution of the following integral equation: $\forall \bar{t} \in [0, \bar{t}],$

$$
\begin{align*}
\begin{cases}
  m^{L}(t) = 1 - F(\int_{0}^{t} \lambda^{L}(m(s)) \cdot ds + \int_{t}^{\bar{t}} \lambda^{H}(m(s)) \cdot ds), \\
  m^{H}(t) = 1 - m^{L}(t) - F(\int_{0}^{t} \lambda^{H}(m(s)) \cdot ds).
\end{cases}
\end{align*}
$$

We have depicted the relationship between a market and pricing policy in equilibrium in Figure 1.

Consider the set $\mathcal{M} \subset \mathcal{M}$ made up of market processes $m = (m(t)  \mid t \in [0, \bar{t}])$ satisfying the requirements that both $m^{L}(t)$ and $m^{H}(t) + m^{H}(t)$ are decreasing in $t$ and Lipschitz continuous in $t$ with coefficient $f \cdot \lambda^{L}(0, 0)$. Recall that $f$ is an upper bound to the initial stock pdf $d_{s}F(x)$. Also, due to assumptions (D2) and (D5), $\lambda^{L}(0, 0)$, being the demand rate to a firm charging the low price when all other firms have sold out, constitutes an upper bound to demand rates under all circumstances.

We may treat the right-hand side of Equation (15) as a map $Z_{M}^{\mathcal{M}}$ from the space of time-improving market processes to itself. Moreover, we show the existence of a solution to Equation (15) through the existence of a fixed point $m \in \mathcal{M}^{0}$ for $Z_{M}^{\mathcal{M}}$. For the proof of the latter, we invoke Tarski’s (1955) fixed point theorem, which says that, if $S$ is a complete lattice and $g : S \to S$ is an isotone mapping, then $g$ has a fixed point. The reader may refer to Birkhoff (1967) and Topkis (1998) for a detailed description of notions such as complete lattices and isotone maps.

To this end, we need to show that (I) $\mathcal{M}^{0}$ is a complete lattice, and (II) $Z_{M}^{\mathcal{M}}$ is an isotone map from $\mathcal{M}^{0}$ to $\mathcal{M}^{0}$, meaning that, for any $m^{1}, m^{2} \in \mathcal{M}^{0}$ satisfying $m^{1} \leq m^{2}$, it will follow that $Z_{M}^{\mathcal{M}}(m^{1}), Z_{M}^{\mathcal{M}}(m^{2}) \in \mathcal{M}^{0}$, and $Z_{M}^{\mathcal{M}}(m^{1}) \leq Z_{M}^{\mathcal{M}}(m^{2})$.

We can indeed prove (I) and (II).

**Proposition 4.** $\mathcal{M}^{0}$ is a complete lattice under the partial order defined through Equations (3) and (4).
Proposition 5. $Z^M_m$ as defined through Equation (15) is an isote map from $M^0$ to $M^0$.

With (I) and (II), Tarski’s theorem immediately leads to the existence of a solution to Equation (15).

Theorem 1. There exists a time-improving market process $m^D = (m^D_L(t), m^D_H(t) \mid t \in [0, \bar{t}])$ as a solution to the implicit integral equation $m = Z^M_m(m)$.

When we plug $m^D$ into the right-hand side of Equation (13), we will get the equilibrium pricing policy $p^D = (x^D_L(t), x^D_H(t) \mid t \in [0, \bar{t}])$. Here, $x^D_L(t)$ specifies the initial inventory level of those firms that should switch their prices from $p^L$ to $p^H$ at time $t$, and $x^D_H(t)$ predicts the initial inventory level of those firms that run out of stock at exactly time $t$. When all firms adopt the policy $p^D$, the consequence is the just-identified market process $m^D$.

2.5. Dynamic Interpretation of the Equilibrium Pricing Policy

Alternatively, given the equilibrium time-improving market process $m^D$, we can obtain the equilibrium pricing policy $p^{Dx} = (p^{Dx}(t) \mid t \in [0, \bar{t}])$ for every $x$-firm:

$$
p^{Dx}(t) = \begin{cases} 
p^L, & \text{when } t \in [0, t^D_{LH}(x)], \\
p^H, & \text{when } t \in [t^D_{LH}(x), t^D_{DH}(x)], \\
p^\infty, & \text{when } t \in [t^D_{DH}(x), \bar{t}].
\end{cases}
$$

Here, $t^D_{LH}(x)$ and $t^D_{DH}(x)$ are defined for the given equilibrium market process $m^D$ in the same fashion in which $t^L(x)$ and $t^H(x)$ were defined earlier for the arbitrary market process $m$ around Equations (11) and (12).

From the $p^{Dx}$, we can generate a dynamic Markovian policy $p^D = (p^D(y, t) \mid y \in [0, +\infty), t \in [0, \bar{t}])$, because we can pretend that the pricing decision depends on a given firm’s inventory level $y$ at the concerned point time $t$. For $t \in [0, \bar{t}]$, we define $x^D(t)$ as the level of stock that can be exactly depleted using the high price under the equilibrium market process; that is,

$$
x^D(t) = \int_{t}^{\bar{t}} \bar{p}^H(m^D(s)) \cdot ds.
$$

We can describe a firm’s equilibrium decision as charging the low price until its inventory level reaches $x^D(t)$ and then charging the high price until running out of stock. Therefore, for every $(y, t)$ pair we have

$$
p^D(y, t) = \begin{cases} 
p^L, & \text{when } y > x^D(t), \\
p^H, & \text{when } 0 < y \leq x^D(t), \\
p^\infty, & \text{when } y = 0.
\end{cases}
$$

From the above description, we see that the policy $p^D$ is inventory-motone. Also, $p^D$ is time-monotone, since $x^D(t)$ as defined in Equation (17) is apparently decreasing in $t$.

We caution that the decrease of the price trajectory $p^{Dx}(t)$ in $t$ and the decrease of the pricing policy $p^D(y, t)$ in $t$ are not contradictory to each other. Let the inventory trajectory of an $x$-firm be $y^* = (y^*(t) \mid t \in [0, \bar{t}])$. We have the important relationship

$$
p^{Dx}(t) = p^D(y^*(t), t).
$$

Note that $y^*(0) = x$ and also that $y^*(t)$ is decreasing in $t$. For our particular situation, $y^*(t)$ decreases in $t$ at least as fast as does the iso-price borderline of the $p^D$ policy because (D1) governs that $\bar{p}^H(m^D(t)) > \bar{p}^H(m^D(t))$. Thus, a time-decreasing policy $p^D(y, t)$ may generate a time-increasing price trajectory $p^{Dx}(t)$.

The $y^*$ process randomly decreases in the stochastic case. Suppose it is, with a high probability, decreasing faster than the iso-price borderline of the underlying Markovian pricing policy. Then, the market will, with a high probability, exhibit time-improving trends.

3. The Stochastic-Demand Case

3.1. The Setting

This case differs from the deterministic case in that, to each firm, demand comes as a Poisson process with a rate that instantaneously responds to the firm’s price and the market condition. Due to the Markovian nature of the demand process, a firm needs only to consider Markovian policies to maximize its profit when some $m \in M$ is given as the anticipated market process. We use $p = (p_n(t) \mid n = 1, 2, \ldots, t \in [0, \bar{t}])$ to denote such a policy, where $p_n(t)$ specifies the price the firm charges at time $t$, when it has $n$ items in stock at that time. As a default, we always suppose that $p_0(t) = p^\infty$, the demand-stopping price the firm is supposed to charge when it is out of stock.

We use probability distribution $f(0) = (f_n(0) \mid n = 1, 2, \ldots)$ with each $f_n(0) \geq 0$ and $\sum_{n=0}^{\infty} f_n(0) \leq 1$ to denote the initial stock distribution of all firms. Here, each $f_n(0)$ stands for the fraction of firms with $n$ items in stock at time 0. Note that $f_0(0) = 1 - \sum_{n=1}^{\infty} f_n(0)$, which is not included in the definition, stands for the fraction of firms with no inventory at the very beginning.

Besides the demand-stopping $p^\infty$, there are two demand-allowing, $p^L$ and $p^H$. The arrival-rate function can be written as $\bar{p}^H(m)$. Here, we suppose this function takes a decomposable form: There exist constants $\bar{z}^L$, $\bar{z}^H$ and function $\beta(m)$ such that

$$
\bar{z}(m) = \bar{z} \cdot \beta(m), \quad \forall i = L, H, \quad m \in M.
$$
Under this assumption, an individual firm’s own-price and outside-market effects on the arrival rate are uncorrelated, and, our earlier assumption (D4) becomes automatic.

The following are our other assumptions:

(S1) \( \tilde{\alpha}_L > \tilde{\alpha}_H > 0 \), so that the lower price entices more demand.

(S2) More than (S1), we have also \( \tilde{p}^H \tilde{\alpha}_L > \tilde{p}^H \tilde{\alpha}_H \); this way, the low price will lead to a higher revenue generation rate.

(S3) \( \beta(m) \) is continuous in the set \( \Delta^M \), where the definition of \( \Delta^M \) follows that in (D3).

(S4) \( \sup_{m \in \Delta^M} \beta(m) > 0 \).

(S1) through (S3) are merely specializations of (D1) through (D3), respectively. Due to (S3) and the fact that \( \Delta^M \) is a compact set, we define positive constant \( \beta^0 \) through

\[
\beta^0 = \sup_{m \in \Delta^M} \beta(m).
\]

When \( \tilde{\alpha}_L \) and \( \tilde{\alpha}_H \) are properly scaled, we may view \( \beta^0 \) as the size of the potential market. Moreover, (S4) is very essential in making a firm’s pricing decision continuous with respect to the market process it anticipates. In view of (S3) and the compactness of \( \Delta^M \), this assumption merely requires that \( \beta(m) > 0 \) for every \( m \in \Delta^M \).

Furthermore, we suppose that \( \sum_{n=1}^{\infty} p(t) = 0 \) for some positive integer \( n \). That is, no firm has more than \( n \) items in the beginning. We find that the only policies that warrant consideration are those that are both inventory- and time-monotone; that is, those \( p \equiv (p_n(t) \mid n = 1, 2, \ldots, n, t \in [0, T]) \) with \( p_n(t) \) being decreasing in both \( n \) and \( t \). In addition, a firm has no need to charge the demand-stopping price before it runs out of stock. With only two demand-allowing prices, we can use an \( n \)-tuple \( \tau = (\tau_n \mid n = 1, 2, \ldots, n) \) satisfying \( 0 = \tau_{n+1} \leq \tau_0 \leq \tau_{n-1} \leq \cdots \leq \tau_1 \leq \tau_0 = t \) to represent such a policy \( p \):

\[
p_n(t) = \begin{cases} \tilde{p}_H, & \text{when } t < \tau_n, \text{ or equivalently, } t \in [\tau_{n+1}, \tau_n] \text{ for some } l = n, n + 1, \ldots, n, \\ \tilde{p}_L, & \text{when } t \geq \tau_n, \text{ or equivalently, } t \in [\tau_{l+1}, \tau_l] \text{ for some } l = 0, 1, \ldots, n - 1. \end{cases}
\]

Note that \( p_l(\bar{t}) \) is immaterial; also, each \( \tau_l \) is the time point beyond which a firm with \( n \) remaining items will switch from the high price to the low price.

A differential equation and its solution prove useful when examining both market-to-decision and decision-to-market directions. Given real number \( g_0 \) as well as piecewise continuous functions \( a \) and \( b \) on interval \([0, +\infty)\), we introduce the following differential equation:

\[
\begin{align*}
g(0) &= g_0, \\
dg(t) &= b(t) - a(t) \cdot g(t), \forall t \in (0, +\infty).
\end{align*}
\]

This differential equation has a unique solution \( g \). For any \( t \in [0, +\infty) \), it follows that

\[
g(t) = g_0 \cdot \exp \left( - \int_0^t a(s) \cdot ds \right) + \int_0^t b(s) \cdot \exp \left( - \int_s^t a(u) \cdot du \right) \cdot ds.
\]

Basically, \( g(t) \) is the sum of two parts. Its first part is the initial value \( g_0 \) after being dissipated at instantaneous rates of \( a(s) \), and its second part is the cumulative effect of the \( b(s) \)'s after being dissipated at instantaneous rates of \( a(s) \) as well. In addition, \( a \) is continuous and piecewise continuously differentiable; when \( a \) and \( b \) are both continuous, \( g \) is further continuously differentiable.

### 3.2. The Market-to-Decision Direction

In the beginning, we allow the set \( P \setminus \{\tilde{p}^\infty\} \) of demand-allowing prices to be an arbitrary compact subset of \([0, \tilde{p}^\infty)\) and the rate function to be of the form \( \tilde{\alpha}(p, m) = \tilde{\alpha}(p) \cdot \beta(m) \). Suppose that firms anticipate a common market process \( m \in M \) that is piecewise continuous in time. Let \( v_n(t) \) be any individual firm’s optimal remaining value function when it has \( n \) remaining items at time \( t \). The value functions satisfy the following HJB equations:

\[
d_t v_n(t) + \left\{ \beta(m(t)) \cdot \sup_{p \in P_n(p)} \left[ \tilde{\alpha}(p) \cdot (p + v_{n-1}(t) - v_n(t)) \right] \right\} \quad \forall 0 = 0, \quad \forall n = 1, 2, \ldots, \bar{n}, \quad t \in (0, \bar{t}),
\]

where the “\( v0 \)” portion of the equation can be ascribed to the demand-stopping option afforded by \( \tilde{p}^\infty \). In addition, there are boundary and terminal conditions, that \( v_0(t) = 0 \) for every \( t \in [0, \bar{t}] \) and \( v_{\bar{n}}(\bar{t}) = 0 \) for every \( n = 1, 2, \ldots, \bar{n} \).

When \( \beta(m(t)) \cdot \sup_{p \in P_n(p)} \left[ \tilde{\alpha}(p) \cdot (p + v_{n-1}(t) - v_n(t)) \right] < 0 \), we let \( p_n(t) = \tilde{p}^\infty \); otherwise, we let \( p_n(t) \) be the lowest \( p \in P \setminus \{\tilde{p}^\infty\} \) that achieves the supremum for \( \tilde{\alpha}(p) \cdot (p + v_{n-1}(t) - v_n(t)) \). Because the firm can earn at most sup \( P \setminus \{\tilde{p}^\infty\} \), we know
\[ v_n(t) - v_{n-1}(t) \leq \sup P \setminus \{ \bar{p}^\infty \}, \] 

wheras the right-hand side is a member of \( P \setminus \{ \bar{p}^\infty \} \) due to its compactness. Hence, the former case is never true, meaning that \( p_n(t) \neq \bar{p}^\infty \) when \( n \geq 1 \). With this understanding, the “\( \forall 0 \)’ portion of Equation (24) can be removed. We can see that the simplified (24) is almost the same as Equation (1) of Zhao and Zheng (2000). The only notable difference lies in that, to conveniently express the arrival rate’s dependence on the potentially time-varying market process, we have let \( t \) be the time that has elapsed instead of the time that still remains.

We have the following:

Fact 1. \( v_n(t) \) is increasing in \( n \), as having more items will not hurt the revenue.

Fact 2. \( v_n(t) \) is decreasing in \( t \), as having less time to sell will not boost the revenue. Just because (S1) is true, Zhao and Zheng (2000) proved the following:

Fact 3. \( v_n(t) \) is concave in \( n \) (their Theorem 1).

Fact 4. \( v_n(t) \) is submodular in \( (n,t) \) (their Theorem 2; note the different \( t \)-definitions).

Fact 5. The optimal pricing policy \( p = (p_n(t) | n = 1, 2, \ldots, n, t \in [0, \bar{t}]) \) exists and for each \( t \in [0, \bar{t}] \), \( p_n(t) \) is decreasing in \( n \) (their Theorem 3).

Now, we suppose that \( P \setminus \{ \bar{p}^\infty \} = \{ \bar{p}^l, \bar{p}^H \} \), \( \bar{L}^{(H)}(m) = \bar{L}^{(H)} \cdot \beta(m) \), and all assumptions (S1) to (S4) apply. For convenience, we introduce the instantaneous stream \( \hat{\beta}(t) = \hat{\beta}(m(t)) \), \( \forall t \in [0, \bar{t}] \),

\[ \hat{\beta}(t) = \hat{\beta}(m(t)), \quad \forall t \in [0, \bar{t}], \tag{26} \]

and denote the above transition as \( \hat{\beta} = Z_{\hat{\beta}}^\infty(m) \). However, now we suppose that the transition between market processes and actual arrival-rate streams firms experience is not exactly instantaneous. The actual stream \( \hat{\beta} = \beta(t) \) for \( t \in [0, \bar{t}] \) is a lagged and time-averaged version of \( \hat{\beta} \). For some \( \theta > 0 \),

\[ \beta(t) = \frac{1}{\theta} \int_{t-\theta}^{t} \hat{\beta}(s) \cdot ds, \quad \forall t \in [0, \bar{t}], \tag{27} \]

where we have let \( \hat{\beta}(t) = \hat{\beta}(0) \) for \( t \in [-\theta,0) \). We note this transition by \( \hat{\beta} = Z_{\hat{\beta}}^\infty(\hat{\beta}) \). That \( Z_{\hat{\beta}}^\infty(\hat{\beta}) \) is not an identity mapping reflects that some time is needed for buyers to react to changes that have occurred in the market and that different buyers may react at different speeds. Our implicit assumption of the uniformly distributed buyer response time is merely an innocuous simplification. Technically, Equation (27) enables us to deal with the linear topological space of time-continuous streams \( \hat{\beta} \), instead of the harder-to-deal-with space of time-discontinuous streams \( \hat{\beta} \).

Given a time-continuous actual stream \( \beta \), Equation (24) becomes

\[ d_v u(t) + \beta(t) \cdot \max \{ \bar{p}^l \bar{L} - \bar{L} \cdot (v_n(t) - v_{n-1}(t)) \}, \tag{28} \]

\[ p^H \bar{L} - \bar{L} \cdot (v_n(t) - v_{n-1}(t)) = 0, \quad \forall n = 1, 2, \ldots, \bar{n}, \quad t \in [0, \bar{t}] \].

By Fact 4, we know that \( v_n(t) - v_{n-1}(t) \) is decreasing over time. Also, (S2) says that \( \bar{p}^l \bar{L} > p^H \bar{L} \). Combining these two, we may define \( \tau_1, \tau_2, \ldots, \tau_n \), such that,

\[ \tau_n = \sup \left\{ t \in [0, \bar{t}] | v_n(t) - v_{n-1}(t) > \frac{\bar{p}^l \bar{L} - p^H \bar{L}}{\bar{L} - \bar{L}^H} \right\}, \tag{29} \]

with the understanding that \( \tau_n = 0 \) when the strict inequality is never achieved. Inside (29), note that \( v_n(t) - v_{n-1}(t) \) is the marginal value of a unit change in the inventory level; meanwhile, \( (\bar{p}^l \bar{L} - p^H \bar{L})/(\bar{L} - \bar{L}^H) \) may be understood as the marginal revenue change brought by one more demand arrival due to the high-to-low price switch. Thus, \( \tau_n \) is the last time point at which staying at the high price is marginally beneficial.

In view of Equation (28), \( p_n(t) \) will be \( \bar{p}^H \) when \( t < \tau_n \) and \( \bar{p}^l \) when \( t \geq \tau_n \). By Fact 3, we know that \( p_0(t) = v_0(t) - v_{n-1}(t) \) is decreasing in \( n \). This and Fact 4 lead us to conclude that \( \tau_n \) as defined by Equation (29) is decreasing in \( n \). For convenience, we let \( \tau_{n+1} = 0 \) and \( \tau_0 = \bar{t} \). Then, \( p_n(t) \) is both inventory- and time-monotone, and it also fits the description in Equation (21).

In Online Appendix S8, we present an iterative procedure involved in obtaining the optimal threshold policy \( p = (p_n(t) | n = 1, 2, \ldots, \bar{n}, \ t \in [0, \bar{t}]) \), which is essentially also \( (\tau_n | n = 1, 2, \ldots, \bar{n}) \), from a given actual stream \( \beta = \beta(t) \) for \( t \in [0, \bar{t}] \). We use \( p = Z_{\hat{\beta}}^\infty(\beta) \) to denote this process. Let us point out that this recursive way of computing for the policy \( p \) is reminiscent of Feng and Xiao (2000a), who faced multiple prices but stationary albeit price-dependent demand patterns.

### 3.3. The Decision-to-Market Direction

We let the continuum of firms be a replica of an underlying probability space that allows Poisson demand arrivals. Now, provided that all firms adopt the same inventory-monotone pricing policy \( p = (p_n(t) | n = 1, 2, \ldots, \bar{n}, \ t \in [0, \bar{t}]) \), we can formulate the consequent market process \( m = (m(t,p) | t \in [0, \bar{t}], p \in P) \) when even the set of prices is a general \( P \). To this end, let \( f = (f(t) | t \in [0, \bar{t}]) = (f_n(t) | n = 1, 2, \ldots, \bar{n}, \ t \in [0, \bar{t}]) \), with every \( f(t) \geq 0 \) and \( \sum_{n=1}^{\bar{n}} f(t) \leq 1 \), be the stock-distribution process of all firms. Here, \( f_n(t) \) is the fraction of firms with
inventory level \( n \) at time \( t \). Note that the initial stock distribution \( f(0) \) is given.

We can establish a general relationship between a given inventory-monotone pricing policy \( p \) and its resultant market process \( m \) through the intermediate stock-distribution process \( f \). First, by the definition (1), we have

\[
m(t, p') = \sum_{n=1}^{\infty} 1(p_n(t) \leq p') \cdot f_n(t), \quad \forall t \in [0, \bar{t}], \quad p' \in P.
\]  

(30)

Because the policy \( p \) is inventory-monotone, we may define \( k(t, p') \) so that

\[
k(t, p') = \max\{k = 1, 2, \ldots |p_k(t) > p'\}, \quad \forall t \in [0, \bar{t}], \quad p' \in P.
\]  

(31)

We can then rewrite Equation (30) as

\[
m(t, p') = \sum_{k=k(t, p')+1}^{\infty} f_k(t), \quad \forall t \in [0, \bar{t}], \quad p' \in P.
\]  

(32)

As demand arrivals are Poissonian, we know that the instantaneous arrival rate to the \( n \)-item crowd is \( \bar{a}(p_{n+1}(t)) \cdot \bar{b}(m(t)) \cdot f_{n+1}(t) \), while the instantaneous departure rate from it is \( \bar{a}(p_n(t)) \cdot \bar{b}(m(t)) \cdot f_n(t) \). Hence, the following dictates the evolution of \( f \): For every \( n = 1, 2, \ldots, \bar{n} \) and \( t \in (0, \bar{t}) \),

\[
d_{fn}(t) = \bar{b}(m(t)) \cdot (\bar{a}(p_{n+1}(t)) \cdot f_{n+1}(t) - \bar{a}(p_n(t)) \cdot f_n(t)).
\]  

(33)

Equations (31), (32), and (33) together determine the transition from a given inventory-monotone policy \( p \) to the resultant market process \( m \).

Let us come back to the case where the price set \( P \) is equal to \( \{p^0, p^1, p^\infty\} \), with \( p^\infty \) being demand-stopping. Let \( p = (\tau_n | n = 1, 2, \ldots, \bar{n}) \) with \( 0 = \tau_{\bar{n}+1} \leq \tau_{\bar{n}} \leq \cdots \leq \tau_1 = \tau_0 = \bar{t} \) be a given inventory- and time-monotone pricing policy. Suppose a stock-distribution process \( f = (f(t) | t \in [0, \bar{t}]) \) has been given, where each \( f(t) \) is in the set \( \Delta^F \) and

\[
\Delta^F = \left\{ (f_1, \ldots, f_{\bar{n}}) \in [0, 1]^{\bar{n}} | \sum_{n=1}^{\bar{n}} f_n \leq 1 \right\}.
\]  

(34)

Then, corresponding to Equation (32), we have

\[
m^L(t) = \sum_{k'=n+1}^{\bar{n}} f_{k'}(t), \quad m^H(t) = \sum_{k=1}^{n} f_{k}(t),
\]  

\[\forall n = 0, 1, \ldots, \bar{n}, \quad t \in [\tau_{n+1}, \tau_n].
\]  

(35)

We view Equation (35) as \( m = Z^M_{\bar{n}}(p, f) \).

Given threshold policy \( p = (\tau_n | n = 1, 2, \ldots, \bar{n}) \) and actual stream \( \beta = (\beta(t) | t \in [0, \bar{t}]) \), we can define arrival-rate stream \( \lambda = Z^A_{\bar{n}}(p, \beta) \) so that \( \lambda = (\lambda_n(t) | n = 1, 2, \ldots, \bar{n}, t \in [0, \bar{t}]) \) satisfies

\[
\lambda_n(t) = \begin{cases} 
2^{\beta}(t) & \text{when } t \in [\tau_n, \bar{t}],
\end{cases}
\]  

(36)

In Online Appendix S9, we illustrate in detail the procedure of obtaining the stock-distribution process \( f \) from a given arrival-rate stream \( \lambda \). In the following, we denote this transition by \( f = Z^A_{\bar{n}}(\lambda) \).

### 3.4. The Existence of an Equilibrium

We leave many technical details, including the definition of the actual-stream space \( Y_B \), to Online Appendix S10. Our overall plan to show the existence of an equilibrium is very simple. First, we define operator \( Z^B_{\bar{n}} \) as a map from the space \( Y_B \) to itself, so that

\[
Z^B_{\bar{n}}(\beta) = Z^B_{\bar{n}}(Z^M_{\bar{n}}(Z^A_{\bar{n}}(Z^P_{\bar{n}}(\beta), \beta), \beta)) \). 
\]  

(37)

An illustration of this composite operator is provided in Figure 2.

Then, we use mainly continuity properties of the operator and a continuity-based fixed point theorem to prove the existence of a fixed point for \( Z^B_{\bar{n}} \) within the space \( Y_B \). The fixed point theorem we use is a version of the Kakutani–Glicksberg–Fan theorem, which is also known as Tychonoff’s fixed point theorem. It says that, if \( S \) is a nonempty compact, convex subset of a locally convex linear topological space and \( g; S \rightarrow S \) is continuous, then \( g \) has a fixed point: There exists an \( x^* \in S \) such that \( x^* = g(x^*) \) (Granas and Dugundji 2003, p. 147). The execution of this plan involves many often un-exciting technical details. For the interest of the reader, we have left these details in Online Appendix S10. Eventually, we reach the following result.

**Theorem 2.** There exist an actual stream \( \beta^\infty \), a threshold pricing policy \( p^\infty \), an arrival-rate stream \( \lambda^\infty \), a stock-distribution process \( f^\infty \), a market process \( m^\infty \), and
an instantaneous stream $\tilde{p}^s$, all in equilibrium, so that $p^S = Z^S_p(\tilde{p}^S)$, $\lambda^S = Z^S_p(\tilde{p}^S)$, $f^S = Z^S_A(\tilde{\lambda}^S)$, $m^S = Z^S_M(\tilde{p}^S, f^S)$, $\tilde{p}^S = Z^S_B(m^S)$, and $\tilde{\lambda}^S = Z^S_B(\tilde{p}^S)$.

That is, all firms may agree on an equilibrium pricing policy $p^S$ that possesses the monopolistic characteristics of inventory- and time-monotonicity; this policy in turn is hinged on the self-fulfilling anticipation of a piecewise-continuous equilibrium market policy $m^S$. It is not known, however, whether individual price pathways will rise over time in some stochastic sense or if the overall market condition will exhibit certain traits. An order-based approach may be more conducive to answering the above question. Our attempt in this regard has achieved only partial results. For the sake of brevity, we omit going into further details.

4. A Computational Study

We use a computational study to show the following:

(i) NG equilibria in our settings, which were purportedly developed for the virtual world with a continuum of firms, can actually be used in finite-firm settings to achieve satisfactory results.

(ii) Results for the deterministic case can help to achieve good performance in the stochastic case when expected sales volumes are sufficiently heavy.

4.1. The Computational Setup

We let the deterministic and stochastic cases share a common setup. Let there be strictly positive constants $p$ and $\eta$, as well as constants $\bar{p}$, $\bar{\eta}$, $\bar{\gamma}$, and $\bar{\delta}$ in $(0,1)$. We let the two demand-allowing prices be such that $p^H = \bar{p}$ and $p^S = \bar{p}$. For demand arrival rates, we let

\[
\begin{align*}
\tilde{\lambda}^L(m^L, m^H) &= \eta \cdot (1 - \bar{\eta} \cdot m^L/2 - \bar{\delta} \cdot (m^L + m^H)/2), \\
\tilde{\lambda}^H(m^L, m^H) &= \bar{\mu} \bar{\eta} \cdot (1 - \bar{\gamma} \cdot m^L/2 - \bar{\delta} \cdot (m^L + m^H)/2).
\end{align*}
\]

(38)

Note that both $\tilde{\lambda}^L(m^L, m^H)$ and $\tilde{\lambda}^H(m^L, m^H)$ are decreasing in $m^L$ and $m^L + m^H$; we have

\[
\frac{\tilde{\lambda}^L(m^L, m^H)}{\tilde{\lambda}^H(m^L, m^H)} = \bar{\nu} < 1;
\]

(39)

and, apparently $\tilde{\lambda}^H(m^L, m^H)$ can be decomposed into two multiplicative factors, the price-dependent $\tilde{\lambda}^H(m^H)$ and the market-dependent $\tilde{\lambda}^H(m^L, m^H)$, with

\[
\tilde{\lambda}^L = \eta, \quad \tilde{\lambda}^H = \bar{\mu} \bar{\eta}, \\
\beta(m^L, m^H) = 1 - \frac{1}{\bar{\gamma}} \cdot m^L - \frac{1}{2} \cdot (m^L + m^H).
\]

(40)

A lower bound of $\beta(m^L, m^H)$ is the strictly positive $\tilde{p}^S = 1 - \bar{\gamma}/2 - \bar{\delta}/2$. Hence, the current demand rate function satisfies all earlier assumptions.

Without loss of generality, we may fix $\tilde{p} = 1$ and $\tilde{t} = 1$. For the deterministic case, we adopt $F(x) = ((x - 1)/2)^+ / (\bar{\eta} + 1/2)) \land 1$ as the initial stock cdf; for the stochastic case, we let $f_1(0) = f_2(0) = \cdots = f_0(0) = 1/\bar{\eta}$ and $f_{n+1}(0) = f_{n+2}(0) = \cdots = 0$. Both reflect uniform distributions. We also fix the relationship that $\eta = \bar{\eta}/2$.

For actual implementation, we partition $[0, \tilde{t}]$ into $I$ intervals $[0, \Delta t], [\Delta t, 2\Delta t], \ldots, [I-1\Delta t, I\Delta t)$, where $\Delta t = \tilde{t}/I$. We have adopted $I = 5000$ or 150 times in order to strike a balance between accuracy and speed. We suppose that $\theta$ for the stochastic case, as defined around Equation (27), is much smaller than $\Delta t$. Hence, $Z^S_\theta$ is effectively the identity map. To reach equilibria for both deterministic and stochastic cases, we use iterative Tatonnement methods, the detailed descriptions of which we have omitted for brevity.

We define the distance between two market processes $m^1$ and $m^2$ as follows:

\[
||m^1 - m^2|| = \max_{i=0}^{I-1} \{ |m^1(i\Delta t) - m^2(i\Delta t)| \}.
\]

(41)

For the deterministic case, we use $||m^1 - m^2|| < \epsilon^D$ for some $\epsilon^D > 0$ as our convergence criterion. We also define the distance between two streams $\beta^1$ and $\beta^2$ as follows:

\[
||\beta^1 - \beta^2|| = \max_{i=0}^{I-1} \{ |\beta^1(i\Delta t) - \beta^2(i\Delta t)| \}.
\]

(42)

For the stochastic case, we use $||\beta^1 - \beta^2|| < \epsilon^S$ for some $\epsilon^S > 0$ as our convergence criterion. Here, we fix $\epsilon^D = 0.0001$ and $\epsilon^S = 0.0001$.

We offer in Online Appendix S2 some discussion on the convergence of our Tatonnement schemes. Basically, there is a theoretical guarantee for the deterministic-case scheme; yet, no such thing seems to exist for that of the stochastic case. Nevertheless, our numerical tests have shown that both schemes can converge within very reasonable numbers of iterations to unique equilibria. From this point on, we let $\bar{\mu} = 0.7, \bar{\nu} = 0.8, \bar{\gamma} = 0.2$, and $\bar{\delta} = 0.2$.

4.2. Finite-Firm Simulations

We first use simulations to investigate whether much accuracy would be lost by taking the NG approximation in the stochastic-demand case. In our simulations, we let there be a finite $Q$ number of firms; and, for $q = 1, 2, \ldots, Q$, firm $q$ has a random initial stock level $n_q$ that is sampled from the uniform distribution on $\{1, 2, \ldots, \bar{n}\}$. First, we let each firm use its optimal pricing policy $p^S$ derived for the infinite-firm setting.
Note that this policy has an associated market process $m^S$ derived for the NG model. At the same time, we record the actual market process $m'$ resulting from the $Q$ firms exercising the $p^S$ policy.

We take some $K$ simulation runs to generate statistical results. Let the actual market process in run $k$ be $m'(k)$. Then, the average distance $A^M = \frac{\sum_{k=1}^K ||m'(k) - m^S||}{K}$ and the standard deviation of the distance $D^M = \sqrt{\frac{\sum_{k=1}^K (||m'(k) - m^S|| - A^M)^2}{(K - 1)}}$ serves as a good measure of how closely the NG setting can approximate the real finite-firm situation.

In the $k$th simulation, suppose firm $q$’s realized revenue is $v'_q(k)$. Then, the realized revenue per firm for the entire run is $v'(k) = \sum_{q=1}^Q v'_q(k)/Q$. However, the expected revenue per firm in the ideal NG setting should be $v^S = \sum_{n=1}^\infty f_n(0) \cdot v(0)$, which is equal to $\sum_{n=1}^\infty v_n(0)/\bar{n}$ under our current setup. We define $R(k) = \frac{|v^S - v'(k)|}{v^S}$ as the percentage revenue difference between the real and ideal situations in run $k$. We also define the average revenue difference $A^R = \frac{\sum_{k=1}^K R(k)}{K}$ and the standard deviation of the difference $D^R = \sqrt{\frac{\sum_{k=1}^K (R(k) - A^R)^2}{(K - 1)}}$.

From our experience, we find that $K = 1,000$ suffices to generate stable results. In Figures 3 and 4, we present the $A^M \pm D^M$ values under various $Q$’s when $\bar{n} = 20$ and 100, respectively; and, in Figures 5 and 6, we present $A^R \pm D^R$ values under various $Q$’s when $\bar{n} = 20$ and 100, respectively.

Figures 3 and 4 clearly indicate that $m^S$ and the $m'(k)$’s get closer as $Q$ increases. Figures 5 and 6 show that the percentage revenue difference diminishes as $Q$ increases. Both groups of figures demonstrate that the converging trend between the multi-firm and infinite-firm settings is not much related to $\bar{n}$. Also, all hint at a turning point around $Q = 30$, whereas, before that point, the pace of change is dramatic, while after it, the pace discernibly slows down. At this point, we note that the percentage difference in market is $9.8 \pm 8.9\%$ and that in revenue is $6.5 \pm 4.9\%$ when $\bar{n} = 20$, whereas these differences are $9.4 \pm 8.2\%$ and $5.9 \pm 4.6\%$, respectively, when $\bar{n} = 100$. So, on average, a firm is not much worse off than what it anticipates from the NG situation.

When the number of firms $Q$ grows to 500, the market and revenue differences become $2.2 \pm 2.3\%$ and $1.0 \pm 0.9\%$, respectively, for $\bar{n} = 20$ and $2.1 \pm 2.2\%$ and $0.9 \pm 0.9\%$, respectively, for $\bar{n} = 100$. At this point, there is not much difference between the finite-firm and NG situations.

4.3. The Heavy-Volume Regime
Consistent with our earlier theory, all deterministic market processes $m^D$ satisfy that both $m^{DL}(i\Delta t)$ and $(m^{DL} + m^{DH})(i\Delta t)$ are decreasing in $i$, and all deterministic(stochastic) policies $p^{DS} = (i^{DS}/\Delta t)|n = 1, 2, \ldots, \bar{n})$ satisfy that $i_n^{DS} \leq i_{n-1}^{DS} \leq \cdots \leq i_1^{DS}$.
For each of the stochastic market processes $m^s$, we always find $(m^{SL} + m^{SH})(i\Delta t)$ to be decreasing in $i$. However, the $m^{SL}(i\Delta t)$’s are not necessarily decreasing in $i$. They often have sharp upward jumps followed by gradual downward slopes, and the jumps become more frequent as $\bar{n}$ increases. This may be caused by the discrete demand arrivals and binary price choices.

Figure 7 depicts both $m^D$ and $m^S$ at $\bar{n} = 200$. In Figure 8, we show $||m^D - m^S||$ as $\bar{n}$ increases.

Figure 7 demonstrates that, at the $\bar{n} = 200$ value, the difference in $m^D$ and $m^S$ is still considerable. From Figure 8, we see that the gap between the two market processes is indeed decreasing in $\bar{n}$ when the latter is small; however, to our surprise, the gap starts to widen when $\bar{n}$ grows further. Computational limitations prevent us from precisely investigating cases with even larger $\bar{n}$ values. At present, it is not conclusive whether and at what rate the two market processes converge.

We conduct one more round of finite-firm simulations. In this round, we try to see if adopting the deterministic policy $p^D$ in a stochastic environment results in revenue differences that decrease with the expected sales volume. An answer in the affirmative will compensate for our inconclusiveness on the convergence of the market processes.

Here, each firm’s initial stock distribution is an independent sample from $f(0)$ prescribed in section 4.1, the uniform distribution on $\{1, 2, \ldots, \bar{n}\}$. Firm 1 adopts policy $p^D$. At the same time, all other $Q-1$ firms adopt policy $p^S$. Expectedly, firm 1 on average collects lower revenues than the other firms, as the policy adopted by the latter is already near optimal. In Figure 9, we record the percentage revenue differences $A^R \pm D^R$ for $Q = 100$ and various $\bar{n}$ values.

From this, we conclude that the percentage revenue difference from adopting the deterministic policy is actually not that significant, and this difference dwindles as $\bar{n}$ increases. So, policy $p^D$ is not too much worse than $p^S$ in the finite-firm setting, as long as the expected sales volume is heavy enough.
5. Concluding Remarks

We have shown for a two-price case that the equilibrium pricing policy under intense competition is similar to its monopolistic counterpart. The policy exhibits both inventory- and time-monotone trends. For the deterministic case, we have also gone further to show that the price paths of individual firms tend to rise over time, thanks to the fact that firms’ inventory levels drop over time faster than the iso-price borderline. Consequently, the market condition has the tendency to improve over time.

A natural step forward for modeling the real world more truthfully is that of considering multiple price options. Though we have not touched upon this topic, some of the tools we used or developed here may turn out to be useful. Aside from variations in their starting inventory levels, we let firms be homogenous by using the same demand-rate function for all of them. On the other hand, it would be worthwhile to investigate the more general case where this uniformity is relaxed. Many times, the innate characteristics and local environment of one firm may differ substantially from those of another.

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References


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**Supporting Information**

Additional Supporting Information may be found in the online version of this article:

- Appendix S1: Proof of Proposition 1.
- Appendix S2: Proof of Proposition 2.
- Appendix S3: Proof of Lemma 1.
- Appendix S4: Proof of Lemma 2.
- Appendix S5: Proof of Proposition 3.
- Appendix S7: Proof of Proposition 5.
- Appendix S8: A Recursive Procedure that Facilitates $Z_B^p$.
- Appendix S9: Details concerning the Operator $Z_A^f$.
- Appendix S10: Continuity-based Derivations of Section 3.4.
- Appendix S14: Proof of Lemma 8.
- Appendix S16: Proof of Lemma 10.
- Appendix S17: Proof of Lemma 11.
- Appendix S18: Proof of Lemma 12.
- Appendix S21: Proof of Theorem 2.
- Appendix S22: Discussion on the Tatônnement Schemes in Section 4.

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Online Supplement

A. Proof of Proposition 1: Given interval $[t^1, t^2]$, point $t^S \in [t^1, t^2]$, and price $p = L, H$, suppose a firm charges the demand-stopping price $p^\infty$ in interval $[t^1, t^S]$ first and then the price $p$ in interval $[t^S, t^2]$. Let $x = \int_{t^1}^{t^S} \lambda^p(m(\tau)) \cdot d\tau$ be the sales quantity the firm can make under price $p$. Also, define $\tilde{x}(t)$ for $t \in [t^1, t^2]$, so that

$$\tilde{x}(t) = \int_{t^1}^{t} \lambda^p(m(\tau)) \cdot d\tau,$$

which is the sales quantity the firm can make under price $p$ in the interval $[t^1, t^2]$. Note that $\tilde{x}(\cdot)$ is continuous on $[t^1, t^2]$, and $\tilde{x}(t^1) = 0 \leq x \leq \tilde{x}(t^2)$. Therefore, by the intermediate value theorem, we know there exists $t^0 \in [t^1, t^2]$ with $\tilde{x}(t^0) = x$. This means that, we could just as well let the firm charge price $p$ in interval $[t^1, t^0]$ and then charge $p^\infty$ in interval $[t^0, t^2]$; this would allow the firm to sell the same quantity $x$ and earn the same revenue $p^p \cdot x$ in the entire interval $[t^1, t^2]$.

By the earlier discussion in Section 2.2, we know that $[0, \bar{t})$ is decomposed into countable left-closed-right-continuous intervals in which the firm charges the $p^L$, $p^H$, and $p^\infty$ prices intermittently. Given an interval in which the firm is supposed to charge a single price, we can traverse from its left end point $t^1$ to its right end point $t^2$. This key feature of our set of interval end points is not possessed by the equally countable set of rational numbers: there is no such thing as the smallest rational number strictly greater than a given rational number.

Starting from $t_0 = 0$, we can keep on identifying the next interval end point, and so on and so forth. We name these end points $t_1, t_2, \ldots$. There are two possibilities:

(I) at some finite $n$, we will hit $t_n = \bar{t}$; or,

(II) we can keep on finding distinct $t_n$ numbers strictly below $\bar{t}$.

For case (I), we let $t_m = t_n = \bar{t}$ for $m = n + 1, n + 2, \ldots$. For case (II), the increasing and bounded sequence $\{t_n \mid n = 0, 1, \ldots\}$ will have a limit point, say $t_\infty$. Then, we have two sub-cases:

(IIa) we already have $t_\infty = \bar{t}$, and

(IIb) we still have $t_\infty < \bar{t}$.

In case (I), $\bar{t}$ is indeed also a limit point of the $t_n$-sequence.

We first treat cases (I) and (IIa) together. Let $n = 1, 2, \ldots$ be the smallest index so that the firm charges $p^\infty$ on $[t_{n-1}, t_n)$. Let $s_0 = t_{n-1}$ and $m_0 = n$. Note that the firm charges a demand-allowing price on $[t_{m_0}, t_{m_0} + 1] = [t_n, t_{n+1})$. So, according to our earlier result achieved through the intermediate value theorem, we may identity $s_1 \in [s_0, t_{m_0} + 1] = [t_{n-1}, t_{n+1}]$, so that the firm may as well charge the demand-allowing price on $[s_0, s_1) = [t_{n-1}, s_1)$ and $p^\infty$ on $[s_1, t_{m_0} + 1) = [s_1, t_{n+1})$. If
the firm charges \( \tilde{p}^\infty \) on \( [t_n+1, t_{n+2}] \), we may merge the two \( \tilde{p}^\infty \)-charging intervals to get a combined \( \tilde{p}^\infty \)-charging interval \( [s_1, t_{n+2}] \).

Certainly, we may let \( m_1 = n + 1 \) or \( n + 2 \) depending on whether \( [t_{n+1}, t_{n+2}] \) was a demand-allowing or demand-stopping interval. We reiterate that the firm may as well charge demand-allowing prices on \( [t_0, s_1] \), \( \tilde{p}^\infty \) on \( [s_1, t_{m_1}] \), and original prices on \( [t_{m_1}, \tilde{t}] \); within \( [t_0, s_1] \), the firm may charge original prices on \( [t_0, s_0] \) and use the price formerly for \( [t_{m_0}, t_{m_0+1}] \) on \( [s_0, s_1] \). For the interval \( [s_1, \tilde{t}] \), we may repeat the above step, and obtain some \( s_2 \geq s_1 \) and \( m_2 \geq m_1 + 1 \), so that the firm may as well charge demand-allowing prices on \( [s_1, s_2] \), \( \tilde{p}^\infty \) on \( [s_2, t_{m_2}] \), and original prices on \( [t_{m_2}, \tilde{t}] \).

As long as there are \( \tilde{p}^\infty \)-charging intervals in \( [0, \tilde{t}] \) to begin with, we can keep on repeating the above steps, and using \( s_k \) and \( m_k \) to denote the “current” \( \tilde{p}^\infty \)-charging interval \( [s_k, t_{m_k}] \) after \( k \) steps. For each \( k = 1, 2, \ldots \), the firm may as well charge certain demand-allowing prices on \( [t_0, s_k] \), \( \tilde{p}^\infty \) on \( [s_k, t_{m_k}] \), and original prices on \( [t_{m_k}, \tilde{t}] \). As \( \{s_k : k = 1, 2, \ldots \} \) is an increasing sequence in \( [0, \tilde{t}] \), it must have a limit \( s_\infty \). As \( m_k \) is strictly increasing, and the limit of the \( t_n \)-sequence is \( \tilde{t} \), we must have \( \lim_{k \to +\infty} t_{m_k} = \tilde{t} \).

Therefore, we may as well let the firm charge demand-allowing prices on \( [0, s_\infty] \) and \( \tilde{p}^\infty \) on \( [s_\infty, \tilde{t}] \); in addition, the firm should charge its original prices on \( [0, s_0] \), and now for \( k = 0, 1, \ldots \), charge on interval \( [s_k, s_{k+1}] \) the price formerly charged on \( [t_{m_k}, t_{m_k+1}] \). When \( [0, \tilde{t}] \) originally contains no \( \tilde{p}^\infty \)-charging intervals, we may let \( s_0 = s_1 = \cdots = \tilde{t} \) and \( m_0 = m_1 = \cdots = \infty \). Therefore, for cases (I) and (IIa) where \( t_n \Rightarrow \tilde{t} \), the firm may charge \( \tilde{p}^\infty \) only in the end of the interval \( [0, \tilde{t}] \).

Let us now turn to case (IIb). We may re-label points \( t_0, t_1, \ldots \), and \( t_\infty \) as \( t_{00}, t_{01}, \ldots \) and \( t_{0\infty} \), respectively. Now let \( t_{10} = t_{0\infty} \) and continue our process of labeling consecutively the next interval end points. For this matter, we may create points \( t_{10}, t_{11}, \ldots \) and limit point \( t_{1\infty} \). If \( t_{1\infty} < \tilde{t} \), we may keep on creating \( t_{20}, t_{21}, \ldots \) and \( t_{2\infty} \). Indeed, if we have \( t_{n\infty} < \tilde{t} \), we may keep on creating \( t_{n+1,0}, t_{n+1,1}, \ldots \). If at some point we have \( t_{n\infty} = \tilde{t} \), we may still pretend that there are \( t_{m0}, t_{m1}, \ldots \) and \( t_{m\infty} \) for \( m = n + 1, n + 2, \ldots \) by letting all of them be \( \tilde{t} \). But it may be that we still have \( t_{n\infty} < t_{1\infty} < \cdots < \tilde{t} \). This time, we have to consider \( \lim_{n \to +\infty} t_{n\infty} \). If this limit point is equal to \( \tilde{t} \), we are through.

Otherwise, we go back to re-label earlier points by adding a 0 to the front of the subscript of each of them. For instance, what was termed \( t_{nm} \) is now \( t_{0nm} \), and what was termed \( t_{n\infty} \) is now \( t_{0n\infty} \). Now \( t_{0\infty} \) has the meaning of the above limit of limit points. We let \( t_{100} = t_{0\infty} \), and then
continue the same process of labeling the next end points.

As the above process goes on to the $k$th “layer” for some $k = 2, 3, ..., \ldots$, we will have points $t_{n_1n_2...n_k}$ for $n_1, n_2, \ldots, n_k = 0, 1, \ldots$, and points $t_{n_1n_2...n_r\infty}$ for $r = 1, 2, \ldots, k-1, n_1, n_2, \ldots, n_r = 0, 1, \ldots$, so that

$$t(0)^k = 0,$$  \hspace{1cm} (44)

$$t_{0n_2...n_{k-1}0} \leq t_{0n_2...n_{k-1}1} < t_{0n_2...n_{k-1}2} < \cdots < t_{0n_2...n_{k-1}(k-2)} < t_{0n_2...n_{k-1}(k-1)} < t_{0n_2...n_{k-1}(k)}, \quad \forall n_2, \ldots, n_{k-1} = 0, 1, \ldots,$$  \hspace{1cm} (45)

$$t_{n_1n_2...n_k1} \leq t_{n_1n_2...n_k2} \leq \cdots \leq t_{n_1n_2...n_k(k-1)} \leq t_{n_1n_2...n_k(k)}, \quad \forall n_1 = 1, 2, \ldots, n_2, \ldots, n_k = 0, 1, \ldots,$$  \hspace{1cm} (46)

$$t_{n_1...n_{k-1}\infty} = \lim_{n_k \to +\infty} t_{n_1...n_{k-1}n_k}, \quad \forall n_1, \ldots, n_{k-1} = 0, 1, \ldots,$$  \hspace{1cm} (47)

$$t_{n_1...n_{r-1}\infty} = \lim_{n_r \to +\infty} t_{n_1...n_{r-1}n_r}, \quad \forall r = 2, 3, \ldots, k-1, n_1, \ldots, n_{r-1} = 0, 1, \ldots,$$  \hspace{1cm} (48)

$$t_{n_1...n_{r+1}(0)^{k-r}} = t_{n_1...n_{r}\infty}, \quad \forall r = 1, 2, \ldots, k-1, n_1, \ldots, n_r = 0, 1, \ldots,$$  \hspace{1cm} (49)

$$t_{0\infty} < t_{\infty}.$$  \hspace{1cm} (50)

In (44) and (49), as well as hereafter, we shall use $(a)^l$ to stand for the $l$-long string $a...a$. Now we claim that, at some finite $k$, we must have $\lim_{n \to +\infty} t_{n\infty} = \bar{t}$. If not, then by (45), the set of distinct interval end points will have the power of continuum and will not be countable.

Suppose the finite $k$ has been identified. Then, just like for cases (I) and (IIa), we can identify $s_{(0)^{k-1}\infty} \in \{t(0)^k, t(0)^{k-1}\infty]\}$, so that the firm may as well charge demand-allowing prices on $[t(0)^k, s_{(0)^{k-1}\infty}]$ and $\bar{p}^\infty$ on $[s_{(0)^{k-1}\infty}, t(0)^{k-1}\infty)$. If we treat $[s_{(0)^{k-1}\infty}, t(0)^{k-2}\infty)$ as the earlier $[t_0, \bar{t}]$, we can further identify $[s_{(0)^{k-2}\infty}, t(0)^{k-2}\infty)$, so that the firm may as well charge demand-allowing prices on $[s_{(0)^{k-2}\infty}, s_{(0)^{k-1}\infty}]$ and $\bar{p}^\infty$ on $[s_{(0)^{k-1}\infty}, t(0)^{k-2}\infty)$. Combined with the earlier $[t(0)^k, s_{(0)^{k-1}\infty}]$, the firm is allowed to charge demand-allowing prices on $[t(0)^k, s_{(0)^{k-2}\infty})$. We can keep on going like this to identify points $s_{(0)^{k-2}\infty}$ that are increasing in $m$, so that the firm may as well charge demand-allowing prices on $[t(0)^k, s_{(0)^{k-2}\infty})$ and $\bar{p}^\infty$ on $[s_{(0)^{k-2}\infty}, t(0)^{k-2}\infty)$.

As $m$ tends to $+\infty$, the increasing sequence $s_{(0)^{k-2}\infty}$ will have a limit point $s_{(0)^{k-2}\infty} \in \{t(0)^k, t(0)^{k-2}\infty\}$. Like earlier, the firm may as well charge demand-allowing prices on $[t(0)^k, s_{(0)^{k-2}\infty})$, $\bar{p}^\infty$ on $[s_{(0)^{k-2}\infty}, t(0)^{k-2}\infty)$, and original prices on $[t(0)^{k-2}\infty, \bar{t})$. Now we may treat the interval $[s_{(0)^{k-2}\infty}, t(0)^{k-3}\infty)$ as the earlier $[t_0, \bar{t})$ and generate $s_{(0)^{k-3}\infty} \in [s_{(0)^{k-2}\infty}, t(0)^{k-3}\infty)$, so that the firm may as well charge demand-allowing prices on $[s_{(0)^{k-2}\infty}, s_{(0)^{k-3}\infty})$ and $\bar{p}^\infty$ on the interval $[s_{(0)^{k-3}\infty}, t(0)^{k-3}\infty)$. Since we may let this process go on, we know there exist points $s_{n_1...n\infty}$ for $r = 0, 1, \ldots, k-1, n_1, \ldots, n_r = 0, 1, \ldots$, so that

$$s_{(0)^{k-1}\infty} \text{ is generated by treating } [t(0)^k, t(0)^{k-1}\infty) \text{ as the earlier } [t_0, \bar{t}),$$  \hspace{1cm} (51)
In the end, we are able to conclude that the firm may as well charge demand-allowing prices on
\[ s_{n_1, ..., n_k+1, \infty} \] as generated by treating \( s_{n_1, ..., n_k+1, \infty}, t_{n_1, ..., n_k+1, \infty} \) as the earlier \( [t_0, \bar{t}] \),
\[ \forall n_1, ..., n_k = 0, 1, ... \]
\[ s_{n_1, ..., n_k} = \lim_{n_k \to +\infty} s_{n_1, ..., n_k, \infty} \]
\[ \forall r = 1, 2, ..., k - 1, n_1, ..., n_r = 0, 1, ... \]
\[ s_{n_1, ..., n_k+1, (0)^{k-r} \infty} \] is generated by treating \( s_{n_1, ..., n_k+1, (0)^{k-r} \infty}, t_{n_1, ..., n_k+1, (0)^{k-r} \infty} \)
as the earlier \( [t_0, \bar{t}] \),
\[ \forall r = 2, 3, ..., k - 1, n_1, ..., n_r = 0, 1, ... \]

In the end, we are able to conclude that the firm may as well charge demand-allowing prices on
\( [0, s_\infty) \) and \( \bar{p}^\infty \) on \( [s_\infty, \bar{t}] \). Hence, we are done for case (IIb) as well.

**B. Proof of Proposition 2:** Consider an \( x \)-firm with \( x \leq \bar{x}(0, \bar{t}) = \int_0^\bar{t} \bar{\lambda}^H(m(t)) \cdot dt \). By (D1), this firm will be able to sell its entire stock regardless of the prices it charges. The firm will get \( \bar{p}^H \cdot x \) in revenue if it sells all items at the high price; on the other hand, it will earn less if it ever charges the low price.

Consider an \( x \)-firm with \( x \geq \bar{x}(0, \bar{t}) = \int_0^\bar{t} \bar{\lambda}^L(m(t)) \cdot dt \). By (D1), this firm will use the entire horizon \( [0, \bar{t}] \) to make sales regardless of the prices it charges. The firm will get \( \bar{p}^L \cdot \bar{x}(0, \bar{t}) = \int_0^\bar{t} \bar{p}^L \bar{\lambda}^L(m(t)) \cdot dt \) in revenue if it charges the low price all along; on the other hand, due to (D2), it will earn less if it ever charges the high price.

**C. Proof of Lemma 1:** For an arbitrary policy \( p \), there are four possibilities:

(I) \( s(p) < \bar{t} \) and \( y(p) < x \), so that there is more time to sell more items;

(II) \( s(p) < \bar{t} \) and \( y(p) = x \), so that all items are sold before the horizon ends;

(III) \( s(p) = \bar{t} \) and \( y(p) < x \), so that some items are left unsold in the end; and,

(IV) \( s(p) = \bar{t} \) and \( y(p) = x \), so that items are sold exactly in time.

We show that \( p \) can still be improved if it leads to any of the first three possibilities.

Suppose (I) is true, and hence \( \left| T^\infty(p) \right| = \bar{t} - s(p) > 0 \). Then, the firm can expand either \( T^L(p) \) or \( T^H(p) \) to make more sales and earn higher revenues.

Suppose (II) is true, and hence \( \left| T^\infty(p) \right| = \bar{t} - s(p) > 0 \). Then, it must be the case that \( \left| T^L(p) \right| > 0 \). That is, the firm must have charged the low price. Otherwise, since \( x > \bar{x}(0, \bar{t}) = \int_0^\bar{t} \bar{\lambda}^H(m(t)) \cdot dt \), the firm would not have sold off all \( x \) items in time. Based on the above, the hypothesis that \( s(p) < \bar{t} \), and (D1), we can identify intervals \( T^1 \subset T^L(p) \) and \( T^2 \subset T^\infty(p) \), so that \( \int_{T^1} (\bar{\lambda}^L(m(t)) - \bar{\lambda}^H(m(t))) \cdot dt = \int_{T^2} \bar{\lambda}^H(m(t)) \cdot dt > 0 \), or equivalently,

\[ \Delta y = \int_{T^1} \bar{\lambda}^L(m(t)) \cdot dt = \int_{T^1 \cup T^2} \bar{\lambda}^H(m(t)) \cdot dt > 0. \]
Now, we may construct policy \( p' \) with \( T^L(p') = T^L(p) \setminus T^1 \), \( T^H(p') = T^1 \cup T^H(p) \cup T^2 \), and automatically, \( T^\infty(p') = T^\infty(p) \setminus T^2 \). We can verify that

\[
s(p') = |T^L(p')| + |T^H(p')| = |T^L(p)| + |T^H(p)| + |T^2| = s(p) + |T^2| \leq s(p) + |T^\infty(p)| \leq \bar{\ell};
\]

which is equal to \( x \) according to (55); and,

\[
y(p') = x - \int_{T^1} \bar{\lambda}^L(m(t)) \cdot dt + \int_{T^1 \cup T^2} \bar{\lambda}^H(m(t)) \cdot dt,
\]

where the second and last equalities are due to (55). Basically, policy \( p' \) lets the firm charge the high price for a longer time and still allows it to sell all items in time; at the same time, it lets the firm sell \( \Delta y \) more items at the high rather than low price.

Suppose (III) is true. Then, it must be the case that \(|T^H(p)| > 0\). That is, the firm must have charged the high price. Otherwise, since \( x < \pi(0, \bar{\ell}) = \int_0^\bar{\ell} \bar{\lambda}^L(m(t)) \cdot dt \), the firm would have sold off all \( x \) items in time. Due to the above fact, the hypothesis that \( y(p) < x \), and (D1), we can identify \( T' \subset T^H(p) \) with \(|T'| > 0\), so that

\[
\Delta y = \int_{T'} (\bar{\lambda}^L(m(t)) - \bar{\lambda}^H(m(t))) \cdot dt \in (0, x - y(p)].
\]

Now, we may construct policy \( p' \) with \( T^L(p') = T^L(p) \cup T', T^H(p') = T^H(p) \setminus T', \) and automatically, \(|T^\infty(p')| = |T^\infty(p)| = 0\). We can check that

\[
s(p') = |T^L(p')| + |T^H(p')| = |T^L(p)| + |T^H(p)| = s(p) = \bar{\ell};
\]

where the last two relations are from (59); and,

\[
r(p') = r(p) + \int_{T'} (\bar{p}^L \bar{\lambda}^L(m(t)) - \bar{p}^H \bar{\lambda}^H(m(t)) \cdot dt,
\]

which is strictly greater than \( r(p') \) due to (D2) and the fact that \(|T'| > 0\). Basically, policy \( p' \) allows the firm to charge the low price more and the high price less, to sell more items in the designated time interval, and to earn a higher revenue.

Hence, for \( p \) to be an optimal response to the given market process \( m \), it should lead to no other possibility than (IV).
D. Proof of Lemma 2: As mentioned earlier, the arrival-rate stream \( \lambda = (\bar{\lambda}_L(m(t)), \bar{\lambda}_H(m(t))) \mid t \in [0, \bar{t}] \) is time-continuous due to (D3) and the time-continuity of the market process \( m \). Therefore, both \( \int_t^T \bar{\lambda}_L(m(s)) \cdot ds \) and \( \int_t^{t^2} \bar{\lambda}_L(m(s)) \cdot ds \), when viewed as functions of \( t \), are continuous and continuously differentiable.

Define \( \tau_0^2 \) so that
\[
x^L_B = \int_{\tau_0^2}^{t^2} \bar{\lambda}_L(m(t)) \cdot dt, \tag{63}
\]
whose existence is guaranteed by the intermediate value theorem. For \( y \in [0, x^L_B] \), define strictly increasing functions \( \tau^1(y) \) and \( \tau^2(y) \) so that
\[
y = \int_{\tau_1^1(y)}^{\tau^1(y)} \bar{\lambda}_L(m(t)) \cdot dt = \int_{\tau_0^2}^{\tau^2(y)} \bar{\lambda}_L(m(t)) \cdot dt. \tag{64}
\]
We know that \( \tau^1(0) = t^1 \), \( \tau^1(x^L_B) = t^L_H \) in view of (8), \( \tau^2(0) = \tau_0^2 \), and \( \tau^2(x^L_B) = t^2 \) in view of (63). From (64), we have
\[
x^L_B - y = \int_{\tau^1(y)}^{\tau^1(x^L_B)} \bar{\lambda}_L(m(t)) \cdot dt = \int_{\tau^2(y)}^{\tau^2(x^L_B)} \bar{\lambda}_L(m(t)) \cdot dt. \tag{65}
\]
On the other hand, due to (D5) and the fact that the market process \( m \) is time-improving, we know that
\[
\int_{\tau^1(y)}^{\tau^1(x^L_B)} \bar{\lambda}_L(m(t)) \cdot dt \leq \int_{\tau^1(y)}^{\tau^1(x^L_B) + t^2 - t^L_H} \bar{\lambda}_L(m(t + t^2 - t^L_H)) \cdot dt
= \int_{\tau^1(y) + t^2 - t^L_H}^{\tau^1(y) + t^2 - t^L_H} \bar{\lambda}_L(m(t)) \cdot dt = \int_{\tau^1(y) + t^2 - t^L_H}^{\tau^2(y)} \bar{\lambda}_L(m(t)) \cdot dt. \tag{66}
\]
Comparing (65) and (66), we may obtain
\[
\tau^2(y) \geq \tau^1(y) + t^2 - t^L_H > \tau^1(y). \tag{67}
\]
Due to (67), the hypothesis about the time-improving market process, and (D4), we have, for every \( y \in [0, x^L_B] \),
\[
\frac{\bar{\lambda}_H(m(\tau^2(y)))}{\bar{\lambda}_L(m(\tau^2(y)))} \geq \frac{\bar{\lambda}_H(m(\tau^1(y)))}{\bar{\lambda}_L(m(\tau^1(y)))}. \tag{68}
\]
From the definitions in (64), we can check that
\[
d_y \tau^1(y) = \frac{1}{\bar{\lambda}_L(m(\tau^1(y)))}, \quad d_y \tau^2(y) = \frac{1}{\bar{\lambda}_L(m(\tau^2(y)))}. \tag{69}
\]
Now by (64), (68), and (69), we have
\[
\int_{\tau^2(0)}^{\tau^2(0)} \bar{\lambda}_L(m(t)) \cdot dt = \int_0^{x^L_B} \bar{\lambda}_L(m(\tau^2(y))) \cdot d_y \tau^2(y) \cdot dy
= \int_0^{x^L_B} \frac{\bar{\lambda}_H(m(\tau^2(y)))}{\bar{\lambda}_L(m(\tau^2(y)))} \cdot d_y \tau^2(y) \cdot dy
\geq \int_0^{x^L_B} \frac{\bar{\lambda}_H(m(\tau^1(y)))}{\bar{\lambda}_L(m(\tau^1(y)))} \cdot d_y \tau^1(y) \cdot dy
= \int_0^{x^L_B} \bar{\lambda}_H(m(\tau^1(y))) \cdot d_y \tau^1(y) \cdot dy = \int_0^{t^L_H} \bar{\lambda}_H(m(t)) \cdot dt. \tag{70}
\]
Therefore, we can derive
\[
\int_{t^1_{HL}}^{t^2_{HL}} \tilde{\lambda}^L(m(t)) \cdot dt = \int_{t^1_{HL}}^{t^2_{HL}} \tilde{\lambda}^H(m(t)) \cdot dt + (\int_{t^1_{HL}}^{t^2_{HL}} \tilde{\lambda}^H(m(t)) \cdot dt - \int_{t^2(0)}^{t^2_{HL}} \tilde{\lambda}^H(m(t)) \cdot dt) \geq x - x^H_G - x^L_B \geq 0.
\] (71)
where the first equality is an identity, the second equality is due to (8), and the last inequality comes from (70).

Now, we have
\[
\int_{t^1_{HL}}^{t^2_{HL}} (\tilde{\lambda}^L(m(t)) - \tilde{\lambda}^H(m(t))) \cdot dt = (\int_{t^1_{HL}}^{t^2_{HL}} \tilde{\lambda}^H(m(t)) \cdot dt + \int_{t^1_{HL}}^{t^2_{HL}} \tilde{\lambda}^L(m(t)) \cdot dt) - (\int_{t^1(0)}^{t^2_{HL}} \tilde{\lambda}^H(m(t)) \cdot dt + \int_{t^2(0)}^{t^2_{HL}} \tilde{\lambda}^L(m(t)) \cdot dt) = x - x^H_G - x^L_B = 0.
\] (72)
where we have taken the convention that \(\int_a^b c(x) \cdot dx\) means \(-\int_b^a c(x) \cdot dx\) in case \(a > b\), the first equality is an identity, the second equality is due to (8) and the definition of \(\tau_0^2 = \tau^2(0)\) through (63), the only inequality is from (71), and the last equality is due to (8) again. By (D1), we must have \(t^1_{HL} \leq \tau^2(0)\). Hence, by (8) and (63), we have
\[
x^L_G = \int_{t^1_{HL}}^{t^2_{HL}} \tilde{\lambda}^L(m(t)) \cdot dt \geq \int_{t^2(0)}^{t^2_{HL}} \tilde{\lambda}^L(m(t)) \cdot dt = x^L_B.
\] (73)
By (8) again, we therefore have \(x^H_G - x^H_B = x^L_G - x^L_B \geq 0\).

E. Proof of Proposition 3: By earlier discussion in Section 2.2 and Lemma 1, we may suppose that the concerned \(x\)-firm adopts a policy \(p\) as follows: For some \(t_{\infty} \in [0, \bar{t}]\), the interval \([0, t_{\infty}]\) is decomposed into countable left-closed-right-continuous intervals in which the firm charges the low and high prices intermittently, and in \([t_{\infty}, \bar{t}]\), the firm is out of stock. Given an interval in which the firm is supposed to charge a single price, we can traverse from its left end point \(t^1\) to its right end point \(t^2\).

Starting from \(t_0 = 0\), we can identify the next interval end point \(t_1\). If the firm is to charge the high price on \([t_0, t_1]\), we may re-define \(t_1\) so that it equals \(t_0 = 0\). From \(t_1\) on, we can keep on identifying the next interval end point, and so on and so forth. We name these end points \(t_2, t_3, \ldots\). There are two possibilities:

(I) at some finite \(n\), we will hit \(t_n = t_{\infty}\); or,

(II) we can keep on finding distinct \(t_n\) numbers strictly below \(t_{\infty}\).

For case (I), we let \(t_m = t_n = t_{\infty}\) for \(m = n + 1, n + 2, \ldots\). For case (II), the increasing and bounded sequence \(\{t_n \mid n = 0, 1, \ldots\}\) will have a limit point, say \(t_{0\infty}\). Then, we have two sub-cases:
We first treat cases (I) and (IIa) together. We may describe these cases in this fashion: there is an increasing sequence of points \( \{t_n \mid n = 0, 1, \ldots\} \) in \([0, \bar{t}]\) with \( t_0 = 0 \), such that \( p \) recommends the low price on intervals \([t_{2n}, t_{2n+1}]\) and the high price on intervals \([t_{2n+1}, t_{2n+2}]\) for \( n = 0, 1, \ldots \). The firm will be out of stock exactly on the interval \([t_\infty, \bar{t}]\). Note that any occasion where \( t_n = t_{n+1} \) merely entails an empty \([t_n, t_{n+1}]\), which will not warrant a separate treatment in our ensuing analysis.

For any \((t^1, t^{LH}, t^2)\) satisfying \(0 \leq t^1 \leq t^{LH} \leq t^2 \leq \bar{t}\), we use \( \tilde{r}(t^1, t^{LH}, t^2) \) to denote the revenue made by a firm in the interval \([t^1, t^2]\), when it charges the low price in \([t^1, t^{LH}]\) and the high price in \([t^{LH}, t^2]\). Note that \( \tilde{r}(t, t, t) = 0 \) for any \( t \); and, in the presence of the finite revenue-generation bound \( \bar{p}L \cdot \lambda \cdot (0, 0) \), the revenue function \( \tilde{r}(-, -,-) \) is Lipschitz continuous, whereas the metric we use for \((t^1, t^{LH}, t^2)\) or its corresponding Lipschitz coefficient is irrelevant for this proof.

Let policy \( p_n \) be such that, for some \( s_n \in [t_0, t_{2n}] \), the firm is to set the low price in the interval \([t_0, s_n]\) and the high price in the interval \([s_n, t_{2n}]\), to the effect that the total sales in the combined interval \([0, t_{2n}]\) are the same as under policy \( p \); and, the policy is the same as \( p \) from point \( t_{2n} \) onward. Apparently, \( p_1 \) is just \( p \) with \( s_1 = t_1 \), which helps the firm earn \( r(p_1) \equiv \sum_{n=0}^{+\infty} \tilde{r}(t_{2n}, t_{2n+1}, t_{2n+2}) \).

For \( n = 1, 2, \ldots, \) suppose \( p_n \) is achievable. We can construct \( p_{n+1} \) with \( s_{n+1} \geq s_n \) and earning a higher revenue than \( p_n \) does. Indeed, by Lemma 3, we can identify some \( s_{n+1} \in [s_n, t_{2n+1}] \), so that by letting the firm charge the low price in \([s_n, s_{n+1}]\) and the high price in \([s_{n+1}, t_{2n+1}]\), the firm can sell the same quantity as \( p_n \) does in the interval \([s_n, t_{2n+1}]\) while making a higher earning \( \tilde{r}(s_n, s_{n+1}, t_{2n+1}) \). By making the same low-price recommendation in interval \([0, s_{n+1}]\) and the same high-price recommendation in interval \([s_{n+1}, t_{2n+2}]\), policy \( p_{n+1} \) differs from policy \( p_n \) only in its recommendation for interval \([s_n, t_{2n+1}]\). According to the above, it helps the firm earn a higher revenue:

\[
\begin{align*}
    r(p_{n+1}) & \equiv \tilde{r}(0, s_{n+1}, t_{2n+2}) + \sum_{m=n+1}^{+\infty} \tilde{r}(t_{2m}, t_{2m+1}, t_{2m+2}) \\
    & \geq r(p_n) \equiv \tilde{r}(0, s_n, t_{2n}) + \sum_{m=n}^{+\infty} \tilde{r}(t_{2m}, t_{2m+1}, t_{2m+2}).
\end{align*}
\]

(74)

Since \( \{s_n \mid n = 1, 2, \ldots\} \) is an increasing sequence in \([0, \bar{t}]\), it must converge to a limit point \( s_\infty \). The policy \( p_\infty \) of letting the firm charge the low price in the interval \([0, s_\infty]\) and the high price in \([s_\infty, t_\infty]\) will sell all items at time \( t_\infty \) and make a higher revenue \( r(p_\infty) \) than the \( r(p) \) made by \( p \):

\[
    r(p_\infty) \equiv \tilde{r}(0, s_\infty, t_\infty) = \lim_{n \to +\infty} \left[ \tilde{r}(0, s_n, t_{2n}) + \sum_{m=n}^{+\infty} \tilde{r}(t_{2m}, t_{2m+1}, t_{2m+2}) \right] = \lim_{n \to +\infty} r(p_n),
\]

(75)
which is greater than \( r(p) = r(p_1) \) by (74). In (75), the first equality is due to the limiting properties of the \( t_n \)- and \( s_n \)-sequences, as well as the Lipschitz continuity of \( \tilde{r}(\cdot, \cdot, \cdot) \) and the fact that \( \tilde{r}(t, t, t) = 0 \).

Therefore, we are done for cases (I) and (IIa): there is a policy \( p_\infty \) which, for some \( s_\infty \), lets the firm charge the low price on \([t_0, s_\infty] \) and the high price on \([s_\infty, t_\infty] \), and which lets it earn more money than \( p \) allows. Let us now turn to case (IIb). Like in the proof of Proposition 1, we may identify \( k = 1, 2, \ldots \), and points \( t_{n_1} \ldots t_{n_k} \) for \( n_1, n_2, \ldots, n_k = 0, 1, \ldots \), and \( t_{n_1} \ldots t_{n_r} \) for \( r = 1, 2, \ldots \), \( k-1, n_1, n_2, \ldots, n_r = 0, 1, \ldots \), so that (44) to (50) are satisfied, all interval end points are in the form of \( t_{n_1} \ldots t_{n_k} \), and \( \lim_{n \rightarrow +\infty} t_{n_k} = t_\infty \).

Then, just like for cases (I) and (IIa), on each interval \([t_{n_1} \ldots t_{n_k-1}, t_{n_1} \ldots t_{n_k-2}] \), we can identify policy \( p_{n_1} \ldots p_{n_k-1} \) that performs better than \( p \) on the same interval; and, for some \( s_{n_1} \ldots s_{n_k-1} \in [t_{n_1} \ldots t_{n_k-2}], t_{n_1} \ldots t_{n_k-1}] \), this policy lets the firm charge the low price on \([t_{n_1} \ldots t_{n_k-1}, s_{n_1} \ldots s_{n_k-1}] \) and the high price on \([s_{n_1} \ldots s_{n_k-1}, t_{n_1} \ldots t_{n_k-1}] \).

Now, each interval \([t_{n_1} \ldots t_{n_k-2}, t_{n_1} \ldots t_{n_k-1}] \) is decomposed into intervals \([t_{n_1} \ldots t_{n_k-2}, s_{n_1} \ldots s_{n_k-1}] \), \([s_{n_1} \ldots s_{n_k-2}, t_{n_1} \ldots t_{n_k-1}] \), \([s_{n_1} \ldots s_{n_k-2}, t_{n_1} \ldots t_{n_k-1}] \), \([s_{n_1} \ldots s_{n_k-2}, t_{n_1} \ldots t_{n_k-1}] \), \( \ldots \), on which the firm should charge the two prices intermittently. Again, just like for cases (I) and (IIa), on each interval \([t_{n_1} \ldots t_{n_k-2}, t_{n_1} \ldots t_{n_k-1}] \), we can identify policy \( p_{n_1} \ldots p_{n_k-2} \) that performs better than the concatenation of the \( p_{n_1} \ldots p_{n_k-2} \) policies for \( n_k = 0, 1, \ldots \) on the same interval; and, for some \( s_{n_1} \ldots s_{n_k-2} \in [t_{n_1} \ldots t_{n_k-2}, t_{n_1} \ldots t_{n_k-2}] \), this newly discovered policy lets the firm charge the low price on \([t_{n_1} \ldots t_{n_k-2}, s_{n_1} \ldots s_{n_k-2}] \) and the high price on \([s_{n_1} \ldots s_{n_k-2}, t_{n_1} \ldots t_{n_k-2}] \).

The above process can go on for \( k \) “layers”. In the end, we will have in hand a policy \( p_\infty \) that performs better than the concatenation of the \( p_{n_1} \)’s, which in turn performs better than the concatenation of the \( p_{n_1} \)’s, and so on and so forth, which can eventually be shown to be better than the original policy \( p \); and, for some \( s_\infty \in [t_0, t_\infty] \), this newly discovered policy lets the firm charge the low price on \([t_0, s_\infty] \) and the high price on \([s_\infty, t_\infty] \). Of course, the firm will remain out of stock on the interval \([t_\infty, t]\).

F. Proof of Proposition 4: Suppose \( I \) is an index set, and for every \( i \in I \), \( m_i = (m_i(t) \mid t \in [0, \bar{t}]) = (m_i^L(t), m_i^H(t) \mid t \in [0, \bar{t}]) \) is a member of \( M^0 \); that is, \( m_i^L(t) \) is decreasing and \( m_i^\infty(t) \equiv 1 - m_i^L(t) - m_i^H(t) \) is increasing, and both functions are Lipschitz continuous with coefficient \( \bar{f} \cdot \bar{\lambda}^L(0, 0) \). Let us consider \( m = \sup_{i \in I} m_i \).

By (3) and (4), we have \( m = (m(t) \mid t \in [0, \bar{t}]) = (m^L(t), m^H(t) \mid t \in [0, \bar{t}]) \), where at every

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\[ t \in [0, \bar{t}], \]
\[
m^L(t) = \inf_{i \in I} m^L_i(t), \quad m^\infty(t) \equiv 1 - m^L(t) - m^H(t) = \sup_{i \in I} m^\infty_i(t). \tag{76}\]

Since \([0,1]\) is a compact set, we know that, for every \(t \in [0, \bar{t}]\), both \(m^\infty(t)\) and \(m^L(t)\) are well defined with values in \([0,1]\). Apparently, we still have \(m^\infty(t) + m^L(t) \leq 1\) for every \(t \in [0, \bar{t}]\). Let us show the monotonicity and Lipschitz continuity of \(m^\infty(\cdot)\) first.

For \(t \in [0, \bar{t}]\) and \(\epsilon > 0\), we conclude from (76) that there exists some \(i \in I\) with \(m^\infty(t) - \epsilon \leq m^\infty_i(t) \leq m^\infty(t)\). Now, for any \(s \in [t, \bar{t}]\), we have

\[
m^\infty(s) \geq m^\infty_i(s) \geq m^\infty_i(t) \geq m^\infty(t) - \epsilon, \tag{77}\]

where the first inequality is due to the definition of \(m^\infty\), the second one is due to the increasingness of \(m^\infty_i\), and the last one is due to the choice of \(i \in I\). As \(\epsilon\) can be arbitrarily small, we have \(m^\infty(s) \geq m^\infty(t)\) for any \(s \geq t\). Hence, \(m^\infty(t)\) is increasing in \(t\).

For \(t, s \in [0, \bar{t}]\) with \(t < s\) and \(\epsilon > 0\), we conclude from (76) that there exists some \(i \in I\) with \(m^\infty(s) - \epsilon \cdot (s - t) \leq m^\infty_i(s) \leq m^\infty(s)\). Now, we have

\[
m^\infty(s) - m^\infty(t) \leq m^\infty_i(s) - m^\infty_i(t) + \epsilon \cdot (s - t) \leq (\tilde{f} \cdot \check{\lambda}_L(0,0) + \epsilon) \cdot (s - t), \tag{78}\]

where the first inequality is due to the choice of \(i \in I\) and the fact that \(m^\infty(t) \geq m^\infty_i(t)\), and the second one is due to the fact that \(m^\infty_i\) is Lipschitz continuous with coefficient \(\tilde{f} \cdot \check{\lambda}_L(0,0)\). As \(\epsilon\) can be arbitrarily small, we have \(m^\infty(s) - m^\infty(t) \leq \tilde{f} \cdot \check{\lambda}_L(0,0) \cdot (s - t)\) for any \(s > t\). Combined with the above, we see that \(m^\infty(t)\) is Lipschitz continuous with coefficient \(\tilde{f} \cdot \check{\lambda}_L(0,0)\).

We can similarly show that \(m^L(t)\) is increasing and Lipschitz continuous with coefficient \(\tilde{f} \cdot \check{\lambda}_L(0,0)\). Therefore, we can establish \(m = \sup_{i \in I} m_i \in \mathcal{M}^0\). We can symmetrically show \(\inf_{i \in I} m_i \in \mathcal{M}^0\). Since \((m_i \mid i \in I)\) may be any subset of \(\mathcal{M}^0\), the latter must be a complete lattice.

\[ \blacksquare \]

**G. Proof of Proposition 5:** To differentiate left- and right-hand sides, we rewrite (15) as

\[
\begin{align*}
(Z^M_M(m))^L(t) &= 1 - F(\int_0^t \check{\lambda}_L(m(s)) \cdot ds + \int_t^\bar{t} \check{\lambda}_H(m(s)) \cdot ds), \\
(Z^M_M(m))^H(t) &= 1 - (Z^M_M(m))^L(t) - F(\int_0^t \check{\lambda}_H(m(s)) \cdot ds).
\end{align*}
\tag{79}\]

Both \((Z^M_M(m))^L(t)\) and \((Z^M_M(m))^H(t)\) are between 0 and 1, with \((Z^M_M(m))^L(t) + (Z^M_M(m))^H(t) \leq 1\). Due to (D1) and the fact that the cdf \(F(x)\) is increasing, both \((Z^M_M(m))^L(t)\) and \((Z^M_M(m))^L(t) + (Z^M_M(m))^H(t)\) are decreasing in \(t\). They are also Lipschitz continuous with coefficient \(\tilde{f} \cdot \check{\lambda}_L(0,0)\), as \(\tilde{f}\) bounds the pdf \(d_x F(x)\) and \(\check{\lambda}_L(0,0)\) bounds the rates \(\check{\lambda}^{(H)}(m^L, m^H)\).
Suppose \( m^1, m^2 \in \mathcal{M}^0 \) with \( m^1 \leq m^2 \). Due to (D5), we have, for every \( t \in [0, \bar{t}] \),

\[
\bar{\chi}^{(H)}(m^1(t)) \leq \bar{\chi}^{(H)}(m^2(t)).
\]  

(80)

By this, the fact that the cdf \( F(x) \) is increasing, and (79), we have, for every \( t \in [0, \bar{t}] \),

\[
\begin{cases}
(Z_M^M(m^1))^L(t) \geq (Z_M^M(m^2))^L(t), \\
(Z_M^M(m^1))^H(t) + (Z_M^M(m^1))^H(t) \geq (Z_M^M(m^2))^H(t) + (Z_M^M(m^2))^H(t).
\end{cases}
\]

(81)

But by (3) and (4), this is exactly the definition for \( Z_M^M(m^1) \leq Z_M^M(m^2) \).

\[ \blacksquare \]

**H. A Recursive Procedure that Facilitates \( Z^P_B \):** We know that the constant \((\bar{p}^L\bar{\alpha}^L - \bar{p}^H\bar{\alpha}^H)/(\bar{\alpha}^L - \bar{\alpha}^H)\) is well-defined and strictly positive by (S1) and (S2). Let \( n = 1, 2, \ldots, \bar{n} \). By the fact that \( v_{n-1}(\bar{t}) = v_n(\bar{t}) = 0 \), we know that \( \tau_n \) cannot be \( \bar{t} \) and hence must be in \([0, \bar{t}]\). When \( t \in [0, \tau_n) \), the second term inside the maximization of (28) dominates the first term, while when \( t \in (\tau_n, \bar{t}] \), the opposite happens. Therefore, (28) now tells the following:

\[
d_t v_n(t) + \bar{\alpha}^L \beta(t) \cdot (\bar{p}^L + v_{n-1}(t)) - \bar{\alpha}^L \beta(t) \cdot v_n(t) = 0, \quad \forall t \in (\tau_n, \bar{t});
\]

(82)

and

\[
v_n(\tau_n) = v_{n-1}(\tau_n) + \frac{\bar{p}^L\bar{\alpha}^L - \bar{p}^H\bar{\alpha}^H}{\bar{\alpha}^L - \bar{\alpha}^H};
\]

(83)

Using conclusions revolving around (22) and (23) in a time-reversed way and the fact that \( v_n(\bar{t}) = 0 \), we can solve the above to obtain the following: for \( t \in [\tau_n, \bar{t}] \),

\[
v_n(t) = \int_t^\bar{t} \bar{\alpha}^L \beta(s) \cdot (\bar{p}^L + v_{n-1}(s)) \cdot \exp(-\int_t^s \bar{\alpha}^L \beta(u)du) \cdot ds.
\]

(85)

Using integration by parts and the boundary condition \( v_{n-1}(\bar{t}) = 0 \), we can derive from (85) that,

\[
v_n(t) = \int_t^\bar{t} (\bar{p}^L + v_{n-1}(s)) \cdot \partial_s (-\exp(-\int_t^s \bar{\alpha}^L \beta(u)du)) \cdot ds
\]

\[
= (\bar{p}^L + v_{n-1}(s)) \cdot (-\exp(-\int_t^s \bar{\alpha}^L \beta(u)du))(|_{s=\bar{t}} - |_{s=t})
\]

\[
- \int_t^\bar{t} d_t v_{n-1}(s) \cdot (-\exp(-\int_t^s \bar{\alpha}^L \beta(u)du)) \cdot ds
\]

\[
= v_{n-1}(t) + v_n^L(t),
\]

(86)

where

\[
v_n^L(t) = \bar{p}^L \cdot (1 - \exp(-\int_t^\bar{t} \bar{\alpha}^L \beta(s) \cdot ds)) + \int_t^\bar{t} d_t v_{n-1}(s) \cdot \exp(-\int_t^s \bar{\alpha}^L \beta(u)du) \cdot ds.
\]

(87)
By Fact 4, \( v^L_n(t) \) must be decreasing in \( t \) for \( t \in [\tau_n, \bar{\ell}] \); and, according to (29), \( \tau_n \) is the first time that \( v^L_n(t) \geq (\hat{p}^L \hat{\alpha}^L - \hat{p}^H \hat{\alpha}^H)/ (\tilde{\alpha}^L - \tilde{\alpha}^H) \) as \( t \) moves leftward from \( \bar{\ell} \) on. When \( v^L_n(0) \geq (\hat{p}^L \hat{\alpha}^L - \hat{p}^H \hat{\alpha}^H)/ (\tilde{\alpha}^L - \tilde{\alpha}^H) \), it will happen that

\[
v^L_n(\tau_n) = \frac{\hat{p}^L \hat{\alpha}^L - \hat{p}^H \hat{\alpha}^H}{\tilde{\alpha}^L - \tilde{\alpha}^H}.
\]  

(88)

Otherwise, we need to force \( \tau_n = 0 \).

To move on to \( \tau_{n+1} \), we need to know values of \( v_n(t) \) for \( t \in [0, \tau_n] \) as well. For this, we can use results revolving around (22) and (23) in a time-reversed fashion to solve (84). Then, for \( t \in [0, \tau_n] \), we have

\[
v_n(t) = v_n(\tau_n) \cdot \exp(- \int_{\tau_n}^t \tilde{\alpha}^H \beta(s) \cdot ds) + \int_{\tau_n}^t \tilde{\alpha}^H \beta(s) \cdot (\hat{p}^H + v_{n-1}(s)) \cdot \exp(- \int_{t}^s \lambda^H \beta(u)du) \cdot ds. \]

(89)

Given actual stream \( \beta \), we first have \( v_0(t) = 0 \) for any \( t \in [0, \bar{\ell}] \). Suppose for some \( n = 1, 2, \ldots, \bar{n} \), we have obtained the \( v_{n-1} \) function. Then, we can use (86) and (87) to obtain \( v_n(t) \) for any \( t \) that is large enough for (88) not to be true. We may use (88) to determine \( \tau_n \). For any \( t \) that is below \( \tau_n \), we can use (89) to determine \( v_n(t) \).

I. Details concerning the Operator \( Z^F_n \): We can rewrite the stock-distribution evolution equation (33) as

\[
d_tf_n(t) = -\lambda_n(t) \cdot f_n(t) + \lambda_{n+1}(t) \cdot f_{n+1}(t), \quad \forall n = 1, 2, \ldots, \bar{n}, \ t \in (0, \bar{\ell}).
\]  

(90)

Actually, we can use conclusions related to (22) and (23) to solve (90) in order to obtain an iterative relation between \( f_n(\cdot) \) and \( f_{n+1}(\cdot) \):

\[
f_n(t) = f_n(0) \cdot \exp(- \int_0^t \lambda_n(s) \cdot ds) + \int_0^t f_{n+1}(s) \cdot \lambda_{n+1}(s) \cdot \exp(- \int_s^t \lambda_n(u)du) \cdot ds.
\]  

(91)

We can exploit (91) to obtain closed-form solutions to the \( f_n(\cdot) \)'s. Given sequence \( (a_n \mid n = 1, 2, \ldots) \) of piecewise continuous functions on \([0, \bar{\ell}]\) and real number \( \bar{r} \in [0, \bar{\ell}] \), we may define sequence \( (G_n[a_1, \ldots, a_n, r] \mid n = 1, 2, \ldots) \) of continuous functions on \([\bar{r}, \bar{\ell}]\), by letting

\[
G_1[a_1, r](t) = \exp(- \int_r^t a_1(s) \cdot ds), \quad \forall t \in [\bar{r}, \bar{\ell}];
\]  

(92)

and, iteratively, for \( n = 1, 2, \ldots \) and \( t \in [\bar{r}, \bar{\ell}] \),

\[
G_{n+1}[a_1, \ldots, a_n, a_{n+1}, r](t) = \int_r^t G_n[a_1, \ldots, a_n, r](s) \cdot a_n(s) \cdot \exp(- \int_s^t a_{n+1}(u)du) \cdot ds.
\]  

(93)

\[12\]
From (92) and (93), we can use induction to deduce that

\[
G_n[a_1, ..., a_n, r](t) = \int_t^s ds_{n-1} \cdot f_{s_{n-1}}^0 ds_{n-2} \cdots f_{s_2}^0 ds_1 \cdot a_1(s_1) \cdot a_2(s_2) \cdots a_{n-1}(s_{n-1}) \times \\
\times \exp(-\int_r^s a_1(u)du_1 + \sum_{q=2}^{n-1} \int_{s_{q-1}}^{s_q} a_q(u)du_q + \int_{s_{n-1}}^{t} a_n(u)du_n)).
\]

(94)

Note that (91) can be expressed as

\[
f_n(t) = f_n(0) \cdot G_1[\lambda, 0](t) + \int_0^t f_{n+1}(s) \cdot \lambda_{n+1}(s) \cdot \exp(-\int_s^t \lambda_n(u)du) \cdot ds.
\]

(95)

So, from (93) and (95), we may use an induction from \( n = \bar{n} \) down to \( n = 1 \) to show that

\[
f_n(t) = f_n(0) \cdot G_1[\lambda, 0](t) + f_{n+1}(0) \cdot G_2[\lambda_{n+1}, \lambda_n, 0](t) + \cdots \\
= \sum_{l=0}^{\bar{n}-n} f_{n+l}(0) \cdot G_{l+1}[\lambda_{n+l}, \lambda_{n+l-1}, ..., \lambda_n, 0](t).
\]

(96)

We can write this solution as \( f = Z_{\mathbb{A}}^F(\lambda) \).

**J. Continuity-based Derivations of Section 3.4:** For any strictly positive integer \( k \), we introduce norm \( || \cdot ||^k \) for the linear \( k \)-dimensional real space \( \mathbb{R}^k \), so that for any \( x = (x_{k'} \mid k' = 1, 2, ..., k) \in \mathbb{R}^k \),

\[
|| x ||^k = \max_{k' = 1}^{k} | x_{k'} |.
\]

(97)

Of course, \( || \cdot ||^1 \) is simply \( | \cdot | \). When left and right limits \( \bar{t}^L \) and \( \bar{t}^R \) are clear from the context, we may introduce the norm \( || \cdot ||^k_\infty \) for \( \mathbb{R}^k \)-valued functions \( g \) defined on \( [\bar{t}^L, \bar{t}^R] \), so that

\[
|| g ||^k_\infty = \sup_{t \in [\bar{t}^L, \bar{t}^R]} || g(t) ||^k.
\]

(98)

For convenience, we may denote \( || \cdot ||^1_\infty \) by \( || \cdot ||_\infty \).

Let \( X^k \) be the space of \( \mathbb{R}^k \)-valued functions defined on \( [-\theta, \bar{t}] \) that are right continuous with left limits, and let \( Y^k \) be the space of \( \mathbb{R}^k \)-valued functions on \( [0, \bar{t}] \) that are continuous. We may identify any \( f \in Y^k \) as a member of \( X^k \) by agreeing on the convention that \( f(t) = f(0) \) for any \( t \in [-\theta, 0] \).

We also have the occasion to use the Skorohod topology. Let \( K \) denote the class of strictly increasing, continuous mappings of \( [-\theta, \bar{t}] \) onto itself, and let \( I \) denote the identity map on \( [-\theta, \bar{t}] \). For \( f, g \in X^k \), we define

\[
\rho^k(f, g) = \inf_{l \in K} \{ || l - I ||_\infty \lor || f - g \circ l ||^k_\infty \}.
\]

(99)

Here, \( g \circ l \) stands for the composition of the two functions \( g \) and \( l \), whose value is \( g(l(t)) \) at any \( t \in [-\theta, \bar{t}] \). We can think of \( l \) as a time-rescaling function to be used so that values \( f(t) \) and \( g(l(t)) \)
can be more closely matched. Note that $\rho^k$ is a metric for $X^k$, which in turn induces the Skorohod topology.

Among the five processes, the arrival-rate stream $\lambda$, market process $m$, and instantaneous stream $\tilde{\beta}$ are merely piecewise continuous, while the actual stream $\beta$ and stock-distribution process $f$ are continuous. We shall define piecewise continuous processes on $[-\theta, \bar{t}]$ and continuous processes on $[0, \bar{t}]$.

To this end, we need to make slight extensions to earlier definitions. Now, the definition of the market-to-instantaneous-stream operator $Z^T_M$ relies on (26) for $t \in [-\theta, \bar{t}]$, where $m(t) = m(0)$ results in $\tilde{\beta}(t) = \tilde{\beta}(0)$ for $t \in [-\theta, 0)$; the definition of the instantaneous-to-actual-stream operator $Z^B_M$ relies on (27) for $t \in [0, \bar{t}]$; the definition of the actual-stream-to-policy operator $Z^P_B$ relies on the process from (86) to (89); the definition of the policy-stock-distribution-to-market operator $Z^M_{PF}$ relies on (35) for $t \in [0, \bar{t}]$ and the convention that $m(t) = m(0)$ for $t \in [-\theta, 0)$; the definition of the policy-actual-stream-to-arrival operator $Z^A_{PB}$ relies on (36) for $t \in [0, \bar{t}]$ and the convention that $\lambda(t) = \lambda(0)$ for $t \in [-\theta, 0)$; and, the definition of the arrival-to-stock-distribution operator $Z^F_A$ relies on (96) for $t \in [0, \bar{t}]$.

To facilitate processes related to arrival-rate streams, let $X^k_A$ be the subset of $X^k$, with each of whose members $a = (a(t) | t \in [-\theta, \bar{t}]) = (a_1(t), ..., a_k(t) | t \in [-\theta, \bar{t}])$ satisfying the following: (A-I) $a$ has at most $n$ discontinuities, (A-II) $a(t) = a(0)$ for $t \in [-\theta, 0)$, and (A-III) $a(t) \in [0, \alpha L \beta^0]^k$ for every $t \in [-\theta, \bar{t}]$, where $\beta^0$ is defined in (20).

To facilitate market processes, let $X^2_M$ be the subset of $X^2$, with each of whose members $m = (m(t) | t \in [-\theta, \bar{t}]) = (m_L(t), m_H(t) | t \in [-\theta, \bar{t}])$ satisfying the following: (M-I) $m$ has at most $\bar{n}$ discontinuities, (M-II) $m(t) = m(0)$ for $t \in [-\theta, 0)$, and (M-III) for every $t \in [-\theta, \bar{t}]$, the value $m(t)$ is inside $\Delta^M$, the subset of $R^2$ defined at the time when assumption (S3) is introduced.

To facilitate instantaneous streams, let $X^1_B$ be the subset of $X$, with each of whose members $\tilde{\beta} = (\tilde{\beta}(t) | t \in [-\theta, \bar{t}])$ satisfying the following: (B'-I) $\tilde{\beta}$ has at most $\bar{n}$ discontinuities, (B'-II) $\tilde{\beta}(t) = \tilde{\beta}(0)$ for $t \in [-\theta, 0)$, and (B'-III) $\tilde{\beta}(t) \in [\underline{\beta}^0, \overline{\beta}^0]$ for every $t \in [-\theta, \bar{t}]$.

To facilitate stock-distribution processes, let $Y^\bar{n}_F$ be the subset of $Y^\bar{n}$, with each of whose members $f = (f(t) | t \in [0, \bar{t}])$ satisfying the following: (F-I) for every $t \in [0, \bar{t}]$, the value $f(t)$ is in $\Delta^F$, the set defined in (34).

To facilitate actual streams, let $Y_B$ be the subset of $Y$, with each of whose members $\beta = (\beta(t) | t \in [0, \bar{t}])$ satisfying the following: (B-I) $\beta(t) \in [\underline{\beta}^0, \overline{\beta}^0]$ for every $t \in [0, \bar{t}]$, and (B-II) $\beta$ is Lipschitz continuous with coefficient $2\beta^0/\theta$. 

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It can be verified that actual streams $\beta = (\beta(t) \mid t \in [0, \bar{t}])$ come from $Y_B$; pricing policies $p = (\tau_n \mid n = 1, 2, ..., \bar{n})$ come from $\Delta^P$, where $\Delta^P = \{(\tau_1, ..., \tau_{\bar{n}}) \in [0, \bar{t}]^{\bar{n}} \mid 0 \leq \tau_{\bar{n}} \leq \cdots \leq \tau_1 \leq \bar{t}\}$ is a simplex in $R^{\bar{n}}$; arrival-rate streams $\lambda = (\lambda_n(t) \mid n = 1, 2, ..., \bar{n}, t \in [-\theta, \bar{t}])$ come from $X_{\bar{n}}^\lambda$; stock-distribution processes $f = (f_n(t) \mid n = 1, 2, ..., \bar{n}, t \in [0, \bar{t}])$ come from $Y_{\bar{n}}^f$; market processes $m = (m(t) \mid t \in [-\theta, \bar{t}])$ come from $X_M^2$; and, instantaneous streams $\tilde{\beta} = (\tilde{\beta}(t) \mid t \in [\theta, \bar{t}])$ come from $X_{B'}^\theta$.

In particular, we can verify that $Z_{B'}^B$ is a map from $X_{B'}$ to $Y_B$. Suppose $\tilde{\beta} \in X_{B'}$ is given. For any $t \in [0, \bar{t})$ and $\Delta t \in (0, (\bar{t} - t) \wedge \theta)$, we have, by (27),

$$
\frac{|(Z_{B'}^B(\tilde{\beta}))(t + \Delta t) - (Z_{B'}^B(\tilde{\beta}))(t)|}{\Delta t} = \int_{t-\Delta t}^{t+\Delta t} \beta(s) \cdot ds - \int_{t}^{t+\Delta t} \beta(s) \cdot ds \leq 2 \beta \beta \int \Delta t / \theta \leq 2 \beta \beta \Delta t / \theta,
$$

which entails that $Z_{B'}^B(\beta)$ is Lipschitz continuous with coefficient $2 \beta \beta / \theta$.

As said, we may define composite operator $Z_{B}^B$ through (37). We now embark on showing the existence of an inventory- and time-monotone pricing policy $p^\theta$ through showing the existence of a fixed point to operator $Z_{B}^B$. With the help of a continuity-based fixed point theorem, we achieve this mainly by showing the continuity of $Z_{B}^B$ as a map from $(Y_B, || \cdot ||_\infty)$ to itself. The latter we accomplish through showing the various continuity properties of the operator’s components. In our proofs, we will use $\bar{x}^0$ to denote a strictly positive constant, such that, for any $x \in [0, \bar{x}^0]$, it follows that $\exp(x/2) - 1 \leq x$.

Now we embark on showing the continuity of the various components of $Z_{B}^B$ as shown in (37). First, we can show the continuity of $Z_{M}^B$.

**Lemma 4** The operator $Z_{B'}^B$ is uniformly continuous from $(X_M^2, \rho^2)$ to $(X_{B'}^\theta, \rho)$.

To show that $Z_{B'}^B$ is continuous, we need only follow the proof of Lemma 6 in Yang, Xia, and Qi (2007).

**Lemma 5** The operator $Z_{B'}^B$ is continuous from $(X_{B'}^\theta, \rho)$ to $(Y_B, || \cdot ||_\infty)$.

To show the continuity of $Z_{B}^B$ requires quite some preparation. First, due in large part to (S4), we have the following important result.

**Lemma 6** Each $v_n$ is continuously differentiable. In addition, more than Fact 4, each $d_tv_n(t) - d_tv_{n-1}(t)$ is strictly negative.
From the proof of Lemma 6, we may learn that \( \tau_n \) is determined by the \( d_tv_{n-1} \) function: it is the unique solution for
\[
v_n^L(\tau_n) = \int_{\tau_n}^t (\bar{p}^L \bar{\alpha}^L \beta(s) + d_tv_{n-1}(s)) \cdot \exp(-\int_{\tau_n}^s \bar{\alpha}^L \beta(u) du) \cdot ds = \frac{\bar{p}^L \bar{\alpha}^L - \bar{p}^H \bar{\alpha}^H}{\bar{\alpha}^L - \bar{\alpha}^H}, \tag{101}
\]
when the solution exists in \((0, \bar{t})\); otherwise, \( v_n^L(0) < (\bar{p}^L \bar{\alpha}^L - \bar{p}^H \bar{\alpha}^H)/(\bar{\alpha}^L - \bar{\alpha}^H) \) and \( \tau_n = 0 \). From the same proof, we may also establish an iterative relationship between \( d_tv_n \) and \( d_tv_{n-1} \):
\[
d_tv_n(t) = -\bar{p}^L \bar{\alpha}^L \cdot \beta(t) + \bar{\alpha}^L \beta(t) \cdot \int_t^\bar{t} (\bar{p}^L \bar{\alpha}^L \beta(s) + d_tv_{n-1}(s)) \cdot \exp(-\int_t^s \bar{\alpha}^L \beta(u) du) \cdot ds,
\]
when \( t \in (\tau_n, \bar{t}) \), and
\[
d_tv_n(t) = -\bar{p}^H \bar{\alpha}^H \cdot \beta(t) + \bar{\alpha}^H \beta(t) \cdot \int_t^{\tau_n} (\bar{p}^H \bar{\alpha}^H \beta(s) + d_tv_{n-1}(s)) \cdot \exp(-\int_t^{\tau_n} \bar{\alpha}^H \beta(u) du) \cdot ds
+ \bar{\alpha}^H \beta(t) \cdot \exp(-\int_t^{\tau_n} \bar{\alpha}^H \beta(s) ds) \cdot v_n^L(\tau_n),
\]
when \( t \in (0, \tau_n) \).

Now, we may denote the process (101) from a given pair \((d_tv_{n-1}, \beta)\) to a \( \tau_n \) by \( \tau_n[d_tv_{n-1}, \beta] \). Relying on Lemma 6, we can derive the continuity of the \( \tau_n \) function.

**Lemma 7** The function \( \tau_n[d_tv_{n-1}, \beta] \) is jointly continuous in \((d_tv_{n-1}, \beta)\) as a mapping from \((Y, || \cdot ||_\infty) \times (Y_B, || \cdot ||_\infty)\) to \(([0, \bar{t}], | \cdot |)\).

We may also denote the process involving (101), (102), and (103), from a given pair \((d_tv_{n-1}, \beta)\) to the triplet \((\tau_n[d_tv_{n-1}, \beta], d_tv_{n-1}, \beta)\) and then from the triplet to a \( d_tv_n \), by \( d_tv_n[d_tv_{n-1}, \beta] \). For this process, we have the following continuity result that is related to Lemma 7.

**Lemma 8** The function \( d_tv_n[d_tv_{n-1}, \beta] \) is jointly continuous in \((d_tv_{n-1}, \beta)\) as a mapping from \((Y, || \cdot ||_\infty) \times (Y_B, || \cdot ||_\infty)\) to \((Y, || \cdot ||_\infty)\).

Joining Lemmas 7 and 8, we may finally obtain the desired continuity of \( Z^P_B \).

**Lemma 9** The \( Z^P_B \) operator is continuous from \((Y_B, || \cdot ||_\infty)\) to \((\Delta^P, || \cdot ||^p)\).

We show the continuity of \( Z^M_F \) and \( Z^A_{PB} \) in the following two lemmas.

**Lemma 10** The operator \( Z^M_F \), as a map from \((\Delta^P, || \cdot ||^p) \times (Y^p_B, || \cdot ||^p_\infty)\) to \((X^p_M, \rho^2)\), is uniformly continuous in \( p \) at every \( f \in Y^p_B \) and Lipschitz continuous in \( f \) with a coefficient independent of \( p \in \Delta^P \).
Lemma 11 The operator \( Z^F_{PB} \), as a map from \( (\Delta^P, \| \cdot \|_n) \times (Y_B, \| \cdot \|_\infty) \) to \( (X_A^n, \rho^n) \), is uniformly continuous in \( p \) at every \( \beta \in Y_B \) and Lipschitz continuous in \( \beta \) with a coefficient independent of \( p \in \Delta^P \).

The proof of \( Z^F_{A} \)'s continuity is again quite involved. First, we need to establish an intermediate result for the \( G_n[a_1, ..., a_n, r] \) functions defined around (92) and (93).

Lemma 12 \( G_n[a_1, ..., a_n, r](t) \) is uniformly continuous in \( t \) in an \( (a_1, ..., a_n, r) \)-equal fashion when \( (a_1, ..., a_n) \in X_A^n \). Also, there is a uniform upper bound say \( \bar{g}_n \), for all the \( G_n[a_1, ..., a_n, r] \)'s. Especially, we may let \( \bar{g}_1 = 1 \).

Then, using Lemma 12, we can show the vital continuity of \( G_n[a_1, ..., a_n, 0] \) in \( (a_1, ..., a_n) \).

Lemma 13 As a function from \( (X_A^n, \rho^n) \) to \( (Y, \| \cdot \|_\infty) \), \( G_n[a_1, ..., a_n, 0] \) is continuous in \( (a_1, ..., a_n) \).

Using (96) and Lemma 13, we can immediately reach the continuity of \( Z^F_{A} \).

Lemma 14 The operator \( Z^F_{A} \) is continuous from \( (X_A^n, \rho^n) \) to \( (Y_F^n, \| \cdot \|_\infty^n) \).

Now we are done with showing the continuity of the various components of \( Z^B_{PB} \). So, joining (37) with Lemmas 4, 5, 9, 10, 11, and 14, we can obtain the continuity of \( Z^B_{PB} \).

Proposition 6 The operator \( Z^B_{PB} \) is continuous from \( (Y_B, \| \cdot \|_\infty) \) to \( (Y_B, \| \cdot \|_\infty) \).

We can follow the proof of Proposition 4 in Yang, Xia, and Qi (2007), which relies on the Ascoli-Arzelà Theorem to show that \( Y_B \) is compact due to its construction through (B-I) and (B-II).

Proposition 7 \( Y_B \) is a nonempty, compact, convex subset of \( (Y, \| \cdot \|_\infty) \).

As \( (Y, \| \cdot \|_\infty) \) is a normed space, we certainly have the following.

Proposition 8 \( (Y, \| \cdot \|_\infty) \) is a locally convex linear topological space.

Now, combining (37), Propositions 6, 7, and 8, as well as the Kakutani-Glicksberg-Fan fixed point theorem, we can show the ultimate equilibrium-existence result Theorem 2.
K. Proof of Lemma 4: By \((S3)\) and the fact that \(\Delta^M\) is a compact set, we know that \(\tilde{\beta}\) is uniformly continuous in \((m^L, m^H)\). So for any \(\varepsilon > 0\), there exists \(\delta^B_M(\varepsilon) > 0\), such that for any \((m^L, m^H), (m'^L, m'^H) \in \Delta^M\) satisfying \(|m^L - m'^L| \vee |m^H - m'^H| < \delta^B_M(\varepsilon)\), we will have

\[
|\tilde{\beta}(m^L, m^H) - \tilde{\beta}(m'^L, m'^H)| < \varepsilon. \tag{104}
\]

For any \(m = (m^L(t), m^H(t)) \mid t \in [-\theta, \bar{t}], m' = (m'^L(t), m'^H(t)) \mid t \in [-\theta, \bar{t}) \in X^2_M\) satisfying \(\rho^2(m, m') < (\delta^B_M(\varepsilon) \wedge \varepsilon)/2\), we can find a time-rescaling function \(l\) such that

\[
\| l - I \|_\infty < \varepsilon, \tag{105}
\]

and for every \(t \in [-\theta, \bar{t}]\),

\[
|m^L(t) - m'^L(l(t))| \vee |m^H(t) - m'^H(l(t))| < \delta^B_M(\varepsilon). \tag{106}
\]

But according to claims revolving around \((104)\), the above \((106)\) will lead to

\[
|\tilde{\beta}(m^L(t), m^H(t)) - \tilde{\beta}(m'^L(l(t)), m'^H(l(t)))| < \varepsilon, \tag{107}
\]

for every \(t \in [-\theta, \bar{t}]\). Combining the definition of \(Z^B_M\) through \((26)\), as well as, \((105)\) and \((107)\), we see that \(\rho(Z^B_M(m), Z^B_M(m')) < \varepsilon\).

L. Proof of Lemma 6: By the comment after \((23)\) and the continuity of the \(\beta\) process, we know that \(v_n\) is continuously differentiable in \((\tau_n, \bar{t})\) as well as in \((0, \tau_n)\). Thus, for the first part, we need only show that \(d_t v_n(\tau_n^+) = d_t v_n(\tau_n^-)\). We first use the same trick used in \((86)\) to obtain from \((89)\) that, for \(t \in [0, \tau_n]\),

\[
v_n(t) = v_{n-1}(t) + v^H_n(t), \tag{108}
\]

where

\[
v^H_n(t) = \bar{p}^H \cdot (1 - \exp(-\int_{\tau_n}^{\bar{t}} \bar{\alpha}^H \beta(s) \cdot ds)) + \int_{\tau_n}^{\bar{t}} d_t v_{n-1}(s) \cdot \exp(-\int_t^s \bar{\alpha}^H \beta(u) du) ds + \exp(-\int_{\tau_n}^{\bar{t}} \bar{\alpha}^H \beta(s) ds) \cdot (v_n(\tau_n) - v_{n-1}(\tau_n)). \tag{109}
\]

Note that we can rewrite \((88)\) as

\[
v^L_n(\tau_n) = v^H_n(\tau_n) = v_n(\tau_n) - v_{n-1}(\tau_n) = \frac{\bar{p}^L \bar{\alpha}^L - \bar{p}^H \bar{\alpha}^H}{\bar{\alpha}^L - \bar{\alpha}^H}. \tag{110}
\]

Now, according to \((82)\) and \((84)\), we have, for \(t \in (\tau_n, \bar{t})\),

\[
d_t v_n(t) = \bar{\alpha}^L \beta(t) \cdot (v^L_n(t) - \bar{p}^L), \tag{111}
\]
and for $t \in (0, \tau_n)$,
\[ d_t v_n(t) = \alpha^H \beta(t) \cdot (v_n^H(t) - \bar{p}^H). \]  
(112)

In view of (110), we have
\[ d_t v_n(\tau_n^+) = \bar{\alpha}^L \beta(\tau_n) \cdot (v_n^L(\tau_n) - \bar{p}^L) = -\beta(\tau_n) \cdot \bar{\alpha}^L \alpha^H \cdot (\bar{p}^H - \bar{p}^L)/(\bar{\alpha}^L - \alpha^H) \]
\[ = \bar{\alpha}^H \beta(\tau_n) \cdot (v_n^H(\tau_n) - \bar{p}^H) = d_t v_n(\tau_n^-). \]  
(113)

Hence, $v_n$ is continuously differentiable.

For the second part, we may first obtain for function $c(t^1, t^2) = \exp(-\int_{t^1}^{t^2} a(s)ds)$ the identity that
\[ \int_{t^1}^{t^2} a(s) \cdot \exp(-\int_{t^1}^{t^2} a(u)du) \cdot ds = -\int_{t^1}^{t^2} \partial_{t^2} c(t^1, s) \cdot ds \]
\[ = c(t^1, t^1) - c(t^1, t^2) = 1 - \exp(-\int_{t^1}^{t^2} a(s)ds). \]  
(114)

Applying this to (87) and (109), we obtain, for $t \in [\tau_n, \bar{t}]$
\[ v_n^L(t) = \int_t^{\bar{t}} (\bar{p}^L \bar{\alpha}^L \beta(s) + d_t v_{n-1}(s)) \cdot \exp(-\int_t^s \bar{\alpha}^L \beta(u)du) \cdot ds, \]  
(115)

and for $t \in [0, \tau_n]$
\[ v_n^H(t) = \int_t^{\tau_n} (\bar{p}^H \bar{\alpha}^H \beta(s) + d_t v_{n-1}(s)) \cdot \exp(-\int_t^s \bar{\alpha}^H \beta(u)du) \cdot ds 
+ \exp(-\int_t^{\tau_n} \bar{\alpha}^H \beta(ds) \cdot (v_n(\tau_n) - v_{n-1}(\tau_n)) 
= \int_t^{\tau_n} (\bar{p}^H \bar{\alpha}^H \beta(s) + d_t v_{n-1}(s)) \cdot \exp(-\int_t^s \bar{\alpha}^H \beta(u)du) \cdot ds 
+ \exp(-\int_t^{\tau_n} \bar{\alpha}^H \beta(ds) \cdot \int_t^{\bar{t}} (\bar{p}^L \bar{\alpha}^L \beta(s) + d_t v_{n-1}(s)) \cdot \exp(-\int_t^s \bar{\alpha}^L \beta(u)du) \cdot ds. \]  
(116)

Now we proceed by induction. From (110), (111), (112), (114), (115), and (116), as well as the fact that $v_0(\cdot) = 0$, we know that,
\[ d_t v_1(t) = \begin{cases} 
-\bar{p}^L \bar{\alpha}^L \beta(t) \cdot \exp(-\int_t^s \bar{\alpha}^L \beta(u)du) \cdot ds, & \forall t \in (\tau_1, \bar{t}), \\
-\bar{\alpha}^L \bar{\alpha}^H \beta(t) \cdot (\bar{p}^H - \bar{p}^L) \cdot \exp(-\int_t^{\tau_1} \bar{\alpha}^H \beta(ds) \cdot (\bar{\alpha}^L - \bar{\alpha}^H), & \forall t \in (0, \tau_1). 
\end{cases} \]  
(117)

By (S4), we see that $d_t v_1(t) - d_t v_0(t) = d_t v_1(t)$ is strictly negative for $t \in (0, \bar{t})$.

For some $n = 1, 2, \ldots$, suppose $d_t v_n(t) - d_t v_{n-1}(t)$ is known to be strictly negative for $t \in (0, \bar{t})$.

Note that $\tau_{n+1} \leq \tau_n$. Now, by combining (111), (112), (115), and (116), we can derive that, for $t \in (\tau_n, \bar{t})$,
\[ d_t v_{n+1}(t) - d_t v_n(t) = \bar{\alpha}^L \beta(t) \cdot \int_t^{\bar{t}} (d_t v_n(s) - d_t v_{n-1}(s)) \cdot \exp(-\int_t^s \bar{\alpha}^L \beta(u)du) \cdot ds, \]  
(118)

which is strictly negative due to (S4) and the induction hypothesis. For $t \in (0, \tau_{n+1})$, we have
\[ d_t v_{n+1}(t) - d_t v_n(t) = \bar{\alpha}^H \beta(t) \cdot (\int_t^{\tau_{n+1}} (d_t v_n(s) - d_t v_{n-1}(s)) \cdot \exp(-\int_t^s \bar{\alpha}^H \beta(u)du) \cdot ds 
+ \exp(-\int_t^{\tau_{n+1}} \bar{\alpha}^H \beta(ds) \cdot (v_{n+1}(\tau_{n+1}) - v_n^H(\tau_{n+1})). \]  
(119)
which is strictly negative due to (S4), the induction hypothesis, and that
\[ v_{n+1}^H(\tau_{n+1}) - v_n^H(\tau_{n+1}) = v_{n+1}(\tau_{n+1}) - 2v_n(\tau_{n+1}) + v_{n-1}(\tau_{n+1}) \leq 0, \]  
which is attributable to Fact 3, that \( v_n(\cdot) \) is concave in \( n \).

As \( \tau_n \) is strictly less than \( \bar{\ell} \), we can see from (118) that \( d_t v_{n+1}(\tau_{n+1}^+) - d_t v_n(\tau_{n+1}^+) \) is strictly negative, and hence, due to the continuous differentiability of the \( v_n \) and \( v_{n+1} \) functions, so is \( d_t v_{n+1}(\tau_n) - d_t v_n(\tau_n) \).

We will be done if \( \tau_{n+1} = \tau_n \). Suppose otherwise, \( \tau_{n+1} < \tau_n \). Due to the definitions of \( \tau_n \) and \( \tau_{n+1} \), we have, for any \( t \in (\tau_{n+1}, \tau_n) \),
\[ v_{n+1}^L(t) < \frac{\tilde{p}^L \alpha^L - \tilde{p}^H \alpha^H}{\hat{\alpha}^L - \hat{\alpha}^H} \leq v_n^H(t). \]  
In view of (111) and (112), we have
\[ d_t v_{n+1}(t) - d_t v_n(t) = \beta(t) \cdot (\tilde{\alpha}^L \cdot v_{n+1}^L(t) - \tilde{\alpha}^H \cdot v_n^H(t)) - (\tilde{p}^L \alpha^L - \tilde{p}^H \alpha^H), \]  
which is strictly negative due to (S4) and (121).

Due to the induction hypothesis, the second inequality in (121) is actually with a strictly positive margin as \( t \) approaches \( \tau_{n+1} \) from the right. Hence, \( d_t v_{n+1}(\tau_{n+1}^+) - d_t v_n(\tau_{n+1}^+) \) is strictly negative, and again due to the continuous differentiability of the \( v_n \) and \( v_{n+1} \) functions, so is \( d_t v_{n+1}(\tau_{n+1}) - d_t v_n(\tau_{n+1}) \). We have thus completed the induction process. \[ \square \]

M. Proof of Lemma 7: For convenience, we shall make explicit the dependence of the \( v_n^L \) function as defined in (87) on \( d_t v_{n-1} \) and \( \beta \), and extend it to the entire \([0, \bar{\ell}]\) interval. Due to (115) in the proof of Lemma 6, we have, for any \( t \in [0, \bar{\ell}] \),
\[ v_n^L[d_t v_{n-1}, \beta](t) = \int_t^\bar{\ell} (\tilde{p}^L \alpha^L \beta(s) + d_t v_{n-1}(s)) \cdot \exp(-\int_t^s \hat{\alpha}^L \beta(u) du) \cdot ds. \]  
We note that, for any \( t \in (0, \bar{\ell}) \),
\[ d_t v_n^L[d_t v_{n-1}, \beta](t) = \hat{\alpha}^L \beta(t) \cdot \int_t^\bar{\ell} (\tilde{p}^L \alpha^L \beta(s) + d_t v_{n-1}(s)) \cdot \exp(-\int_t^s \hat{\alpha}^L \beta(u) du) \cdot ds \]
\[ -\tilde{p}^L \alpha^L \cdot \beta(t) - d_t v_{n-1}(t). \]  
So, each \( v_n^L[d_t v_{n-1}, \beta] \) is clearly continuously differentiable in \( t \). More importantly, from Lemma 6, we know that
\[ d_t v_n^L[d_t v_{n-1}, \beta](\tau_n^+) = d_t v_n(\tau_n) - d_t v_{n-1}(\tau_n) < 0, \]  
\[ \square \]
and hence
\[ d_t v_n^L[d_t v_{n-1}, \beta](\tau_n) = d_t v_n^L[d_t v_{n-1}, \beta](\tau_n^+) < 0. \] (126)

We shall revisit (126) later. For the time being, note that the most a \( \Delta t \)-amount of time is worth to a firm is \( \bar{p}^L \bar{\alpha}^L \bar{\beta}^0 \cdot \Delta t \). So we always have
\[ || d_t v_{n-1} ||_\infty \geq \bar{p}^L \bar{\alpha}^L \bar{\beta}^0. \] (127)

Now, from (123), we can derive that,
\[ || v_n^L[d_t v_{n-1}, \beta] - v_n^L[d_t v_{n-1}, \beta'] ||_\infty \leq \int_0^t | d_t v_{n-1}(s) - d_t v_{n-1}(s) | \cdot ds \leq \bar{t} || d_t v_{n-1}^1 - d_t v_{n-1}^2 ||_\infty; \] (128)
also, we have
\[ || v_n^L[d_t v_{n-1}, \beta'] - v_n^L[d_t v_{n-1}, \beta] ||_\infty \leq \bar{p}^L \bar{\alpha}^L \cdot \int_0^t | \beta^1(s) - \beta^2(s) | \cdot ds \]
\[ + \int_0^t | \bar{p}^L \bar{\alpha}^L \beta^2(s) + d_t v_{n-1}(s) | \cdot ds | \cdot ds | - 1), \] (129)
which, in view of the bound (127) and the fact that \( \bar{p}^L \bar{\alpha}^L \beta^2(s) \) is positive while \( d_t v_{n-1}(s) \) negative, will be smaller than \( \bar{p}^L \bar{\alpha}^L \bar{\beta}^0 \cdot (1 + 2\bar{\alpha}^L \bar{\beta}^0 \bar{t}) \cdot || \beta^1 - \beta^2 ||_\infty \) when \( || \beta^1 - \beta^2 ||_\infty \leq \bar{x}^0/(2\bar{\alpha}^L \bar{t}) \). On the other hand, from (124), we have
\[ || d_t v_n^L[d_t v_{n-1}^1, \beta] - d_t v_n^L[d_t v_{n-1}^2, \beta] ||_\infty \leq \bar{\alpha}^L \cdot || \beta^1 ||_\infty \cdot \int_0^t | d_t v_{n-1}^1(s) - d_t v_{n-1}^2(s) | \cdot ds \]
\[ + || d_t v_{n-1}^1 - d_t v_{n-1}^2 ||_\infty \leq (1 + \bar{\alpha}^L \bar{\beta}^0 \bar{t}) \cdot || d_t v_{n-1}^1 - d_t v_{n-1}^2 ||_\infty; \] (130)
also, we have
\[ || d_t v_n^L[d_t v_{n-1}, \beta^1] - d_t v_n^L[d_t v_{n-1}, \beta^2] ||_\infty \leq \bar{\alpha}^L \cdot || \beta^1 - \beta^2 ||_\infty \cdot \int_0^t | \bar{p}^L \bar{\alpha}^L \beta^1(s) + d_t v_{n-1}(s) | \cdot ds \]
\[ + \bar{p}^L(\bar{\alpha}^L)^2 \cdot || \beta^2 ||_\infty \cdot \int_0^t | \beta^1(s) - \beta^2(s) | \cdot ds \]
\[ + \bar{\alpha}^L \cdot || \beta^2 ||_\infty \cdot \int_0^t | \bar{p}^L \bar{\alpha}^L \beta^2(s) + d_t v_{n-1}(s) | \cdot ds | \cdot ds | - 1) + \bar{p}^L \bar{\alpha}^L \cdot || \beta^1 - \beta^2 ||_\infty; \] (131)
which will be smaller than \( \bar{p}^L \bar{\alpha}^L \cdot (1 + 2\bar{\alpha}^L \bar{\beta}^0 \bar{t} + 2(\bar{\alpha}^L \bar{\beta}^0 \bar{t})^2) \cdot || \beta^1 - \beta^2 ||_\infty \) when \( || \beta^1 - \beta^2 ||_\infty \leq \bar{x}^0/(2\bar{\alpha}^L \bar{t}) \).
From (124), we also see that
\[
|d_tv_n^{\ell}[(d_tv_{n-1})^{\beta}] - d_tv_n^{\ell}[(d_tv_{n-1})^{\beta} + \Delta t] |
\leq \bar{\alpha}^L \cdot |\beta(t) - \beta(t + \Delta t) | \cdot f^\ell \bigg[ p^L \bar{\alpha}^L \beta(s) + d_tv_{n-1} \cdot ds \bigg]
\leq \bar{\alpha}^L \cdot |\beta(t) - \beta(t + \Delta t) | \cdot f^\ell \bigg[ p^L \bar{\alpha}^L \beta(s) + d_tv_{n-1} \cdot ds \bigg]
\leq |d_tv_{n-1}(t) - d_tv_{n-1}(t + \Delta t) | + |d_tv_{n-1}(t) - d_tv_{n-1}(t + \Delta t) |\Delta t,
\]
when \( \Delta t \leq \bar{x}^0/(2\bar{\alpha}^L \beta^0) \).

Now let \( d_tv_{n-1} \) and \( \beta^1 \) be given. By \( \tau_n^1 \), we mean \( \tau_n[d_tv_{n-1}, \beta^1] \). There are two cases: case 1, where \( v_n^{\ell}[(d_tv_{n-1})^{\beta^1}] < (p^L \bar{\alpha}^L - p^H \bar{\alpha}^H)/(\bar{\alpha}^L - \bar{\alpha}^H) \) and \( \tau_n^1 = 0 \); and, case 2, where \( v_n^{\ell}[(d_tv_{n-1})^{\beta^1}] \geq (p^L \bar{\alpha}^L - p^H \bar{\alpha}^H)/(\bar{\alpha}^L - \bar{\alpha}^H) \) and \( \tau_n^1 \geq 0 \).

We focus on case 1 first. Define the strictly positive \( \epsilon \) as the difference between \( v_n^{\ell}[(d_tv_{n-1})^{\beta^1}] \) and \( (p^L \bar{\alpha}^L - p^H \bar{\alpha}^H)/(\bar{\alpha}^L - \bar{\alpha}^H) \). By (128) and (129), we know that, for any \( d_tv_{n-1}^2 \) and \( \beta^2 \) satisfying
\[
|d_tv_{n-1}^1 - d_tv_{n-1}^2| < \frac{\epsilon}{2\ell}.
\]
and
\[
|\beta^1 - \beta^2| < \frac{\epsilon}{2p^L \bar{\alpha}^L \ell \cdot (1 + 2\bar{\alpha}^L \beta^0)} \wedge \frac{\bar{x}^0}{2\bar{\alpha}^L \ell}.
\]
we will have
\[
v_n^{\ell}[(d_tv_{n-1})^{\beta^2}] \leq v_n^{\ell}[(d_tv_{n-1})^{\beta^1}] + |v_n^{\ell}[(d_tv_{n-1})^{\beta^1}] - v_n^{\ell}[(d_tv_{n-1})^{\beta^2}]| \leq v_n^{\ell}[(d_tv_{n-1})^{\beta^1}] + \epsilon = (p^L \bar{\alpha}^L - p^H \bar{\alpha}^H)/(\bar{\alpha}^L - \bar{\alpha}^H).
\]
Thus, we will have \( \tau_n[(d_tv_{n-1})^{\beta^1}] = 0 \), which is the same as \( \tau_n^1 \).

Let us focus on case 2 then. Note that \( d_tv_{n-1}^1 \) is continuous in \( t \), and hence uniformly continuous on the compact interval \([0, \bar{t}]\). So, for any \( \epsilon > 0 \), there exists some \( \delta^v(\epsilon, d_tv_{n-1}^1) > 0 \), so that, for any \( t, t + \Delta t \in [0, \bar{t}] \), we will have \( |d_tv_{n-1}(t) - d_tv_{n-1}(t + \Delta t)| < \epsilon \) as long as \( \Delta t < \delta^v(\epsilon, d_tv_{n-1}^1) \). For the time-continuous \( \beta^1 \), we can similarly define the function \( \delta^b(\epsilon, \beta^1) \).

For convenience, define \( \gamma \) so that
\[
\gamma = -\frac{d_tv_n^{\ell}[(d_tv_{n-1})^{\beta^1}](\tau_n^1)}{3},
\]
which, according to (126), is strictly positive. Also, define strictly positive constant \( \overline{\Delta t} \) so that
\[
\overline{\Delta t} = (\gamma/(3p^L \bar{\alpha}^L \beta^0)^2 \cdot (1 + 2\bar{\alpha}^L \beta^0)\ell) \wedge \bar{\delta}^v(\gamma, 9, d_tv_{n-1}^1) \wedge \bar{\delta}^b(\gamma/(9p^L \bar{\alpha}^L \cdot (1 + \bar{\alpha}^L \beta^0)\ell), \beta^1).
\]
For any $\epsilon \in (0, \overline{\Delta}t)$, suppose $d_tv_{n-1}^2$ satisfies
\[
\| d_tv_{n-1}^1 - d_tv_{n-1}^2 \|_\infty < \frac{\epsilon \gamma}{2t} \land \frac{\gamma}{2 \cdot (1 + \alpha^L \beta^0 \overline{t})} \land \frac{\gamma}{9},
\]
while $\beta^2$ satisfies
\[
\| \beta^1 - \beta^2 \|_\infty < (\epsilon \gamma/(2\overline{\rho}^L \alpha^L \overline{t} \cdot (1 + 2\alpha^L \beta^0 \overline{t})) \land (\overline{\rho}^0/(2\alpha^L \overline{t})) \land \\
(\gamma/(2\overline{\rho}^L \alpha^L \cdot (1 + 2\alpha^L \beta^0 \overline{t} + 2(\overline{\alpha}^L \beta^0 \overline{t})^2)) \land (\gamma/(9\overline{\rho}^L \alpha^L \cdot (1 + \overline{\alpha}^L \beta^0 \overline{t}))).
\]

From (128), (129), (138), and (139), we see that
\[
| v_n^L[d_tv_{n-1}^1, \beta^1](\tau_n^1) - v_n^L[d_tv_{n-1}^2, \beta^2](\tau_n^1) | < \epsilon \gamma.
\]
From (130), (131), (138), and (139), we see that
\[
| d_tv_n^L[d_tv_{n-1}^1, \beta^1](\tau_n^1) - d_tv_n^L[d_tv_{n-1}^2, \beta^2](\tau_n^1) | < \gamma,
\]
which, in view of (136), leads to
\[
d_tv_n^L[d_tv_{n-1}^2, \beta^2](\tau_n^1) < -2\gamma.
\]

For any $\Delta t$ satisfying $\tau_n^1 + \Delta t \in (0, \overline{t})$ and $| \Delta t | < \overline{\Delta}t$, we note that
\[
| d_tv_{n-1}^2(\tau_n^1) - d_tv_{n-1}^2(\tau_n^1 + \Delta t) | \leq 2 \cdot | d_tv_{n-1}^1 - d_tv_{n-1}^2 |_\infty + | d_tv_{n-1}^1(\tau_n^1) - d_tv_{n-1}^1(\tau_n^1 + \Delta t) |,
\]
which is smaller than $\gamma/3$ due to the definition of $\overline{\delta}^o(\cdot, d_tv_{n-1}^1)$, (137), and (138); also,
\[
| \beta^2(\tau_n^1) - \beta^2(\tau_n^1 + \Delta t) | \leq 2 \cdot | \beta^1 - \beta^2 |_\infty + | \beta^1(\tau_n^1) - \beta^1(\tau_n^1 + \Delta t) |,
\]
which is smaller than $\gamma/(3\overline{\rho}^L \alpha^L \cdot (1 + \overline{\alpha}^L \beta^0 \overline{t}))$ due to the definition of $\overline{\delta}^b(\cdot, \beta^1)$, (137), and (139).

Using (132), we have
\[
| d_tv_n^L[d_tv_{n-1}^2, \beta^2](\tau_n^1) - d_tv_n^L[d_tv_{n-1}^2, \beta^2](\tau_n^1 + \Delta t) |
\leq| d_tv_{n-1}^2(\tau_n^1) - d_tv_{n-1}^2(\tau_n^1 + \Delta t) | + \overline{\rho}^L \alpha^L \cdot (1 + \overline{\alpha}^L \beta^0 \overline{t}) \cdot | \beta^2(\tau_n^1) - \beta^2(\tau_n^1 + \Delta t) |
\]
\[
+ \overline{\rho}^L \cdot (\overline{\alpha}^L \beta^0)^2 \cdot (1 + 2\overline{\alpha}^L \beta^0 \overline{t}) \cdot \Delta t,
\]
which, according to (137), (134), and (144), is smaller than $\gamma$. Combining this with (142), we see that
\[
d_tv_n^L[d_tv_{n-1}^2, \beta^2](\tau_n^1 + \Delta t) < -\gamma.
\]
This means that $d_tv_n^L[d_tv_{n-1}^2, \beta]$ is below $-\gamma$ in a neighborhood of $\tau_n^1$ whose half-width is more than $\epsilon$, while (140) means that $v_n^L[d_tv_{n-1}^1, \beta^1](\tau_n^1)$ and $v_n^L[d_tv_{n-1}^2, \beta^2](\tau_n^1)$ are less than $\epsilon \gamma$ away from each
other. Hence, by the fact that \( \tau_n \) satisfies \( v_n^L[d_tv_{n-1}^1, \beta^1](\tau_n) = (\tilde{p}^L \bar{\alpha}^L - \tilde{p}^H \bar{\alpha}^H) / (\bar{\alpha}^L - \bar{\alpha}^H) \) and the requirement that \( \tau_n[d_tv_{n-1}^2, \beta^2] - \tau_n^1 \) be the \( \Delta t \) that renders \( v_n^L[d_tv_{n-1}^2, \beta^2](\tau_n + \Delta t) \) as close to \( (\tilde{p}^L \bar{\alpha}^L - \tilde{p}^H \bar{\alpha}^H) / (\bar{\alpha}^L - \bar{\alpha}^H) \) as possible, we see that

\[
| \tau^2[d_tv_{n-1}^2, \beta^2] - \tau_n^1 | < \epsilon. \tag{147}
\]

We have thus shown that \( \tau_n[d_tv_{n-1}^2, \beta] \) is jointly continuous in \( (d_tv_{n-1}, \beta) \).

\[ \blacksquare \]

**N. Proof of Lemma 8:** First, by (127) in the proof of Lemma 7, we know that all the \( \| d_tv_n \|_\infty \)'s are bounded by \( \tilde{p}^L \bar{\alpha}^L \beta^0 \). Also, we may use \( v_n^L[d_tv_{n-1}^1, \beta] \) to denote the same function as defined through (123) in the proof of Lemma 7. From the lemma, we know the existence of strictly positive and increasing functions \( \tilde{\delta}^v(\cdot, \cdot) \) and \( \tilde{\delta}^b(\cdot, \cdot) \) such that, for any given \( \epsilon' \) and \( (d_tv_{n-1}, \beta) \), it will follow that \( | \tau_n[d_tv_{n-1}, \beta] - \tau_n[d_tv_{n-1}', \beta] | < \epsilon' \) as long as \( d_tv_{n-1} \) and \( \beta \) satisfy \( \| d_tv_{n-1} - d_tv_{n-1}' \|_\infty < \tilde{\delta}^v(\epsilon', d_tv_{n-1}, \beta) \) and \( \| \beta - \beta' \|_\infty < \tilde{\delta}^b(\epsilon', d_tv_{n-1}, \beta) \).

We may think of \( d_tv_{n}[d_tv_{n-1}, \beta] \) as the composition of \( d_tv_n[\tau_n, d_tv_{n-1}, \beta] \), denoting the conversion related to (102) and (103), and \( \tau_n[d_tv_{n-1}, \beta] \), denoting the conversion related to (101). In the definition of the former, the \( v_n^L(\tau_n) \) in (103) should be understood as \( v_n^L[d_tv_{n-1}, \beta](\tau_n) \) defined through (123) in the proof of Lemma 7, which is equal to \( (\tilde{p}^L \bar{\alpha}^L - \tilde{p}^H \bar{\alpha}^H) / (\bar{\alpha}^L - \bar{\alpha}^H) \) when \( \tau_n = \tau_n[d_tv_{n-1}, \beta] \), while not necessarily so when \( \tau_n \neq \tau_n[d_tv_{n-1}, \beta] \).

Now, let \( d_tv_{n-1}^1, \beta^1, \) and \( \epsilon > 0 \) be given. We are to show that there are strictly positive numbers \( \delta^v \) and \( \delta^b \), so that \( \| d_tv_n[d_tv_{n-1}^1, \beta^1] - d_tv_n[d_tv_{n-1}^2, \beta^2] \|_\infty < \epsilon \) as long as \( d_tv_{n-1}^2 \) and \( \beta^2 \) are such that \( \| d_tv_{n-1}^2 - d_tv_{n-1}^1 \|_\infty < \delta^v \) and \( \| \beta^2 - \beta^1 \|_\infty < \delta^b \). By \( \tau_n^1 \), we mean \( \tau_n[d_tv_{n-1}^1, \beta^1] \).

When \( d_tv_{n-1}^2, \beta^2 \) is clear from the context, we shall use \( \tau_n^2 \) to denote \( \tau_n[d_tv_{n-1}^2, \beta^2] \). We first show that \( \| d_tv_n[\tau_n^1, d_tv_{n-1}^1, \beta^1] - d_tv_n[\tau_n^2, d_tv_{n-1}^2, \beta^1] \|_\infty \) can be kept small when \( d_tv_{n-1}^2, \beta^2 \) is close enough to \( d_tv_{n-1}^1, \beta^1 \). There are two cases: case 1, where \( v_n^L[d_tv_{n-1}^1, \beta^1](0) < (\tilde{p}^L \bar{\alpha}^L - \tilde{p}^H \bar{\alpha}^H) / (\bar{\alpha}^L - \bar{\alpha}^H) \) and \( \tau_n^1 = 0 \); and, case 2, where \( v_n^L[d_tv_{n-1}^1, \beta^1](0) \geq (\tilde{p}^L \bar{\alpha}^L - \tilde{p}^H \bar{\alpha}^H) / (\bar{\alpha}^L - \bar{\alpha}^H) \) and \( \tau_n^1 \geq 0 \).

For case 1, we may learn from the arguments in the proof of Lemma 7 revolving around (133) to (135) that, \( \tau_n^2 = 0 \) when

\[
\| d_tv_{n-1}^1 - d_tv_{n-1}^2 \|_\infty < \frac{(\tilde{p}^L \bar{\alpha}^L - \tilde{p}^H \bar{\alpha}^H) / (\bar{\alpha}^L - \bar{\alpha}^H) - v_n^L[d_tv_{n-1}^1, \beta^1](0)}{2\tilde{t}}. \tag{148}
\]

and

\[
\| \beta^1 - \beta^2 \|_\infty < \frac{(\tilde{p}^L \bar{\alpha}^L - \tilde{p}^H \bar{\alpha}^H) / (\bar{\alpha}^L - \bar{\alpha}^H) - v_n^L[d_tv_{n-1}^1, \beta^1](0)}{2\tilde{p}^L \bar{\alpha}^L \cdot (1 + 2\bar{\alpha}^L \beta^0 \tilde{t})} \wedge \frac{\bar{x}^0}{2\bar{\alpha}^L \tilde{t}}. \tag{149}
\]

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Thus, (148) and (149) will ensure that

$$|| d_t v_n[\tau_n^1, d_t v_n^1, \beta^1] - d_t v_n[\tau_n^2, d_t v_n^1, \beta^1] ||_\infty = 0. \quad (150)$$

Now we focus on case 2. When $\tau_n^1 = \tau_n^2$, there is nothing to prove. Let us suppose $\tau_n^1 < \tau_n^2$ first.

From (102) and (103), we see that, for $t \in (\tau_n^2, t)$,

$$| d_t v_n[\tau_n^1, d_t v_n^1, \beta^1](t) - d_t v_n[\tau_n^2, d_t v_n^1, \beta^1](t) | = 0. \quad (151)$$

For $t \in (\tau_n^1, \tau_n^2)$, we have

$$| d_t v_n[\tau_n^1, d_t v_n^1, \beta^1](t) - d_t v_n[\tau_n^2, d_t v_n^1, \beta^1](t) |
\leq \tilde{\alpha}^L \beta^1(t) \cdot \int_{\tau_n^1}^{\tau_n^2} \tilde{\alpha}^L \beta^1(s) + d_t v_n^1(s) | \cdot ds
+ \tilde{\alpha}^H \beta^1(t) \cdot \int_{\tau_n^1}^{\tau_n^2} \tilde{\alpha}^H \beta^1(s) + d_t v_n^1(s) | \cdot ds
+ \beta^1(t) \cdot | \tilde{\alpha}^L \cdot \exp(- \int_{\tau_n^1}^{\tau_n^2} \tilde{\alpha}^L \beta^1(s) ds) \cdot v_n^1[\tau_n^1, \beta^1](\tau_n^2)
- \tilde{\alpha}^H \cdot \exp(- \int_{\tau_n^1}^{\tau_n^2} \tilde{\alpha}^H \beta^1(s) ds) \cdot (\tilde{p}^L \tilde{\alpha}^L - \tilde{p}^H \tilde{\alpha}^H) / (\tilde{\alpha}^L - \tilde{\alpha}^H)
+ \tilde{p}^H \tilde{\alpha}^H - \tilde{p}^L \tilde{\alpha}^L | \ . \quad (152)$$

We will have $T_1 < \epsilon/12$ when

$$\tau_n^2 - \tau_n^1 < \frac{\epsilon}{24 \tilde{\alpha}^L (\tilde{\alpha}^L \beta^0)^2}. \quad (154)$$

Also, we have $v_n^L[\tau_n^1, \beta^1](\tau_n^2) < v_n^L[\tau_n^1, \beta^1](\tau_n^1) = (\tilde{p}^L \tilde{\alpha}^L - \tilde{p}^H \tilde{\alpha}^H) / (\tilde{\alpha}^L - \tilde{\alpha}^H)$ as $\tau_n^2 > \tau_n^1$.

Hence, we will have $T_2 < \epsilon/12$ when

$$\tau_n^2 - \tau_n^1 < \frac{(\tilde{\alpha}^L - \tilde{\alpha}^H) \cdot \epsilon}{24 (\tilde{\alpha}^L \beta^0)^2 \cdot (\tilde{p}^L \tilde{\alpha}^L - \tilde{p}^H \tilde{\alpha}^H) \wedge \frac{\tilde{\alpha}^0}{2 \tilde{\alpha}^L \beta^0}}. \quad (155)$$

As for $T_3$, note that

$$v_n^L[\tau_n^1, \beta^1](\tau_n^1) - v_n^L[\tau_n^1, \beta^1](\tau_n^2) \leq || d_t v_n^L[\tau_n^1, \beta^1] ||_\infty \cdot (\tau_n^2 - \tau_n^1)
= || d_t v_n[\tau_n^1, \beta^1] - d_t v_n^L[\tau_n^1, \beta^1] ||_\infty \cdot (\tau_n^1 - \tau_n^2)
\leq \tilde{p}^L \tilde{\alpha}^L \beta^0 \cdot (\tau_n^2 - \tau_n^1) \ , \quad (156)$$
where the equality comes from the fact that $d_{t}v_{n}^{L}[d_{t}v_{n-1}^{1},\beta^{1}](\tau_{n}^{2}) = d_{t}v_{n}[d_{t}v_{n-1}^{1},\beta^{1}](\tau_{n}^{2}) - v_{n-1}^{1}(\tau_{n}^{2})$ as $\tau_{n}^{2} > \tau_{n}^{1}$, the ensuing inequality comes from the fact that both $d_{t}v_{n}[d_{t}v_{n-1}^{1},\beta^{1}] - d_{t}v_{n-1}^{1}$ and $d_{t}v_{n-1}^{1}$ are negative, and the last inequality is due to the universal upper bound of $\| d_{t}v_{n} \|_{\infty}$.

Hence, we will have $T_{3} < \epsilon/12$ when (154) is true. Moreover, we will have $T_{4} < \epsilon/12$ when (155) is satisfied. Finally, we can verify that $T_{5} = 0$, as $v_{n}^{1}[d_{t}v_{n-1}^{1},\beta^{1}](\tau_{n}^{1}) = (\bar{p}^{L}\alpha^{L} - \bar{p}^{H}\alpha^{H})/(\bar{\alpha}^{L} - \bar{\alpha}^{H})$.

For $t \in (0, \tau_{n}^{1})$, we have

$$
| d_{t}v_{n}[\tau_{n}^{1},d_{t}v_{n-1}^{1},\beta^{1}](t) - d_{t}v_{n}^{2}[\tau_{n}^{2},d_{t}v_{n-1}^{1},\beta^{1}](t) |
\leq \bar{\alpha}^{H}\beta^{1}(t) \cdot \int_{\tau_{n}^{1}}^{\tau_{n}^{2}} | \bar{p}^{L}\bar{\alpha}^{L}\beta^{1}(s) + d_{t}v_{n-1}^{1}(s) | \cdot ds
+ \bar{\alpha}^{H}\beta^{1}(t) \cdot \int_{\tau_{n}^{1}}^{\tau_{n}^{2}} | \bar{p}^{H}\bar{\alpha}^{H}\beta^{1}(s) + d_{t}v_{n-1}^{1}(s) | \cdot ds
+ \bar{\alpha}^{H}\beta^{1}(t) \cdot v_{n}[d_{t}v_{n-1}^{1},\beta^{1}](\tau_{n}^{2}) \cdot (\exp(-\int_{\tau_{n}^{1}}^{\tau_{n}^{2}} \bar{\alpha}^{H}\beta^{1}(s)ds) - \exp(-\int_{\tau_{n}^{1}}^{\tau_{n}^{2}} \bar{\alpha}^{L}\beta^{1}(s)ds))
\leq T_{6} + T_{7},
$$

where

$$
\left\{
\begin{array}{l}
T_{6} = 2\bar{p}^{L}(\bar{\alpha}^{L}\bar{\beta}^{0})^{2} \cdot (\tau_{n}^{2} - \tau_{n}^{1}),
T_{7} = \bar{\alpha}^{H}\bar{\beta}^{0} \cdot v_{n}[d_{t}v_{n-1}^{1},\beta^{1}](\tau_{n}^{1}) \cdot (\exp(\int_{\tau_{n}^{1}}^{\tau_{n}^{2}} (\bar{\alpha}^{L} - \bar{\alpha}^{H})\beta^{1}(s)ds) - 1).
\end{array}
\right.
$$

We will have $T_{6} < \epsilon/6$ when (154) is satisfied. For a similar reason as that revolving around $T_{2}$, we will have $T_{7} < \epsilon/6$ when (155) happens.

Let us suppose $\tau_{n}^{2} < \tau_{n}^{1}$ next. From (102) and (103), we see that, for $t \in (\tau_{n}^{1}, \bar{t})$, (151) is true.

For $t \in (\tau_{n}^{2}, \tau_{n}^{1})$, we have

$$
| d_{t}v_{n}[\tau_{n}^{2},d_{t}v_{n-1}^{1},\beta^{1}](t) - d_{t}v_{n}^{2}[\tau_{n}^{2},d_{t}v_{n-1}^{1},\beta^{1}](t) |
\leq \bar{\alpha}^{H}\beta^{1}(t) \cdot \int_{\tau_{n}^{2}}^{\tau_{n}^{1}} | \bar{p}^{L}\bar{\alpha}^{L}\beta^{1}(s) + d_{t}v_{n-1}^{1}(s) | \cdot ds
+ \bar{\alpha}^{H}\beta^{1}(t) \cdot \int_{\tau_{n}^{2}}^{\tau_{n}^{1}} | \bar{p}^{H}\bar{\alpha}^{H}\beta^{1}(s) + d_{t}v_{n-1}^{1}(s) | \cdot ds
+ \bar{\alpha}^{H}\beta^{1}(t) \cdot v_{n}[d_{t}v_{n-1}^{1},\beta^{1}](\tau_{n}^{2}) \times
(\bar{\alpha}^{H}, \exp(-\int_{\tau_{n}^{2}}^{\tau_{n}^{1}} \bar{\alpha}^{H}\beta^{1}(s)ds) - \bar{\alpha}^{L}, \exp(-\int_{\tau_{n}^{2}}^{\tau_{n}^{1}} \bar{\alpha}^{L}\beta^{1}(s)ds))/ (\bar{\alpha}^{L} - \bar{\alpha}^{H}) + 1 |
\leq T_{8} + T_{9} + T_{10},
$$

where

$$
\left\{
\begin{array}{l}
T_{8} = 2\bar{p}^{L}(\bar{\alpha}^{L}\bar{\beta}^{0})^{2} \cdot (\tau_{n}^{1} - \tau_{n}^{2}),
T_{9} = \bar{\alpha}^{H}\bar{\beta}^{0} \cdot (\bar{p}^{L}\bar{\alpha}^{L} - \bar{p}^{H}\bar{\alpha}^{H}) \cdot (\exp(\int_{\tau_{n}^{2}}^{\tau_{n}^{1}} \bar{\alpha}^{H}\beta^{1}(s)ds) - 1)/(\bar{\alpha}^{L} - \bar{\alpha}^{H}),
T_{10} = \bar{\alpha}^{L}\bar{\beta}^{0} \cdot (\bar{p}^{L}\bar{\alpha}^{L} - \bar{p}^{H}\bar{\alpha}^{H}) \cdot (\exp(\int_{\tau_{n}^{2}}^{\tau_{n}^{1}} \bar{\alpha}^{L}\beta^{1}(s)ds) - 1)/(\bar{\alpha}^{L} - \bar{\alpha}^{H}).
\end{array}
\right.
$$

We will have $T_{8} < \epsilon/9$ when

$$
\tau_{n}^{1} - \tau_{n}^{2} < \frac{\epsilon}{18\bar{p}^{L}(\bar{\alpha}^{L}\bar{\beta}^{0})^{2}}.
$$

Also, we will have $T_{9} \leq T_{10} < \epsilon/9$ when

$$
\tau_{n}^{1} - \tau_{n}^{2} < \frac{(\bar{\alpha}^{L} - \bar{\alpha}^{H}) \cdot \epsilon}{18(\bar{\alpha}^{L}\bar{\beta}^{0})^{2} \cdot (\bar{p}^{L}\bar{\alpha}^{L} - \bar{p}^{H}\bar{\alpha}^{H})} \land \frac{\bar{\beta}^{0}}{2\bar{\alpha}^{L}\bar{\beta}^{0}}.
$$

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For $t \in (0, \tau^2_n)$, we have

$$
| d_t v_n[\tau^1_n, d_t v^1_{n-1}, \beta^1](t) - d_t v_n[\tau^2_n, d_t v^1_{n-1}, \beta^1](t) |
\leq \alpha^H \beta^1(t) \cdot \int_{\tau^2_n}^{\tau^1_n} \left| \tilde{p}^L \alpha^L \beta^1(s) + d_t v^1_{n-1}(s) \right| ds + \alpha^H \beta^1(t) \cdot (\tilde{p}^L \alpha^L - \tilde{p}^H \alpha^H) \times (\exp(-\int_{\tau^2_n}^{\tau^1_n} \alpha^L \beta^1(s) ds) - \exp(-\int_{\tau^2_n}^{\tau^1_n} \alpha^L \beta^1(s) ds))/((\alpha^L - \alpha^H))
$$

(163)

(164)

We will have $T_{11} < \epsilon/6$ when (161) is satisfied. Also, we will have $T_{12} < \epsilon/6$ when (162) happens.

Combining (154), (155), (161), and (162), as well as the definitions of $\delta^v(\cdot, \cdot)$ and $\delta^b(\cdot, \cdot)$, we have, for case 2, that

$$
\| d_t v_n[\tau^1_n, d_t v^1_{n-1}, \beta^1] - d_t v_n[\tau^2_n, d_t v^1_{n-1}, \beta^1] \|_\infty < \frac{\epsilon}{3},
$$

(165)

when

$$
\| d_t v^1_{n-1} - d_t v^2_{n-1} \|_\infty < \delta^v\left(\frac{\epsilon}{24 \tilde{p}^L (\alpha^L \beta^0)^2} \wedge \frac{(\alpha^L - \alpha^H) \cdot \epsilon}{24 (\alpha^L \beta^0)^2 \cdot (\tilde{p}^L \alpha^L - \tilde{p}^H \alpha^H)} \wedge \frac{\tilde{x}^0}{2 \alpha^L \beta^0} \cdot d_t v^1_{n-1}, \beta^1\right),
$$

(166)

and

$$
\| \beta^1 - \beta^2 \|_\infty < \delta^b\left(\frac{\epsilon}{24 \tilde{p}^L (\alpha^L \beta^0)^2} \wedge \frac{(\alpha^L - \alpha^H) \cdot \epsilon}{24 (\alpha^L \beta^0)^2 \cdot (\tilde{p}^L \alpha^L - \tilde{p}^H \alpha^H)} \wedge \frac{\tilde{x}^0}{2 \alpha^L \beta^0} \cdot d_t v^1_{n-1}, \beta^1\right).
$$

(167)

By (102) and (103), we have, for $t \in (\tau^2_n, t)$,

$$
| d_t v_n[\tau^2_n, d_t v^1_{n-1}, \beta^1](t) - d_t v_n[\tau^2_n, d_t v^1_{n-1}, \beta^1](t) |
\leq \alpha^L \beta^1(t) \cdot \int_{t}^{\tau^2_n} \left| d_t v^1_{n-1}(s) - d_t v^1_{n-1}(s) \right| ds \leq \alpha^L \beta^0 \cdot \| d_t v^1_{n-1} - d_t v^2_{n-1} \|_\infty.
$$

(168)

Also, for $t \in (0, \tau^2_n)$, we have

$$
| d_t v_n[\tau^2_n, d_t v^1_{n-1}, \beta^1](t) - d_t v_n[\tau^2_n, d_t v^1_{n-1}, \beta^1](t) |
\leq \alpha^H \beta^1(t) \cdot \int_{t}^{\tau^2_n} \left| d_t v^1_{n-1}(s) - d_t v^1_{n-1}(s) \right| ds \leq \alpha^L \beta^0 \cdot \| d_t v^1_{n-1} - d_t v^2_{n-1} \|_\infty.
$$

(169)

Therefore, we will have

$$
\| d_t v_n[\tau^2_n, d_t v^1_{n-1}, \beta^1] - d_t v_n[\tau^2_n, d_t v^1_{n-1}, \beta^1] \| < \frac{\epsilon}{3},
$$

(170)
when
\[
\| d_t v^1_{n-1} - d_t v^2_{n-1} \|_\infty < \frac{\epsilon}{3\bar{\alpha} \beta 0 \bar{\ell}}.
\] (171)

By (102) and (103), we have, for \( t \in (\tau^0_{n}, \bar{\ell}) \),
\[
| d_t v_n[\tau^0_n, d_t v^2_{n-1}, \beta^1](t) - d_t v_n[\tau^0_n, d_t v^2_{n-1}, \beta^2](t) | \\
\leq \bar{p}^L \bar{\alpha} \bar{H} \cdot | \beta^1(t) - \beta^2(t) | + \bar{\alpha} \bar{H} \cdot | \beta^1(t) - \beta^2(t) | \times \int_{\tau^0_n}^{\bar{\ell}} | \bar{p}^L \bar{\alpha} \bar{H} \beta^1(s) + d_t v^2_{n-1}(s) | \cdot ds \\
+ \bar{p}^L (\bar{\alpha} \bar{H})^{2} \beta^2(\bar{\ell}) \cdot \int_{\tau^0_n}^{\bar{\ell}} | \beta^1(s) - \beta^2(s) | \cdot ds \\
+ \bar{\alpha} \bar{H} \beta^2(\bar{\ell}) \cdot \int_{\tau^0_n}^{\bar{\ell}} \left( \int_{\tau^0_n}^{\bar{\ell}} | \bar{p}^L \bar{\alpha} \bar{H} \beta^1(s) + d_t v^2_{n-1}(s) | \cdot ds \right) \times (\exp(\int_{\tau^0_n}^{\bar{\ell}} | \bar{p}^L \bar{\alpha} \bar{H} | \beta^1(s) - \beta^2(s) | ds - 1)) \\
+ \bar{\alpha} \bar{H} \beta^2(t) \cdot \int_{\tau^0_n}^{\bar{\ell}} (\exp(\int_{\tau^0_n}^{\bar{\ell}} | \bar{p}^L \bar{\alpha} \bar{H} | \beta^1(s) - \beta^2(s) | ds - 1)) \times \left( \int_{\tau^0_n}^{\bar{\ell}} | \bar{p}^L \bar{\alpha} \bar{H} \beta^1(s) + d_t v^2_{n-1}(s) | \cdot ds \right) \\
+ \bar{\alpha} \bar{H} \beta^2(t) \cdot \int_{\tau^0_n}^{\bar{\ell}} (\exp(\int_{\tau^0_n}^{\bar{\ell}} | \bar{p}^L \bar{\alpha} \bar{H} | \beta^1(s) - \beta^2(s) | ds - 1)) \times \left( \int_{\tau^0_n}^{\bar{\ell}} | \bar{p}^L \bar{\alpha} \bar{H} \beta^1(s) + d_t v^2_{n-1}(s) | \cdot ds \right)
\] (173)

which will be smaller than \( \bar{p}^L \bar{\alpha} \bar{H} \cdot (1 + 4\bar{\alpha} \bar{H} \beta 0 \bar{\ell} + 6(\bar{\alpha} \bar{H} \beta 0 \bar{\ell})^2) \cdot \| \beta^1 - \beta^2 \|_\infty \) when \( \| \beta^1 - \beta^2 \|_\infty \leq \bar{x}/(2\bar{\alpha} \bar{H} \bar{\ell}) \). Therefore, we will have
\[
\| d_t v_n[\tau^0_n, d_t v^2_{n-1}, \beta^1] - d_t v_n[\tau^0_n, d_t v^2_{n-1}, \beta^2] \| < \frac{\epsilon}{3},
\] (174)

when
\[
\| \beta^1 - \beta^2 \|_\infty < \frac{\epsilon}{3 \bar{p}^L \bar{\alpha} \bar{H} \cdot (1 + 4\bar{\alpha} \bar{H} \beta 0 \bar{\ell} + 6(\bar{\alpha} \bar{H} \beta 0 \bar{\ell})^2) \wedge \bar{x}/2\bar{\alpha} \bar{H} \bar{\ell}.
\] (175)

In view of (148), (149), and (150) for case 1, (165), (166), and (167) for case 2, as well as (170), (171), (174), and (175), we see that, for both cases,
\[
\| d_t v_n[\tau^0_n, d_t v^1_{n-1}, \beta^1] - d_t v_n[\tau^0_n, d_t v^2_{n-1}, \beta^2] \|_\infty < \epsilon,
\] (176)

when \((d_t v^2_{n-1}, \beta^2)\) is close enough to the given \((d_t v^1_{n-1}, \beta^1)\).
O. Proof of Lemma 9: We may make the dependence of the $d_tv_n$’s and $\tau_n$’s on $\beta$ direct by letting $d_tv_0[\beta] = 0$, and for $n = 1, 2, ..., \bar{n}$, iteratively letting $d_tv_n[\beta] = d_tv_n[d_tv_{n-1}][\beta]$ and $\tau_n[\beta] = \tau_n[d_tv_{n-1}][\beta]$. Note that each $\tau_n[\beta]$ is exactly $(Z^P_B(\beta))_n$.

We may use induction to show that all the $d_tv_n[\beta]$’s are continuous in $\beta$, and along the way, that all the $\tau_n[\beta]$’s are continuous in $\beta$. First, $d_tv_0[\beta] = 0$ is certainly continuous in $\beta$. Now, suppose for some $n = 1, 2, ..., \bar{n}$, we know that $d_tv_{n-1}[\beta]$ is continuous in $\beta$. Then, by Lemma 7, that $\tau_n[d_tv_{n-1}, \beta]$ is continuous in $(d_tv_{n-1}, \beta)$, we know that $\tau_n[\beta]$ is continuous in $\beta$; meanwhile, by Lemma 8, that $d_tv_n[d_tv_{n-1}, \beta]$ is continuous in $(d_tv_{n-1}, \beta)$, we know that $d_tv_n[\beta]$ is continuous in $\beta$. We have thus completed the induction.

By the continuity in $\beta$ of its every component, we know that $Z^P_B(\beta)$ is continuous in $\beta$.  

P. Proof of Lemma 10: Let $p \in \Delta^P$ and $f \in Y^P_\beta$ be given. First, $f$ is uniformly continuous on $[0, \bar{t}]$ as the interval is compact. So for any $\epsilon > 0$, there is $\delta_F(\epsilon, f) > 0$, such that

$$|| f(t) - f(t') ||^n < \epsilon, \quad (177)$$

for any $t, t' \in [0, \bar{t}]$ satisfying $| t - t' | < \delta_F(\epsilon, f)$.

Now, for any $p' \in \Delta^P$ satisfying $|| p - p' ||^n < \delta_F(\epsilon/\bar{n}, f) \wedge \epsilon$, we may introduce time-rescaling function $l$, so that $l(t) = t$ for $t \in [-\theta, 0)$; $l(\tau_n) = \tau'_n$ for $n = 0, 1, ..., \bar{n} + 1$; and, $l(t)$ is linear between any two adjacent $\tau_n$ points. From this construction, we see that

$$|| l - I ||_\infty \leq || p - p' ||^n < \delta_F(\frac{\epsilon}{\bar{n}}, f) \wedge \epsilon. \quad (178)$$

From the result revolving around (177), we see that, for any $t \in [0, \bar{t}]$,

$$|| f(t) - f(l(t)) ||^n < \frac{\epsilon}{\bar{n}}. \quad (179)$$

From (35), (179), and the convention that $m(t) = m(0)$ for $t \in [-\theta, 0)$, we see that, for every $t \in [-\theta, \bar{t}]$,

$$|| (Z^M_{p,F}(p, f))(t) - (Z^M_{p,F}(p', f))(l(t)) ||^2 < \epsilon. \quad (180)$$

Combining (178) and (180), we obtain

$$\rho^2(Z^M_{p,F}(p, f), Z^M_{p,F}(p', f)) < \epsilon. \quad (181)$$

As for the other continuity, we have, from (35),

$$\rho^2(Z^M_{p,F}(p, f), Z^M_{p,F}(p, f')) \leq \bar{n} \cdot || f - f' ||^n_\infty, \quad (182)$$
for any $f' \in Y^0_F$.

**Q. Proof of Lemma 11:** Let $p \in \Delta^P$ and $\beta \in Y_B$ be given. First, $\beta$ is uniformly continuous on $[0, \bar{t}]$ as the interval is compact. So for any $\epsilon > 0$, there is $\bar{\delta}_B(\epsilon, \beta) > 0$, such that

$| \beta(t) - \beta(t') | < \epsilon,$  \hspace{1cm} (183)

for any $t, t' \in [0, \bar{t}]$ satisfying $| t - t' | < \bar{\delta}_B(\epsilon, \beta)$.

Now, for any $p' \in \Delta^P$ satisfying $|| p - p' ||^n < \bar{\delta}_B(\epsilon/\bar{\alpha}L, \beta) \wedge \epsilon$, we may introduce time-rescaling function $l$, so that $l(t) = t$ for $t \in [-\theta, 0)$; $l(\tau_n) = \tau'_n$ for $n = 0, 1, \ldots, \bar{n} + 1$; and, $l(t)$ is linear between any two adjacent $\tau_n$ points. From this construction, we see that

$| | | l - I | |_\infty \leq || p - p' ||^n < \bar{\delta}_B(\epsilon/\bar{\alpha}L, \beta) \wedge \epsilon.$  \hspace{1cm} (184)

From the result revolving around (183), we see that, for any $t \in [0, \bar{t}]$,

$| \beta(t) - \beta(l(t)) | < \frac{\epsilon}{\bar{\alpha}L}.$  \hspace{1cm} (185)

From (36), (185), and the convention that $\lambda(t) = \lambda(0)$ for $t \in [-\theta, 0)$, we see that, for every $t \in [-\theta, \bar{t}]$,

$|| (Z_{PB}^A(p, \beta))(t) - (Z_{PB}^A(p', \beta))(l(t)) ||^n < \epsilon.$  \hspace{1cm} (186)

Combining (184) and (186), we obtain

$\rho^n(Z_{PB}^A(p, \beta), Z_{PB}^A(p', \beta)) < \epsilon.$  \hspace{1cm} (187)

As for the other continuity, we have, from (36),

$\rho^n(Z_{PB}^A(p, \beta), Z_{PB}^A(p, \beta')) \leq \bar{\alpha}L \cdot || \beta - \beta' ||_\infty,$  \hspace{1cm} (188)

for any $\beta' \in Y_B$.

**R. Proof of Lemma 12:** We prove by induction. From (92), we have

$| G_1[a_1, r](t + \Delta t) - G_1[a_1, r](t) | = | \exp(- \int_t^{t+\Delta t} a_1(s) \cdot ds) - \exp(- \int_t^t a_1(s) \cdot ds) |
\leq \exp(\int_t^{t+\Delta t} a_1(s) \cdot ds) - 1,$  \hspace{1cm} (189)

which, as $a_1 \in X_A$, is smaller than $2\bar{\alpha}L \bar{\beta}_0 \cdot \Delta t$ when $\Delta t \leq \bar{x}^0/(2\bar{\alpha}L \bar{\beta}_0)$. It is also clear from (92) that $\bar{g}_1 = 1$ is a uniform upper bound for all the $G_1[a_1, r]$'s.
Suppose for some \( n = 1, 2, \ldots \), we know that \( G_n[a_1, \ldots, a_n, r] \) is uniformly continuous in \( t \) in an \((a_1, \ldots, a_n, r)\)-equal fashion. That is, for any \( \epsilon > 0 \), there exists \( \delta(\epsilon) > 0 \), such that, for any \((a_1, \ldots, a_n) \in X_A^n, r \in [0, \bar{t}] \), and \( t, t + \Delta t \in [r, \bar{t}] \), we have, as long as \( \Delta t \in [0, \delta(\epsilon)] \),

\[
| G_n[a_1, \ldots, a_n, r](t + \Delta t) - G_n[a_1, \ldots, a_n, r](t) | < \epsilon. \tag{190}
\]

Also, suppose that there is a uniform upper bound \( \bar{g}_n \), so that

\[
G_n[a_1, \ldots, a_n, r](t) \leq \bar{g}_n, \quad \forall(a_1, \ldots, a_n) \in X_A^n, r \in [0, \bar{t}], \ t \in [r, \bar{t}]. \tag{191}
\]

Then, by (93), we have

\[
| G_{n+1}[a_1, \ldots, a_n, a_{n+1}, r](t + \Delta t) - G_{n+1}[a_1, \ldots, a_n, a_{n+1}, r](t) |
\]

\[
= | \int_t^{t + \Delta t} G_n[a_1, \ldots, a_n, r](s) \cdot a_n(s) \cdot \exp(- \int_s^{t + \Delta t} a_{n+1}(u) du) \cdot ds

- \int_t^{t + \Delta t} G_n[a_1, \ldots, a_n, r](s) \cdot a_n(s) \cdot \exp(- \int_s^{t} a_{n+1}(u) du) \cdot ds |
\]

\[
\leq f_{L}^{t + \Delta t} G_n[a_1, \ldots, a_n, r](s) \cdot a_n(s) \cdot \exp(f_{L}^{t + \Delta t} a_{n+1}(u) du) - 1) \cdot ds

+ f_{L}^{t + \Delta t} G_n[a_1, \ldots, a_n, r](s) \cdot a_n(s) \cdot ds,
\]

which, as \((a_1, \ldots, a_n) \in X_A^n\) and hence \(a_n \in X_A\), is smaller than \((2\bar{L} - \tilde{\beta} \cdot \tilde{t} + 1) \cdot \bar{g}_n \cdot \bar{L} \cdot \tilde{\beta} - \Delta t \) when \( \Delta t \leq \bar{t}/(2\bar{L} - \tilde{\beta}) \). Also by (93) and again the fact that \(a_n \in X_A\), we may see that \( \bar{g}_{n+1} = \bar{L} - \tilde{\beta} \cdot \tilde{t} \cdot \bar{g}_n \) is a uniform upper bound for the \( G_{n+1}[a_1, \ldots, a_n, a_{n+1}, r]'s. \)

**S. Proof of Lemma 13:** We prove by induction. Given \( a_1, a_1' \in X_A \), there will be a time-rescaling function \( l \), so that for every \( t \in [-\theta, \bar{t}] \), both

\[
| l(t) - t | < 2\rho(a_1, a_1'), \tag{193}
\]

and

\[
| a_1'(t) - a_1(l(t)) | < 2\rho(a_1, a_1'). \tag{194}
\]

Then, by (92), we have

\[
| G_1[a_1, 0](t) - G_1[a_1', 0](t) |
\]

\[
= | \exp(- \int_0^t a_1(s) \cdot ds) - \exp(- \int_0^t a_1'(s) \cdot ds) |
\]

\[
\leq \exp(\int_0^t | a_1'(s) - a_1(s) | \cdot ds) - 1 \leq \exp(T_1 + T_2 + T_3) - 1, \tag{195}
\]

where

\[
T_1 = \int_0^t | a_1'(s) - a_1(l(s)) | \cdot ds, \tag{196}
\]

\[
T_2 = \int_{l'}^t | a_1(l(s)) - a_1(s) | \cdot ds, \tag{197}
\]

\[
T_3 = \int_{l'}^t | a_1'(s) - a_1(l(s)) | \cdot ds,
\]

\[
T_1 + T_2 + T_3
\]

\[
\leq \int_0^t | a_1'(s) - a_1(l(s)) | \cdot ds + \int_{l'}^t | a_1(l(s)) - a_1(s) | \cdot ds + \int_{l'}^t | a_1'(s) - a_1(l(s)) | \cdot ds.
\]
and

\[ T_3 = \int_{[0,t] \setminus W} | a_1(l(s)) - a_1(s) | \cdot ds, \tag{198} \]

while \( W \) is a subset of \([0,t]\) defined as follows:

\[ W = \{ s \in [0,t] \mid s \text{ is more than } 2\rho(a_1,a_1') \text{ away from any discontinuity point of } a_1 \}. \tag{199} \]

By the fact that \( a_1 \in X_A \) and (A-I), \( W \) is a finite union of intervals, and hence is Lebesgue measurable. By (194) and (196), we know that

\[ T_1 < 2\bar{t} \cdot \rho(a_1,a_1'). \tag{200} \]

By (193) and (199), we know that any \( s \in W \) satisfies that, \( s \) and \( l(s) \) belong to the same continuity segment of \( a_1 \). By (A-I), we know that \( a_1 \) has at most \( \bar{n} \) discontinuities in \( t \). In addition, \([-\theta, \bar{t}]\) is a compact set and all continuous segments of \( a_1 \) may be deemed as closed intervals. Hence, for any \( \epsilon > 0 \), there will be some \( \tilde{\delta}_1(\epsilon, a_1) > 0 \), such that

\[ | a_1(t) - a_1(t') | < \epsilon, \tag{201} \]

as long as \( t, t' \in [-\theta, \bar{t}] \) belong to the same continuity segment of \( a_1 \) and \( | t - t' | < \tilde{\delta}_1(\epsilon, a_1) \).

Therefore, as long as

\[ \rho(a_1,a_1') \leq \frac{\tilde{\delta}_1((\epsilon \wedge \bar{x}^0)/(6\bar{t}), a_1)}{2}, \tag{202} \]

we will have \( | a_1(l(s)) - a_1(s) | < (\epsilon \wedge \bar{x}^0)/(6\bar{t}) \) for any \( s \in W \), and hence, by (197),

\[ T_2 < \frac{\epsilon \wedge \bar{x}^0}{6}. \tag{203} \]

By (A-I) and (199) again, we know that the Lebesgue measure \( | [0,t] \setminus W | \leq 4\bar{n} \cdot \rho(a_1,a_1') \). Using also (A-III), that each \( a_1(t) \) is in \([0, \bar{\alpha} L^{\bar{\beta}^0}]\), as well as (198), we may derive that

\[ T_3 < 4\bar{n}\bar{\alpha} L^{\bar{\beta}^0} \cdot \rho(a_1,a_1'). \tag{204} \]

Summarizing (195), (200), (202), (203), and (204), we see that, for any \( \epsilon > 0 \), it will follow that

\[ \| G_1[a_1,0] - G_1[a_1',0] \|_\infty < \exp(\frac{\epsilon \wedge \bar{x}^0}{2}) - 1 \leq \epsilon, \tag{205} \]

whenever

\[ \rho(a_1,a_1') < \frac{\epsilon \wedge \bar{x}^0}{12\bar{t}} \wedge \frac{\tilde{\delta}_1((\epsilon \wedge \bar{x}^0)/(6\bar{t}), a_1)}{2} \wedge \frac{\epsilon \wedge \bar{x}^0}{24\bar{n}\bar{\alpha} L^{\bar{\beta}^0}}. \tag{206} \]
Suppose for some \( n = 1, 2, \ldots \), we know that \( G_n[a_1, \ldots, a_n, 0] \) is continuous in \( (a_1, \ldots, a_n) \). That is, for any \( \epsilon > 0 \) and \( a = (a_1, \ldots, a_n) \in X^n_A \), there exists some \( \delta_n(\epsilon, a) > 0 \), such that, for any \( a' = (a'_1, \ldots, a'_n) \in X^n_A \) satisfying \( \rho^n(a, a') < \delta_n(\epsilon, a) \), we have

\[
|| G_n[a_1, \ldots, a_n, 0] - G_n[a'_1, \ldots, a'_n, 0] ||_{\infty} < \epsilon. \tag{207}
\]

Now given \( (a, a_{n+1}) \), \( (a', a'_{n+1}) \in X^{n+1}_A \), we can check that \( a, a' \in X^n_A \) and \( a_{n+1}, a'_{n+1} \in X_A \). By (93), we have

\[
| G_{n+1}[a_1, \ldots, a_n, a_{n+1}, 0](t) - G_{n+1}[a'_1, \ldots, a'_n, a'_{n+1}, 0](t) | \leq T_1 + T_2 + T_3, \tag{208}
\]

where

\[
T_1 = | \int_0^t G_n[a_1, \ldots, a_n, 0](s) \cdot a_n(s) \cdot \exp(-\int_s^t a_{n+1}(u)du) \cdot ds - \int_0^t G_n[a'_1, \ldots, a'_n, 0](s) \cdot a_n(s) \cdot \exp(-\int_s^t a_{n+1}(u)du) \cdot ds |, \tag{209}
\]

\[
T_2 = | \int_0^t G_n[a'_1, \ldots, a'_n, 0](s) \cdot a_n(s) \cdot \exp(-\int_s^t a_{n+1}(u)du) \cdot ds - \int_0^t G_n[a'_1, \ldots, a'_n, 0](s) \cdot a'_n(s) \cdot \exp(-\int_s^t a_{n+1}(u)du) \cdot ds |, \tag{210}
\]

and

\[
T_3 = | \int_0^t G_n[a'_1, \ldots, a'_n, 0](s) \cdot a'_n(s) \cdot \exp(-\int_s^t a_{n+1}(u)du) \cdot ds - \int_0^t G_n[a'_1, \ldots, a'_n, 0](s) \cdot a'_n(s) \cdot \exp(-\int_s^t a'_{n+1}(u)du) \cdot ds |. \tag{211}
\]

And, we can pick a time-rescaling function \( l \) so that, for any \( t \in [-\theta, \theta] \), both

\[
| l(t) - t | < 2\rho^{n+1}((a, a_{n+1}), (a', a'_{n+1})), \tag{212}
\]

and

\[
\max_{i=1}^{n+1} | a'_i(t) - a_i(l(t)) | < 2\rho^{n+1}((a, a_{n+1}), (a', a'_{n+1})). \tag{213}
\]

From (A-III) and (209), we have

\[
T_1 \leq \alpha L \bar{\beta} \bar{t} \cdot || G_n[a_1, \ldots, a_n, 0] - G_n[a'_1, \ldots, a'_n, 0] ||_{\infty}, \tag{214}
\]

which, according to the inductive hypothesis around (207), will be smaller than \( \epsilon/3 \) when

\[
\rho^n(a, a') \leq \rho^{n+1}((a, a_{n+1}), (a', a'_{n+1})) < \bar{\delta}_n(\frac{\epsilon}{3\alpha L \bar{\beta} \bar{t}}, a). \tag{215}
\]

From (210), we have

\[
T_2 \leq T_{21} + T_{22} + T_{23}, \tag{216}
\]

where

\[
T_{21} = \int_0^t G_n[a'_1, \ldots, a'_n, 0](s) \cdot G_1[a_{n+1}, s](t) \cdot | a'_n(s) - a_n(l(s)) | \cdot ds. \tag{217}
\]
\[
T_{22} = \int_{W} G_n[a_1', \ldots, a_n', 0](s) \cdot G_1[a_{n+1}, s](t) \cdot |a_n(l(s)) - a_n(s)| \cdot ds, \tag{218}
\]
and
\[
T_{23} = \int_{[0,t] \setminus W} G_n[a_1', \ldots, a_n', 0](s) \cdot G_1[a_{n+1}, s](t) \cdot |a_n(l(s)) - a_n(s)| \cdot ds, \tag{219}
\]
while \(W\) is a subset of \([0,t]\) defined as follows:
\[
W = \{ s \in [0,t] \mid s \text{ is more than } 2\rho^{n+1}((a, a_{n+1}), (a', a'_{n+1})) \text{ away from any discontinuity point of } (a, a_{n+1}) \}. \tag{220}
\]
Here, as \((a, a_{n+1}) \in X_n^{A+1}\) and (A-I) applies, \(W\) is again a finite union of intervals, and hence is Lebesgue measurable. By Lemma 12, (213), and (217), we know that
\[
T_{21} < 2\bar{g}_n \cdot \rho^{n+1}((a, a_{n+1}), (a', a'_{n+1})). \tag{221}
\]
By (212) and (220), we know that any \(s \in W\) satisfies that, \(s\) and \(l(s)\) belong to the same continuity segment of \((a, a_{n+1})\). By (A-I), we know that \((a, a_{n+1})\) has at most \(\bar{n}\) discontinuities in \(t\). In addition, \([-\theta, \bar{t}]\) is a compact set and all continuous segments of \((a, a_{n+1})\) may be deemed as closed intervals. Hence, for any \(\epsilon > 0\), there will be some \(\delta_{n+1}(\epsilon, (a, a_{n+1})) > 0\), such that
\[
|| (a, a_{n+1})(t) - (a, a_{n+1})(t') ||^{n+1} < \epsilon, \tag{222}
\]
as long as \(t, t' \in [-\theta, \bar{t}]\) belong to the same continuity segment of \((a, a_{n+1})\) and \(| t - t' | < \delta_{n+1}(\epsilon, (a, a_{n+1}))\). Therefore, as long as
\[
\rho^{n+1}((a, a_{n+1}), (a', a'_{n+1})) \leq \frac{\delta_{n+1}(\epsilon/9\bar{g}_n \bar{t}), (a, a_{n+1})}{2}, \tag{223}
\]
we will have \(| a_n(l(s)) - a_n(s) | < \epsilon/(9\bar{g}_n \bar{t})\) for any \(s \in W\), and hence, by Lemma 12 again and (218),
\[
T_{22} < \frac{\epsilon}{\bar{g}}. \tag{224}
\]
By (A-I) and (220), we know that the Lebesgue measure \(| [0, t] \setminus W | \leq 4\bar{n} \cdot \rho^{n+1}((a, a_{n+1}), (a', a'_{n+1}))\). Using also (A-III), that each \((a, a_{n+1})(t)\) is in \([0, \bar{\alpha}L \bar{\beta}_0]^{n+1}\), as well as Lemma 12 and (219), we may derive that
\[
T_{23} < 4\bar{n} \bar{g}_n \bar{\alpha}L \bar{\beta}_0 \cdot \rho^{n+1}((a, a_{n+1}), (a', a'_{n+1})). \tag{225}
\]
Due to (92), (A-III), Lemma 12, the inductive hypothesis around (207), and (211), we shall have
\[
T_3 \leq \int_{0}^{t} G_n[a_1', \ldots, a_n', 0](s) \cdot a_n'(s) \cdot | G_1[a_{n+1}, s](t) - G_1[a'_{n+1}, s](t) | \cdot ds < \frac{\epsilon}{3}, \tag{226}
\]
as long as
\[ \rho(a_{n+1}, a'_{n+1}) \leq \rho^{n+1}(a, a_{n+1}, (a', a'_{n+1})) < \delta_1(\epsilon, a_{n+1}). \] (227)

Combining (215), (221), (223), (224), (225), and (227), we see that, for any \( \epsilon > 0 \), we will have
\[ || G_n[a_1, ..., a_n, a_{n+1}, 0] - G_{n+1}[a_1', ..., a'_{n+1}, 0] ||_\infty < \epsilon, \] (228)
as long as
\[ \rho^{n+1}(a, a_{n+1}, (a', a'_{n+1})) < \text{the minimum of the } (\epsilon, a_{n+1}) - \text{dependent } \delta_1 \text{ through } \delta_5, \] (229)
where \( \delta_1 = \delta_n(\epsilon/(3\tilde{a}_n\tilde{\beta}_0t), a) \), \( \delta_2 = \epsilon/(18\tilde{g}_n\tilde{t}) \), \( \delta_3 = \tilde{\delta}_n(\epsilon/(9\tilde{g}_n\tilde{t}), (a, a_{n+1}))/2 \), \( \delta_4 = \epsilon/(36\tilde{g}_n\tilde{a}_n\tilde{\alpha}\tilde{\beta}_0) \), and \( \delta_5 = \delta_1(\epsilon/(3\tilde{g}_n\tilde{a}_n\tilde{\alpha}\tilde{\beta}_0), a_{n+1}) \).

We have thus completed the induction process.

\[ \square \]

**T. Proof of Proposition 6:** From Lemma 4, we know that, for any \( \epsilon > 0 \), there exists \( \delta^{\tilde{B}'}_M(\epsilon) > 0 \), such that, for any \( m^1, m^2 \in X^2_M \) satisfying \( \rho^2(m^1, m^2) < \delta^{\tilde{B}'}_M(\epsilon) \), it follows that
\[ \rho(Z^{\tilde{B}'}_M(m^1), Z^{\tilde{B}'}_M(m^2)) < \epsilon. \] (230)

From Lemma 5, we know that, for any \( \epsilon > 0 \) and \( \tilde{B}^1 \in X_{\tilde{B}'} \), there exists \( \delta^{\tilde{B}'}_{\tilde{B}'}(\epsilon, \tilde{B}^1) > 0 \), such that, for any \( \tilde{B}^2 \in X_{\tilde{B}'} \) satisfying \( \rho(\tilde{B}^1, \tilde{B}^2) < \delta^{\tilde{B}'}_{\tilde{B}'}(\epsilon, \tilde{B}^1) \), it follows that
\[ || Z^{\tilde{B}'}_{\tilde{B}'}(\tilde{B}^1) - Z^{\tilde{B}'}_{\tilde{B}'}(\tilde{B}^2) ||_\infty < \epsilon. \] (231)

From Lemma 9, we know that, for any \( \epsilon > 0 \) and \( \tilde{B}^1 \in Y_{\tilde{B}} \), there exists \( \delta^{\tilde{B}'}_{\tilde{B}}(\epsilon, \tilde{B}^1) > 0 \), such that, for any \( \tilde{B}^2 \in Y_{\tilde{B}} \) satisfying \( || \tilde{B}^1 - \tilde{B}^2 ||_\infty < \delta^{\tilde{B}}_{\tilde{B}}(\epsilon, \tilde{B}^1) \), it follows that
\[ || Z^{\tilde{B}}_{\tilde{B}}(\tilde{B}^1) - Z^{\tilde{B}}_{\tilde{B}}(\tilde{B}^2) ||_\infty < \epsilon. \] (232)

From Lemma 10, we know that, for any \( \epsilon > 0 \) and \( f^1 \in Y^\tilde{B}_{\tilde{F}} \), there exists \( \delta^{\tilde{M}}_{\tilde{P}\tilde{F}}(\epsilon, f^1) > 0 \), such that, for any \( p^1, p^2 \in \Delta^\tilde{P} \) satisfying \( || p^1 - p^2 ||_\infty < \delta^{\tilde{M}}_{\tilde{P}\tilde{F}}(\epsilon, f^1) \), it follows that
\[ \rho^2(Z^{\tilde{M}}_{\tilde{P}\tilde{F}}(p^1, f^1), Z^{\tilde{M}}_{\til\til{P}\til{F}}(p^2, f^1)) < \epsilon. \] (233)

From the same lemma, we know that, there exists \( \tilde{c}^{\til{M}}_{\til{P}\til{F}} > 0 \), such that, for any \( p \in \Delta^\til{P} \) and \( f^1, f^2 \in Y^\til{B}_{\til{F}} \), it follows that
\[ \rho^2(Z^{\til{M}}_{\til{P}\til{F}}(p, f^1), Z^{\til{M}}_{\til{P}\til{F}}(p, f^2)) < \tilde{c}^{\til{M}}_{\til{P}\til{F}} \cdot || f^1 - f^2 ||_\infty. \] (234)
From Lemma 11, we know that, for any $\epsilon > 0$ and $\beta^1 \in Y_B$, there exists $\delta_{PB}^A(\epsilon, \beta^1) > 0$, such that, for any $p^1, p^2 \in \Delta^P$ satisfying $\| p^1 - p^2 \|_n < \delta_{PB}^A(\epsilon, \beta^1)$, it follows that

$$\rho^A(Z_{PB}^A(p^1, \beta^1), Z_{PB}^A(p^2, \beta^1)) < \epsilon. \quad (235)$$

From the same lemma, we know that, there exists $\bar{c}_{PB} > 0$, such that, for any $p \in \Delta^P$ and $\beta^1, \beta^2 \in Y_B$, it follows that

$$\rho^A(Z_{PB}^A(p, \beta^1), Z_{PB}^A(p, \beta^2)) < \bar{c}_{PB} \cdot \| \beta^1 - \beta^2 \|_\infty. \quad (236)$$

From Lemma 14, we know that, for any $\epsilon > 0$ and $\lambda^1 \in X^A_n$, there exists $\delta_{PB}^F(\epsilon, \lambda^1) > 0$, such that, for any $\lambda^2 \in X^A_n$ satisfying $\rho^A(\lambda^1, \lambda^2) < \delta_{PB}^F(\epsilon, \lambda^1)$, it follows that

$$\| Z_{FA}^F(\lambda^1) - Z_{FA}^F(\lambda^2) \|_\infty < \epsilon. \quad (237)$$

Suppose that $\epsilon > 0$ and $\beta^1 \in Y_B$ are given. For convenience, let $p^1 = Z_{PB}^P(\beta^1)$, $\lambda^1 = Z_{PB}^A(p^1, \beta^1)$, $f^1 = Z_{PB}^F(\lambda^1)$, $m^1 = Z_{PF}^M(p^1, f^1)$, and $\bar{B}^1 = Z_{PM}^B(m^1)$. By (37), we know that $Z_{B}^B(\beta^1) = Z_{B}^B(\bar{B}^1)$.

Now, define $\delta_{PB}^B(\epsilon, \beta^1) > 0$ as follows:

$$\delta_{PB}^B(\epsilon, \beta^1) = \delta_{PB}^P(\epsilon, \beta^1) \wedge \frac{\delta_{PB}^F(\epsilon, \beta^1)}{2\bar{c}_{PB}^{M}}, \quad (238)$$

where

$$\delta_{PB}^P(\epsilon, \beta^1) = \delta_{PB}^M(\frac{\delta_{PB}^M(\epsilon, \beta^1)}{2}, f^1) \wedge \delta_{PB}^A(\frac{\delta_{PB}^M(\epsilon, \beta^1)}{2}, \bar{B}^1), \quad (239)$$

and in turn,

$$\delta_{PB}^B(\epsilon, \beta^1) = \delta_{PB}^B(\epsilon, \bar{B}^1). \quad (240)$$

For any $\bar{B}^2 \in Y_B$ satisfying $\| \beta^1 - \beta^2 \|_\infty < \delta_{PB}^B(\epsilon, \beta^1)$, we let $p^2 = Z_{PB}^P(\beta^2)$, $\lambda^2 = Z_{PB}^A(p^2, \beta^2)$, $f^2 = Z_{PB}^F(\lambda^2)$, $m^2 = Z_{PF}^M(p^2, f^2)$, and $\bar{B}^2 = Z_{PM}^B(m^2)$. By (37), we know that $Z_{B}^B(\beta^2) = Z_{B}^B(\bar{B}^2)$.

By the definition of $\delta_{PB}^P(\cdot, \cdot)$ around (232) and the definition of $\delta_{PB}^B(\cdot, \cdot)$ around (238), we know that

$$\| p^1 - p^2 \|_n = \| Z_{PB}^P(\beta^1) - Z_{PB}^P(\beta^2) \|_n < \delta_{PB}^B(\epsilon, \beta^1). \quad (241)$$

By the definition of $\delta_{PB}^A(\cdot, \cdot)$ around (235), the definition of $\bar{c}_{PB}^A$ around (236), the definition of $\delta_{PB}^B(\cdot, \cdot)$ around (238), the definition of $\delta_{PB}^P(\cdot, \cdot)$ around (239), as well as (241), we know that

$$\rho^A(\lambda^1, \lambda^2) \leq \rho^A(Z_{PB}^P(p^1, \beta^1), Z_{PB}^P(p^2, \beta^1)) + \rho^A(Z_{PB}^A(p^1, \beta^1), Z_{PB}^A(p^2, \beta^2))$$

$$< 2 \cdot \frac{\delta_{PB}^F(\epsilon, \beta^1)}{2\bar{c}_{PB}^{M}}, \lambda^1) + \frac{\delta_{PB}^F(\epsilon, \beta^1)}{2\bar{c}_{PB}^{M}}(\lambda^1, \lambda^1)) = \frac{\delta_{PB}^A(\bar{B}^1, \beta^1)}{(2\bar{c}_{PB}^{M})}. \quad (242)$$

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From the definition of $\delta_A^F(\cdot,\cdot)$ around (237), as well as (242), we know that
\[
\| f^1 - f^2 \|_\infty = \| Z_A^F(\lambda^1) - Z_A^F(\lambda^2) \|_\infty^\beta < \frac{\beta_M^B(\epsilon, \beta^1)}{2c_{PF}^M}.
\] (243)

From the definition of $\delta_{PF}^M(\cdot,\cdot)$ around (233), the definition of $\delta^M_{PF}(\cdot)$ around (234), the definition of $\delta_B^P(\cdot,\cdot)$ around (239), as well as (241) and (243), we know that
\[
\rho^2(m^1, m^2) < \rho^2(Z_{PF}^M(p^1, f^1), Z_{PF}^M(p^2, f^1)) + \rho^2(Z_{PF}^M(p^2, f^1), Z_{PF}^M(p^2, f^2)) < 2 \cdot \delta_{PF}^M(\epsilon, \beta^1)/2 = \delta_B^P(\epsilon, \beta^1).
\] (244)

From the definition of $\delta_{PF}^M(\cdot)$ around (230), the definition of $\delta_{PF}^M(\cdot,\cdot)$ around (240), as well as (244), we know that
\[
\rho(\beta^1, \beta^2) < \rho(Z_{PF}^M(m^1), Z_{PF}^M(m^2)) < \delta_B^P(\epsilon, \beta^1).
\] (245)

From the definition of $\delta_B^P(\cdot)$ around (231), as well as (245), we know that
\[
\| Z_{B'}^P(\beta^1) - Z_{B'}^P(\beta^2) \|_\infty < \epsilon.
\] (246)

Therefore, $Z_B^P$ is a continuous mapping from $(Y_B, \| \cdot \|_\infty)$ to $(Y_B, \| \cdot \|_\infty)$. \hfill $\blacksquare$

**U. Proof of Theorem 2:** Due to Propositions 6, 7, and 8, we can use the point-map version of the Kakutani-Glicksberg-Fan fixed point theorem to establish the existence of an actual stream $\beta^S \in Y_B$ satisfying $\beta^S = Z_B^P(\beta^S)$. We may define inventory- and time-monotone policy $p^S = Z_B^P(\beta^S)$, arrival-rate stream $\lambda^S = Z_{PF}^A(p^S, \beta^S)$, stock-distribution process $f^S = Z_{PF}^F(\lambda^S)$, market process $m^S = Z_{PF}^M(p^S, f^S)$, and instantaneous stream $\beta^S = Z_{B'}^P(m^S)$. These and the definition of $Z_B^P$ through (37) will lead to $\beta^S = Z_{B'}^P(\beta^S)$. \hfill $\blacksquare$

**V. Discussion on the Tatonnement Schemes in Section 4:** We first focus on the deterministic case. We may associate a given complete lattice $S$ with the interval topology, in which sets of forms $[\inf S, x] \equiv \{ y \in S \mid y \leq x \}$ and $[x, \sup S] \equiv \{ y \in S \mid y \geq x \}$ constitute a sub-basis of closed sets. For a totally ordered chain $C \subset S$, both $\inf C$ and $\sup C$ are limit points of $C$ under this topology. We have the following.

**Proposition 9** Suppose complete lattice $\mathcal{M}^0$ and isotope map $Z_{\mathcal{M}}^M$ satisfy conditions imposed in Theorem 1. Furthermore, $Z_{\mathcal{M}}^M$ is continuous under the interval topology of $\mathcal{M}^0$. Let $m^0 \in \mathcal{M}^0$ be the smallest member of $\mathcal{M}^0$, which necessarily satisfies $m^{0L}(t) = 0$ and $m^{0H}(t) = 0$ for every $t \in [0, \bar{t}]$. For $k = 0, 1, \ldots$, define $m^{k+1} = Z_{\mathcal{M}}^M(m^k)$. Then, $C \equiv \{m^0, m^1, \ldots\}$ forms a totally ordered chain within $\mathcal{M}^0$ and $m^\infty \equiv \sup C \in \mathcal{M}^0$ qualifies as an equilibrium $m^D$ identified in Theorem 1.
Proof: As $Z^M_M$ is an isotone map from $\mathcal{M}^0$ to $\mathcal{M}^0$ by Proposition 5 and $m^0$ is clearly the smallest member of $\mathcal{M}^0$, we have $m^0 \leq m^1 \leq m^2 \leq \cdots$ and the chain $C \equiv \{m^0, m^1, \ldots\} \subset \mathcal{M}^0$. As $\mathcal{M}^0$ is a complete lattice by Proposition 4, we know $m^\infty \equiv \sup C \in \mathcal{M}^0$.

Note that $m^\infty = \lim_{k \to +\infty} m^k$ under the interval topology of $\mathcal{M}^0$. By the continuity of $Z^M_M$ under the same topology, we have

$$Z^M_M(m^\infty) = Z^M_M(\lim_{k \to +\infty} m^k) = \lim_{k \to +\infty} Z^M_M(m^k) = \lim_{k \to +\infty} m^{k+1} = m^\infty. \quad (247)$$

Therefore, we can let the $m^D$ in Theorem 1 be $m^\infty$.

Under time discretization, the counterpart to $\mathcal{M}^0$ is a compact subset of a multi-dimensional real space. On it, the interval topology is equivalent to the Euclidean topology. From the continuity of the initial-stock cdf $F(\cdot)$, (D3) which specifies the continuity of $\bar{\lambda}^L(\cdot)$ and $\bar{\lambda}^H(\cdot)$, and (15) whose right-hand side helps to define $Z^M_M$, we may clearly see that the counterpart to the operator $Z^M_M$ under time discretization is continuous under the Euclidean topology. Thus, Proposition 9 would guarantee the convergence of our Tatônnement scheme for the deterministic case.

For the stochastic case, Theorem 2 is reached through the use of Tychonoff's fixed point theorem. The latter may be regarded as a generalization of Brouwer's fixed point theorem. It is not generally known whether a Tatônnement scheme will converge to a fixed point under the latter theorem's conditions, unless the concerned function possesses special properties such as being a contraction mapping. Due partially to the involvement of the demand-to-decision mapping $Z^P_B$, which itself comes from a complex iterative process, it is unlikely that the composite operator $Z^P_B$ involved in Theorem 2 will be contracting or in possession of any nice properties other than continuity. Therefore, a theoretical guarantee to our Tatônnement scheme for the stochastic case is unlikely to be had.

On the other hand, our extensive numerical tests have demonstrated that iterative methods for both cases converge quickly and always to unique equilibria. In these tests, we have let the number of firms $\bar{n}$ assume values 5, 10, 30, 50, and 100, and at each $\bar{n}$, have randomly generated 1,000 problem instances by sampling the problem-defining parameters $\bar{\mu}$, $\bar{v}$, $\bar{\gamma}$, and $\bar{\delta}$ from the uniform distribution on $(0, 1)$. At each problem instance and for either case, we initiate the corresponding Tatônnement scheme from ten different starting points.

We always find the scheme for either case to converge to the same equilibrium regardless of the starting point. For the deterministic case, the number of iterations needed for convergence is less
than 10; for the stochastic case, that number is less than 20. For a better glimpse of the convergence picture, we present in Table 1 the average numbers of iterations needed for convergence for both the deterministic- and stochastic cases under various \( \bar{n} \) values.

Table 1: Average Numbers of Iterations Needed for Convergence

<table>
<thead>
<tr>
<th>Case / ( \bar{n} )</th>
<th>5</th>
<th>10</th>
<th>30</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic</td>
<td>4.72</td>
<td>4.96</td>
<td>5.16</td>
<td>5.23</td>
<td>5.24</td>
</tr>
<tr>
<td>Stochastic</td>
<td>6.58</td>
<td>6.94</td>
<td>7.24</td>
<td>7.45</td>
<td>7.71</td>
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