Renormalization on Toric Varieties

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1 Why toric varieties?

The study of toric varieties is one of the most active area in mathematics for decades. Toric varieties appear almost everywhere in mathematics.

- Algebraic Geometry: structure of moduli space
- Combinatorics: Euler-MacLaurin problem
- Differential Geometry: Kähler-Einstein metric
- Mathematical Physics: Mirror Symmetry
- Number theory: local ζ-functions
- Representation Theory: quiver representations
- Singularity Theory: Frobenius structure
- Symplectic Topology: Orbifold quantum cohomology

2 Goal

To develop a new approach to some of these directions based on the combinatorial nature of toric varieties.

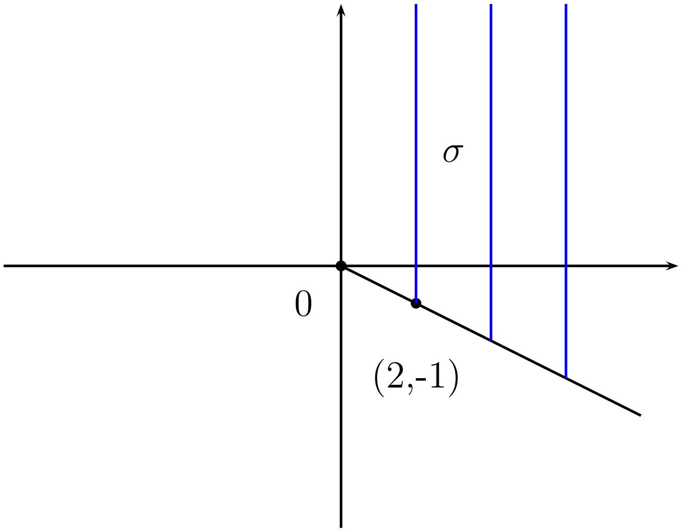
3 What is a toric variety?

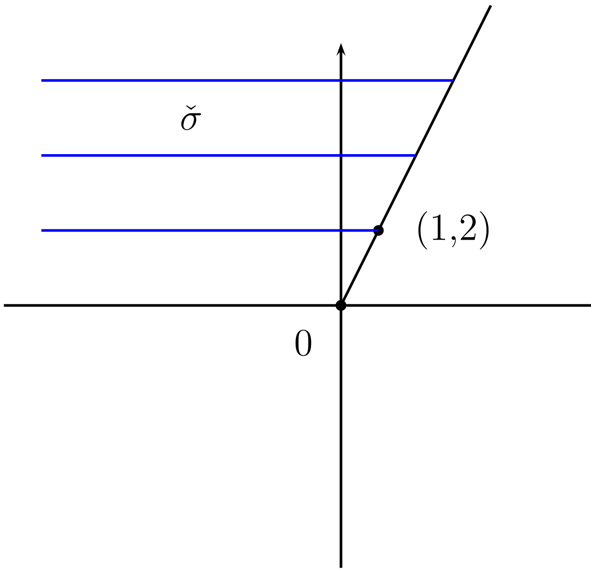
Let $T = (\mathbb{C}^*)^d$ be a rank d (complex) torus, roughly speak, any normal equivariant (partial) compactification of T with finite many orbits is a toric variety.

Let $N = Hom(\mathbb{C}^*, T) \cong \mathbb{Z}^d$ be the set of one parameter subgroups of T, $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $M = Hom(T, \mathbb{C}^*) \cong \mathbb{Z}^d$ be the character set of T. There is a natural pairing between M and N.

Any strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$ with vertex at origin, the dual $\check{\sigma} = \{v \in M| < u, v >\geq 0, \forall u \in \sigma\}$ is a finitely generated monoid, the group ring $\mathbb{C}[\check{\sigma}]$ is a finitely generated algebra, thus corresponds to an affine variety U_σ. It has a natural T action. Moreover if τ is a face of σ, we have a natural embedding $i_{\tau\sigma} : U_\tau \hookrightarrow U_\sigma$.

Example $\sigma \subset \mathbb{R}^2$.



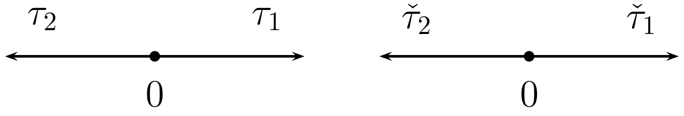


$$Spec\mathbb{C}[\check{\sigma}] = \{xy = z^2\}$$

A fan is a set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ with vertex at origin, which is closed under intersection and taking faces.

For a fan Σ, the corresponding toric variety $X_\Sigma = \sqcup_{\sigma\in\Sigma} U_\sigma / \sim$, where $\sim$ is defined as: $u \sim v$, $u \in U_{\tau_1}$, $v \in U_{\tau_2}$ if there is a cone $\sigma \in \Sigma$ which contains τ_1, τ_2 as faces and $i_{\tau_1\sigma}(u) = i_{\tau_2\sigma}(v)$.

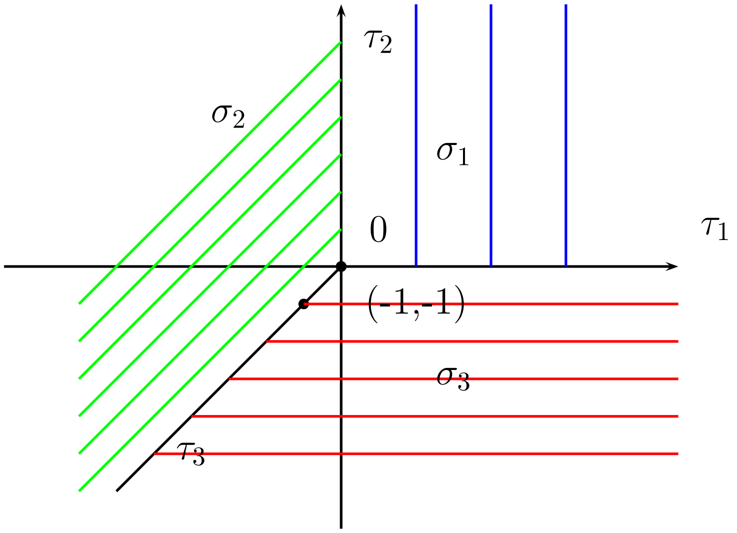
Example $\mathbb{CP}^1$



The set of these 3 cones $\{\{0\}, \tau_1, \tau_2\}$ is the fan for $\mathbb{CP}^1$:

$$\mathbb{C}[\check{\tau}_1] = \mathbb{C}[x],\ Spec\mathbb{C}[\check{\tau}_1] = \mathbb{C}$$
$$\mathbb{C}[\check{\tau}_2] = \mathbb{C}[x^{-1}],\ Spec\mathbb{C}[\check{\tau}_2] = \mathbb{C}$$
$$\mathbb{C}[\{\check{0}\}] = \mathbb{C}[x, x^{-1}],\ Spec\mathbb{C}[\{\check{0}\}] = \mathbb{C}^*$$

Example $\mathbb{CP}^2$, a fan in $\mathbb{R}^2$.



4 Propositions

The fan totally determines the corresponding toric variety.

PROPOSITION 4.1 *X_Σ has a natural T action, the orbits are 1-1 corresponding to the cones in Σ in reverse order.*

PROPOSITION 4.2 *X_Σ is compact iff Σ covers $\mathbb{R}^d$.*

PROPOSITION 4.3 *X_Σ is smooth iff every cone in Σ is spanned by part of basis of N. X_Σ is an orbifold iff every cone in Σ is spanned by linearly independent vectors in $\mathbb{R}^d$.*

For equivariant (co)homology of X_Σ:

PROPOSITION 4.4 *For a smooth X_Σ, $H_T^*(X_\sigma)$ is the ring of piecewise polynomial functions, in general there is a spectral sequence convergent to it.*

PROPOSITION 4.5 *$H_*^T(X_\Sigma)$ is generated by the closure of orbits over $H^*(BT)$.*

PROPOSITION 4.6 *The desingularization of a toric variety can be realized by subdivision of the fan.*

5 Todd Classes

The equivariant homology $H^T_*(X_\Sigma)$ is a module over $H^*(BT) = S^*(M)$. For any $m \in M$, it defines an element e^m of $\bar{S}^*(M)$-the completion of $S^*(M)$.

In $\bar{S}^*(M)$, we introduce the following summation,

DEFINITION 5.1 *$f \in \bar{S}^*(M)$ is summable if there exist $m_1, m_2, \cdots, m_k \in M$, such that $\Pi(1 - e^{m_i})f \in S(M)$. And we define the sum $s(f)$ to be $(\Pi(1 - e^{m_i})f)\Pi(1 - e^{m_i})^{-1}$*

This summation is defined by Brion-Vergne in their paper on equivariant Todd classes for toric varieties.

THEOREM 5.2 (Brylinski-Z.) *For any complete toric variety X_Σ, $\sum_{m \in \check{\sigma}} e^{-m}$ is summable, and if we let*

$$A_\sigma = s(\sum_{m \in \check{\sigma}} e^{-m}),$$

then

$$Td^T(X_\Sigma) = \sum A_\sigma[\bar{O}_\sigma]$$

is the localized equivariant Todd class.

Example For $\mathbb{CP}^1$,

$$Td^T(\mathbb{CP}^1) = \frac{1}{1 - e^{-e^*}}[\bar{O}_{\tau_1}] + \frac{1}{1 - e^{e^*}}[\bar{O}_{\tau_2}]$$

$$e^*[X] = [\bar{O}_{\tau_1}] - [\bar{O}_{\tau_2}]$$

$$[\bar{O}_{\tau_1}][\bar{O}_{\tau_2}] = 0$$

6 Hopf algebra of cones

Let us consider the filtered vector space $\mathbb{R}^\infty = \cup \mathbb{R}^k$, with standard basis $\{e_1, e_2, \cdots, e_k, \cdots\}$. Let Λ be the lattice generated by e_i's and Λ_k be the sublattice generated by $\{e_1, e_2, \cdots, e_k\}$.

We denote the vector space with a basis consisting of convex integral (relatively open) cones in $\mathbb{R}^k$ by V_k.

Let $W_k \subset V_k$ be the subspace generated by two types of elements: i) $C - \sum C_i'$ if $\{C_i'\}$, if $\{C_i'\}$ is a open subdivision of C; ii) C, where C contains a subspace of R^k. Let H^k be the quotient vector space.

By the standard embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$, we have $H^k \hookrightarrow H^{k+1}$, let $H = \cup H^k$.

By subdivision, we see that H is generated by smooth cones.

H has another filtration: $H = \cup H_d$, where H_d =the space generated by cones of dimension $\leq d$. Obviously $H_0 \cong \mathbb{C}$, it is spanned by the 0-dimensional cone $\{\ 0\}$.

DEFINITION 6.1 *For any two cones C_1 and C_2, we define $C_1 \bullet C_2 = \{(x,y) | x \in C_1, y \in C_2\}$ and $[C_1] \bullet [C_2] = [C_1 \bullet C_2]$. It is well-defined in H.*

Following Berline-Vergne,

DEFINITION 6.2 *For any cone C and its face F, we define the transverse cone $t(C,F)$ of F to be the projection of C on $lin^\perp(F)$, where $lin^\perp(F)$ is the orthogonal completion of $lin(F)$ in $lin(C)$.*

DEFINITION 6.3 *For any cone C, we define $\Delta : V_k \to V_k \otimes V_k$ by $\Delta([C]) = \Sigma_{F<C} [F] \otimes [t(C,F)]$.*

LEMMA 6.4 $\Delta(W_k) \subset V_k \otimes W_k + W_k \otimes V_k$

Therefore Δ induces a operator $\Delta : H^k \to H^k \otimes H^k$.

THEOREM 6.1 $(H, \bullet, \Delta)$ *defines a connected filtered Hopf algebra structure on* H.

7 Regularization

We now define an algebra homomorphism from H to $Mer(\mathbb{C}^\infty)$-the algebra of meromophic functions on $\mathbb{C}^\infty$.

DEFINITION 7.1 *A convex cone is called strongly convex if it contains no line.*

For strongly convex cones, we have,

LEMMA 7.2 *If* $C \subset \mathbb{R}^k$ *is a strongly convex cone of rank* k, *then* $\check{C} \cap (\mathbb{R}^k)^*$ *is a cone of rank* k.

For any strongly convex cone C, and $\vec{z} \in (\mathbb{R}^\infty)^*$, let

$$\Phi(C) = \sum_{\vec{n} \in C \cap \Lambda} e^{-\vec{z}(\vec{n})}$$

here we use the convention: $0^z = 1$ for any z.

LEMMA 7.3 *If* $C \subset \mathbb{R}^k$ *is a strongly convex cone, then*

$$\sum_{\vec{n} \in C \cap \Lambda_k} e^{-\vec{z}(\vec{n})}$$

defines an analytic function on $\{\vec{z} \in (\mathbb{R}^k)^* | \vec{z}(\vec{u}) > 0\}$.

In fact, if C is a strongly convex smooth cone, spanned by $\vec{u}_1, \cdots, \vec{u}_m$, where $\vec{u}_i$ is the closest integer vector in $\mathbb{R}_+\vec{u}_1$ to $\vec{0}$, then

$$\sum_{\vec{n} \in C \cap \Lambda_k} e^{-\vec{z}(\vec{n})} = \Pi \frac{e^{-\vec{z}(\vec{v}_i)}}{1 - e^{\vec{z}(\vec{v}_i)}}$$

DEFINITION 7.4 *For any convex rational cone C, let $\{ C_i\}$ be a subdivision of it, with C_i strongly convex rational, we define*

$$\Phi(C) = \sum \Phi(C_i)$$

LEMMA 7.5 *For any strongly convex rational cone C, two definition agree.*

For any rational convex cone, because any two subdivisions have a common smooth subdivision, so by above lemma, we have

LEMMA 7.6 Φ *is well-defined for any rational convex cone. For any convex cone $C \subset \mathbb{R}^k$, $\Phi(C)$ is a meromorphic function on $(z_1, \cdots, z_k)$. If $\{ C_i\}$ is a subdivision of C, then $\Phi(C) = \sum \Phi(C_i)$.*

LEMMA 7.7 *If C contains a line, then $\Phi(C) = 0$*

Therefore, we have a map from the Hopf algebra of cones to meromorphic functions:

$$\Phi : H \to Mer(\mathbb{C}^\infty)$$

PROPOSITION 7.8 *For $C_1 \subset \mathbb{R}^k$, $C_2 \subset \mathbb{R}^l$,*

$$\Phi(C_1 \bullet C_2)(z_1, \cdots z_{k+l}) = \Phi(C_1)(z_1, \cdots, z_k)\Phi(C_2)(z_{k+1}, \cdots, z_{k+l})$$

8 Reormalization

Renormalization in Connes-Kreimer's approach is

THEOREM 8.1 *Let H be a connected filtered Hopf algebra, (R, P) be a Rota-Baxter algebra of weight -1, and $\phi : H \to R$*

be an algebra homomorphism such that $Im\phi$ *is commutative. Then*

$$\phi_-(x) = -P(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x''))$$

and

$$\phi_+(x) = (I - P)(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')).$$

here we have used the notation $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(x)} x' \otimes x''$. *Then* ϕ_- *and* ϕ_+ *are algebra homomorphisms.*

Remark H is not necessary commutative.

Let H' be a Hopf subalgebra of H consisting of cones whose generators doesn't perpendicular to the vector $\vec{1} = (1, \cdots, 1, \cdots)$, then

$$\Psi : H' \to Mer(\mathbb{C}),\ C \mapsto \Phi(C)(z, z, \cdots, z, \cdots)$$

defines an algebra homomorphism from H' to $Mer(\mathbb{C})$.

Or we can use evaluator to renormalize Φ. In any case, let $V(C)$ be the renormalized value of $\Phi(C)$, then

PROPOSITION 8.1 *For a rational cone* C,

$$V(C) \in \mathbb{Q}.$$

If $\{C_i\}$ *is a subdivision of* C,

$$V(C) = \sum V(C_i).$$

If C *contains a line, then*

$$V(C) = 0.$$

For convex cones C_1, C_2,

$$V(C_1 \bullet C_2) = V(C_1)V(C_2).$$

9 Renormalization and Todd classes

To apply to toric varieties, we can modify the above picture a bit. For fixed d, we consider the Hopf subalgebra H_t generated by cones in $\mathbb{R}^n$. There is an algebra homomorphism

$$\Psi_d : H_t \to Mer(\mathbb{C}^\infty)$$

$$\Psi_d(C) = \Phi(\check{C})$$

Applying the renormalization process, let $\mu_d(C)$ to be the renormalized value of $\Psi_d(C)$, then

THEOREM 9.1 *For a complete toric variety X_Σ,*

$$Td(X_\Sigma) = \sum_{\sigma \in \Sigma} \mu_{\dim \sigma}(\sigma)[\bar{O}_\sigma]$$