

Review 2 – Elementary Differential Equations

I. Second order equations – variation of parameters (§3.7): Let $L[y] := y'' + p(t)y' + q(t)y$. If y_1 and y_2 are solutions of $L[y] = 0$ such that $W(y_1, y_2) \neq 0$, then a solution to $L[y] = g(t)$ is $y = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)} dt$.

II. Higher order linear equations:

II.1. Main theorems (§4.1): Let $L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}y' + p_n(x)y$.

(i). (Thm 4.1.1) Consider the IVP $L[y] = g(x), y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$. If there is an open interval containing x_0 on which all $p_1(x), p_2(x), \dots, p_n(x)$ and $g(x)$ are continuous, then the IVP problem has a unique solution and the solution is defined on the largest such open interval.

(ii). (Thm 4.1.2, optional) If $y_1(x), y_2(x), \dots, y_n(x)$ are solutions of $L[y] = 0$ and the Wronskian $W(y_1, \dots, y_n)$ is not zero for at least one x , then the equation has general solution $y(x) = c_1y_1(x) + c_2y_2(x) + \dots + c_ny_n(x)$.

(iii). (Equation (9), p194) Let y_1, y_2, \dots, y_n be as in (ii) and $Y(x)$ be one solution to $L[y] = g(x)$. Then the general solution to $L[y] = g(x)$ is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \dots + c_ny_n(x) + Y(x).$$

II.2 Homogeneous linear equations with constant coefficients (§4.2):

Let $a_0y^{(n)} + a_1y^{(n-1)} + \dots + a_ny = 0$. Let $Z(r) = a_1r^n + a_1r^{n-1} + \dots + a_n$. Then a real root r_1 of $Z(r)$ with multiplicity n_1 gives n_1 fundamental solutions $e^{r_1t}, te^{r_1t}, \dots, t^{n_1-1}e^{r_1t}$; A pair of complex roots $\lambda_1 \pm i\mu_1$ of $Z(r)$

with multiplicity m_1 gives $2m_1$ fundamental solutions $\begin{cases} e^{\lambda_1 t} \cos \mu_1 t, te^{\lambda_1 t} \cos \mu_1 t, \dots, t^{m_1-1} e^{\lambda_1 t} \cos \mu_1 t \\ e^{\lambda_1 t} \sin \mu_1 t, te^{\lambda_1 t} \sin \mu_1 t, \dots, t^{m_1-1} e^{\lambda_1 t} \sin \mu_1 t \end{cases}$

Putting together fundamental solutions from all the roots, we have the general solution of $L[y] = 0$.

II.3. Undetermined coefficients (§3.6, §4.3): Let $a_0y^{(n)} + a_1y^{(n-1)} + \dots + a_ny = g(t)$, where

$g(t) = (b_0t^m + \dots + b_m)e^{\alpha t} \begin{cases} \cos \beta t \\ \sin \beta t \end{cases}$ If the multiplicity of $\alpha + i\beta$ as a root of $Z(t)$ is s , then a particular solution of $L[y] = g(t)$ has the form $Y(t) = t^s [(A_0t^m + \dots + A_m)e^{\alpha t} \cos \beta t + (B_0t^m + \dots + B_m)e^{\alpha t} \sin \beta t]$.

To find $Y(t)$, plug it in the equation and determine the coefficients.

III. Series solutions (§5.1, 2, 4-7)

III.1. Ordinary points (§5.2): Let $P y'' + Q y' + R y = 0$ where P, Q, R are polynomials. If $P(x_0) \neq 0$, then x_0 is called an *ordinary point* for the equation. To find the series solution near an ordinary point $x_0 = 0$,

i) take derivatives of $y = \sum_{n=0}^{\infty} a_n x^n$ and plug them into $P y'' + Q y' + R y = 0$,

ii) index shifting if needed, collect coefficients of x^n and get $\sum E_n x^n$,

iii) solve $E_n = 0$ and get recurrence relations of a_n in terms of a_0 and a_1 ,

iv) put these relations back to $y = \sum a_n x^n$ and rewrite y in the form $y = a_0 y_1 + a_1 y_2$, where y_1 and y_2 are power series. The last expression is called the **series solution** of the differential equation.

III.2. Euler equations (§5.5): For an **Euler equation** $x^2 y'' + \alpha x y' + \beta y = 0$, $r^2 + (\alpha - 1)r + \beta$ is called the **indicial polynomial**. The general solution of the Euler equation is

$$y(x) = \begin{cases} c_1|x|^{r_1} + c_2|x|^{r_2}, & \text{if it has real roots } r_1 \neq r_2, \\ c_1|x|^\lambda \cos(\mu \ln|x|) + c_2|x|^\lambda \sin(\mu \ln|x|), & \text{if it has complex roots } \lambda \pm i\mu, \\ c_1|x|^{r_1} + c_2(\ln|x|)|x|^{r_1}, & \text{if it has real roots } r_1 = r_2. \end{cases}$$

III.3. Series solutions near a regular singular point ($x_0 = 0$) :

i) $x = 0$ is called a **regular singular point** of $y'' + p(x)y' + q(x)y = 0$ if one of $\lim_{x \rightarrow 0} p(x)$ and $\lim_{x \rightarrow 0} q(x)$ does not exist, but $p_0 = \lim_{x \rightarrow 0} xp(x)$ and $q_0 = \lim_{x \rightarrow 0} x^2q(x)$ exist.

ii). $r^2 + (p_0 - 1)r + q_0 = 0$ is called the **indicial equation**. Its real solutions $r_1 \geq r_2$ are called the **exponents of singularities**.

iii). For the general solution $y = c_1y_1 + c_2y_2$, we have $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$. To determine a_n , plug the series in the differential equation and proceed as for ordinary points.