CARTAN'S GENERALIZATIONS OF LIE'S THIRD THEOREM

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Lie's Third Theorem: If L is a finite-dimensional, real Lie algebra, then there exists a Lie algebra homomorphism $\lambda : L \to \mathrm{Vect}(L)$ satisfying

$$\lambda(x)(0) = x \qquad \text{for all } x \in L.$$

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Dual Formulation: Let $\delta : L^* \to \Lambda^2(L^*)$ be a linear map. If its extension $\delta : \Lambda^*(L^*) \to \Lambda^*(L^*)$ as a graded derivation of degree 1 satisfies $\delta^2 = 0$, then there is a DGA homomorphism $\phi : \big(\Lambda^*(L^*), \delta\big) \to \big(\Omega^*(L), \mathrm{d}\big)$ satisfying

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Basis Formulation: If $C^i_{jk} = -C^i_{kj}$ $(1 \le i, j, k \le n)$ are constants, then there exist linearly independent 1-forms ω^i $(1 \le i \le n)$ on $\mathbb{R}^n$ satisfying the structure equations

$$\mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}\, \omega^j \wedge \omega^k$$

if and only if these formulae imply $\mathrm{d}\big(\mathrm{d}\omega^i\big) = 0$.

A geometric problem: Classify those Riemannian surfaces (M^2, g) whose Gauss curvature K satisfies the second order system

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for some functions a and b.

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Analysis Writing $g = {\omega_1}^2 + {\omega_2}^2$, the structure equations yield

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_{12} \wedge \omega_2 & \qquad \mathrm{d}\omega_{12} &= K\, \omega_1 \wedge \omega_2 \\ \mathrm{d}\omega_2 &= \quad \omega_{12} \wedge \omega_1 & \qquad \mathrm{d}K &= K_1\, \omega_1 + K_2\, \omega_2 \end{aligned}$$

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and the condition to be studied is encoded as

$$\begin{pmatrix} \mathrm{d}K_1 \\ \mathrm{d}K_2 \end{pmatrix} = \begin{pmatrix} -K_2 \\ K_1 \end{pmatrix} \omega_{12} + \begin{pmatrix} a(K) + b(K)\,{K_1}^2 & b(K)\,K_1 K_2 \\ b(K)\,K_1 K_2 & a(K) + b(K)\,{K_1}^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

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Applying $\mathrm{d}^2 = 0$ to these two equations yields

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Thus, unless $a'(K) = a(K)b(K) - K$, such metrics have K constant.

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Note: $\mathrm{d}^2 = 0$ is 'formally satisfied' for these structure equations.

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Answer: A theorem of É. Cartan [1904] implies that a 'solution' (N^3, ω) does indeed exist and is determined uniquely (locally near p, up to diffeomorphism) by the 'value' of (K, K_1, K_2) at p.

Cartan's result: Suppose that $C^i_{jk} = -C^i_{kj}$ and F^α_i (with $1 \leq i, j, k \leq n$ and $1 \leq \alpha \leq s$) are real-analytic functions on $\mathbb{R}^s$ such that the equations

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formally satisfy $\mathrm{d}^2 = 0$. Then, for every $b_0 \in \mathbb{R}^s$, there exists an open neighborhood U of $0 \in \mathbb{R}^n$, linearly independent 1-forms η^i on U, and a function $b : U \to \mathbb{R}^s$ satisfying

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Remark 1: Cartan assumed that $F = (F^\alpha_i)$ has constant rank, but it turns out that, for a 'solution' (η, b) with U connected, $F(b) = \big(F^\alpha_i(b)\big)$ always has constant rank anyway.

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Remark 2: Cartan worked in the real-analytic category and used the Cartan-Kähler theorem in his proof, but the above result is now known to be true in the smooth category.

The Hessian equation example:

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Remark: The F-matrix either has rank 0 (when $K_1 = K_2 = a(K) = 0$) or 2 (all other cases). The rank 0 cases have K constant. The rank 2 cases have a 1-dimensional symmetry group and each represents a surface of revolution.

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and a bundle map $\alpha : V \to TA$ that induces a Lie algebra homomorphism on sections and satisfies the compatibility condition

$$\{U, fV\} = \alpha(U)(f)V + f\{U, V\} \qquad \text{for } f \in C^\infty(A) \text{ and } U, V \in \Gamma(V).$$

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A 'solution' is a $b : B^n \to A$ covered by a bundle map $\eta : TB \to V$ of rank n that induces a Lie algebra homomorphism on sections.

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Many more examples drawn from classical differential geometry.

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C^i_{jk}, F^a_i, and $A^a_{i\alpha}$ (where $1 \leq i, j, k \leq n$, $1 \leq a \leq s$, and $1 \leq \alpha \leq r$) are specified functions on a domain $X \subset \mathbb{R}^s$.

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(1) The functions C, F, and A are real analytic.

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$$d\eta^i = -\tfrac{1}{2}C^i_{jk}(h)\,\eta^j \wedge \eta^k \qquad dh^a = \left(F^a_i(h) + A^a_{i\alpha}(h)p^\alpha\right)\eta^i.$$

C^i_{jk}, F^a_i, and $A^a_{i\alpha}$ (where $1 \le i,j,k \le n$, $1 \le a \le s$, and $1 \le \alpha \le r$) are specified functions on a domain $X \subset \mathbb{R}^s$. Assume:

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Remark: The proof is a straightforward modification of Cartan's proof in the case $r = 0$ (i.e., when there are no 'free derivatives' p^α).

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\mathrm{d}\omega_1 &= \phi_2 \wedge \omega_3 - \phi_3 \wedge \omega_2 & \qquad \mathrm{d}\phi_1 &= \phi_2 \wedge \phi_3 + c_1\, \omega_2 \wedge \omega_3 \\
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2^{nd} Bianchi implies that there are functions a_i and b_i so that

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$\mathrm{d}(\mathrm{d}\omega_i) = 0$, then yields 9 equations for $\mathrm{d}a_i$, $\mathrm{d}b_i$. These can be written in the form

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\mathrm{d}a_i &= \big(A_{ij}(a,b) + p_{ij}\big)\, \omega_j \\
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where $q_{ii} = 0$ and $q_{ij} = -p_{jk}/(c_i - c_j)$ when (i, j, k) are distinct.

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This defines an involutive tableau (in the p-variables) of rank $r = 9$ and with characters $s_1 = 6$, $s_2 = 3$, and $s_3 = 0$. Cartan's criteria are satisfied, so the desired metrics depend on three functions of two variables.

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A similar analysis applies when $c_1 = c_2 \neq c_3$, showing that such metrics depend on one function of two variables.

When $c_1 = c_2 = c_3$, the Cartan analysis gives the expected result that the solutions depend on a single constant.

General Holonomy. A torsion-free H-structure on M^n (where $H \subset \mathrm{GL}(\mathfrak{m})$ and $\dim(\mathfrak{m}) = n$) satisfies the first structure equation

$$\mathrm{d}\omega = -\phi \wedge \omega$$

where ϕ takes values in $\mathfrak{h} \subset \mathrm{GL}(\mathfrak{m})$ and the second structure equation

$$\mathrm{d}\phi = -\phi \wedge \phi + R(\omega \wedge \omega)$$

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Ex: For $G_2 \subset \mathrm{GL}(7, \mathbb{R})$, the tableau $K^1(\mathfrak{g}_2)$ has $s_6 = 6 > s_7 = 0$, so the general metric with G_2-holonomy depends on six functions of 6 variables.

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4. (B—, 2008) The solitons for the G_2-flow in dimension 7 depend on 16 functions of 6 variables.