

On the Index and the Order of Quasi-Regular DAE Systems

L. D'Alfonso, G. Jeronimo, G. Massaccesi, P. Solernó

Universidad de Buenos Aires

Setting - Implicit Ordinary DAE System

$$(\Sigma) := \begin{cases} f_1(X_1^{[e_{11}]}, \dots, X_n^{[e_{1n}]}) = 0 \\ \vdots \\ f_r(X_1^{[e_{r1}]}, \dots, X_n^{[e_{rn}]}) = 0 \end{cases}$$

- $X := X_1, \dots, X_n$; $X_j^{[h]} := X_j, \dot{X}_j, \dots, X_j^{(h)}$ $h \in \mathbb{N}_0$,
- $f_i \in \mathcal{K}[X_1^{[e_{i1}]}, \dots, X_n^{[e_{in}]}] \subset \mathcal{K}\{X\}$, \mathcal{K} diff. field $\text{char}(\mathcal{K}) = 0$,
- $e_{ij} :=$ order of the variable X_j in f_i ,
- $e := \max_{ij} \{1, e_{ij}\}$
- $F := f_1, \dots, f_r$.

Hypotheses on the system

For all $\ell \in \mathbb{N}$:

- $\Delta_\ell := (F, \dots, F^{(\ell)}) \subset \mathcal{K}[X^{\ell+e}]$ is a **prime** (polynomial) ideal.
 $\Rightarrow \Delta := [f_1, \dots, f_r]$, the **differential ideal** generated by f_1, \dots, f_r , is a **prime** ideal of $\mathcal{K}\{X\}$.
- the sequence of polynomials $F^{[\ell]} = F, \dot{F}, \dots, F^{(\ell)}$ is **regular**.
 \Rightarrow the ideal $\Delta_\ell \subset \mathcal{K}[X^{\ell+e}]$ has codimension $r(\ell + 1)$.

More general: If the system (Σ) is **quasi-regular**, all the results hold localizing at a minimal differential prime ideal containing Δ .
[Kondratieva-Mikhalev-Pankratiev, 1982], [Ollivier-Sadik, 2006], implicit in [Johnson, 1978].

\Rightarrow The **differential dimension** of Δ is $n - r$.

*For 0-dimensional, first order systems ($e = 1$ and $n = r$),
the differentiation index*

- \approx measures the implicitness of the system;
- is a bound for the number of total derivatives needed to obtain an equivalent explicit system;
- is a bound for the number of derivatives needed to obtain **all** the relations of order $e - 1$ (or any prefixed order) that a solution of the system must verify.

$$\Delta \cap \mathcal{K}[X^{[e-1]}] =? \quad \text{or} \quad \Delta \cap \mathcal{K}[X^{[e]}] = \Delta^{\text{"diff. index"}} \cap \mathcal{K}[X^{[e]}]$$

There are **several** (not necessarily equivalent) notions of indices:

- Brenan-Campbell-Petzold (1996); Kunkel-Mehrmann (2006)
- Rabier-Rheinboldt (1994); Campbell-Gear (1995);
Fliess-Lévine-Martin-Rouchon (1995)
- Seiler (1999), Reid-Lin-Wittkopf (2001); Pritchard-Sit
(2007); Thomas (1996); Lamour (2005); Le Vey (1994)

- Algebraic definition of Differentiation Index for systems (Σ).
- Upper bound for the differentiation index in terms of the parameters of the system (order of the variables, number of variables and equations, etc.).
- Relations between the differentiation index and other algebraic properties of the system.

Algebraic definition of index (Jacobian submatrices)

For each $k \in \mathbb{N}$, $i \in \mathbb{Z}_{\geq e-1}$, let $\mathfrak{J}_{k,i}$ the $kr \times kn$ Jacobian matrix of $F^{(i-e+1)}, \dots, F^{(i-e+k)}$ w.r.t. the variables $X^{(i+1)}, \dots, X^{(i+k)}$

Why the $\mathfrak{J}_{k,i}$'s? Fix $i = e + 1$, $\text{trdeg}_{\mathcal{K}} \mathcal{K}[X^{[e+1]}] / (\Delta_\ell \cap \mathcal{K}[X^{[e+1]}]) = ?$

$$\begin{array}{ccc} B := \mathcal{K}[X^{[e+1]}] / (\Delta_\ell \cap \mathcal{K}[X^{[e+1]}]) & \longrightarrow & C := \mathcal{K}[X^{[e+\ell]}] / \Delta_\ell \\ \uparrow & \nearrow & \\ \mathcal{K} & & \end{array}$$

$$\mathcal{K}[X^{[e+\ell]}] / \Delta_\ell = B[X^{(e+2)}, \dots, X^{(e+\ell)}] / (F^{(2)}, \dots, F^{(\ell)})$$

$$\begin{aligned} \text{trdeg}_{\mathcal{K}}(B) &= \text{trdeg}_{\mathcal{K}}(C) - \text{trdeg}_B(C) \\ &= [(e + \ell + 1)n - (\ell + 1)r] - \dim(\ker(\mathfrak{J}_{\ell-1, e+1})) \\ &= (n - r)(\ell + 1) + er - \dim(\ker(\mathfrak{J}_{\ell-1, e+1}^t)) \end{aligned}$$

The μ -sequence

$\mathbb{L} := \text{Frac}(\mathcal{K}\{X\}/\Delta)$. For each $k \in \mathbb{N}_0$, $i \in \mathbb{N}_{\geq e-1}$ the sequence $\mu_{k,i} \in \mathbb{N}_0$ is defined as follows:

- $\mu_{0,i} := 0$ for $i \geq e - 1$,
- $\mu_{k,i} := \dim_{\mathbb{L}} \ker(\mathfrak{J}_{k,i}^t)$ for $k \geq 1$, $i \geq e - 1$.

Hypothesis

For any pair k, i , the rank of the matrix $\mathfrak{J}_{k,i}$ over the integral domains $\mathcal{K}[X^{[\ell]}]/\Delta_{\ell-e+1}$ does not depend on ℓ .

Results:

- For every $i \in \mathbb{N}_{\geq e-1}$, $\mu_{k,i} = \mu_{k,i+1} =: \mu_k$
(the sequence does not depend on i).
- There exists $\sigma \in \mathbb{N}_0$ such that:

$$\mu_0 < \mu_1 < \cdots < \mu_\sigma = \mu_{\sigma+1} = \cdots = \mu_k = \cdots.$$

$\sigma :=$ **differentiation index** of the system (Σ)

- $\text{trdeg}_{\mathcal{K}}(\mathcal{K}[X^{[i]}] / \Delta_{i-e+1+k} \cap \mathcal{K}[X^{[i]}]) = (n-r)(i+1) + er - \mu_k$
 $\forall k \in \mathbb{N}_0$ and $i \in \mathbb{N}_{\geq e-1}$
- $\sigma = \min\{h \in \mathbb{N}_0 : \Delta_{i-e+1+h} \cap \mathcal{K}[X^{[i]}] = \Delta \cap \mathcal{K}[X^{[i]}\}$.

If $i = e - 1$, this gives an alternative definition:

$$\sigma = \min\{h \in \mathbb{N}_0 : \Delta_h \cap \mathcal{K}[X^{[e-1]}] = \Delta \cap \mathcal{K}[X^{[e-1]}\}$$

Similar results: Sadik, 2000.

- If $g \in \mathcal{K}\{X\}$, does $g \in \Delta$? $N := \sigma + \max\{-1, \text{ord}(g) - e\}$

$$g \in \Delta \Leftrightarrow g \in \Delta_{N+1} \subset \mathcal{K}[X^{[N+e]}].$$

Moreover, from [Dickenstein, Fitchas, Giusti, Sessa, 1991], all degrees are bounded by $\deg(g) + (\max_i \deg(f_i))^{r(N+1)}$.

Bounds for the order in a more general setting (PDE) are given by [Golubitsky, Kondratieva, Ovchinnikov, Szanto, 2008]

The Hilbert function of Δ . The order.

$$H_{\Delta} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$$

$$H_{\Delta}(i) := \text{trdeg}_{\mathcal{K}}(\text{Frac}(\mathcal{K}[X^{[i]})/(\Delta \cap \mathcal{K}[X^{[i]}])).$$

Theorem (Kolchin). $\exists i_0, s \in \mathbb{N}_0$ such that

$$H_{\Delta}(i) = (n - r)(i + 1) + s, \text{ for all } i \geq i_0.$$

s is the **order of the ideal** = $\text{ord}(\Delta)$.

Both i_0 and $\text{ord}(\Delta)$ may be determined by characteristic sets.

Results

- $\text{ord}(\Delta) = er - \mu_{\sigma}$
- $i_0 \leq e - 1$

Upper bound for the order and the index

Let

$$\mathcal{E} := \begin{pmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & \ddots & \vdots \\ e_{r1} & \cdots & e_{rn} \end{pmatrix}, \quad e_{ij} := \begin{cases} \text{ord}_{X_j}(f_i) & \text{if } X_j \text{ appears in } f_i \\ -\infty & \text{otherwise.} \end{cases}$$

The Jacobi number of \mathcal{E}

$$J(\mathcal{E}) := \max \left\{ \sum_{i=1}^s e_{i\tau(i)} \mid \tau : \{1, \dots, s\} \hookrightarrow \{1, \dots, m\} \right\}.$$

$$\text{Jacobi's Bound} \quad \text{ord}(\Delta) \leq J(\mathcal{E}).$$

Introduced (~ 1836) for $n \times n$ systems. Proved by:

- * [Ritt, 1935] (linear systems),
- * [Lando, 1970] (first order systems, weak version)
- * [Kondratieva-Mikhalev-Pankratiev, 1982] (q-r systems).

[Ollivier-Sadik, 2006] (extended to q-r syst. of dimension > 0).

[Tomasovic, 1976]) and [Kondratieva-Mikhalev-Pankratiev, 2007] (PDE versions).

Moreover, if

$$\mathcal{E}_0 := \begin{pmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & \ddots & \vdots \\ e_{r1} & \cdots & e_{rn} \end{pmatrix}, \quad e_{ij} := \begin{cases} \text{ord}_{X_j}(f_i) & \text{if } X_j \text{ appears in } f_i \\ 0 & \text{otherwise.} \end{cases}$$

and (Σ) satisfies all the hypotheses, then

$$\sigma + \text{ord}(\Delta) \leq J(\mathcal{E}_0) + \max\{e_{ij}\} - \min\{e_{ij}\}.$$

A classical example: Hessenberg form of size n

$$(\Sigma) := \begin{cases} \dot{X}_1 &= f_1(X_1, \dots, X_n) \\ \dot{X}_2 &= f_2(X_1, \dots, X_{n-1}) \\ &\vdots \\ \dot{X}_i &= f_i(X_{i-1}, X_i, \dots, X_{n-1}) \quad 2 \leq i \leq n-1 \\ &\vdots \\ \dot{X}_{n-1} &= f_{n-1}(X_{n-2}, X_{n-1}) \\ 0 &= f_n(X_{n-1}) \end{cases}$$

with $\left(\frac{\partial f_n}{\partial X_{n-1}}\right) \cdot \left(\frac{\partial f_{n-1}}{\partial X_{n-2}}\right) \cdots \left(\frac{\partial f_2}{\partial X_1}\right) \cdot \left(\frac{\partial f_1}{\partial X_n}\right) \neq 0$ and Δ prime.

$$\mu_k = k, \text{ for } k = 0, \dots, n \Rightarrow \sigma = n$$

$$\text{ord}(\Delta) = 1 \cdot n - \mu_n = n - n = 0.$$

$$\mathcal{E}_0 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$n+0 = \sigma + \text{ord}(\Delta) = J(\mathcal{E}_0) + \max\{\epsilon_{ij}\} - \min\{\epsilon_{ij}\} = n-1+1-0 = n.$$

$$\mathcal{E} = \begin{pmatrix} 1 & \star & \star & \dots & \star & \star & 0 \\ 0 & 1 & \star & \dots & \star & \star & -\infty \\ -\infty & 0 & 1 & \dots & \star & \star & -\infty \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\infty & -\infty & -\infty & \dots & 1 & \star & -\infty \\ -\infty & -\infty & -\infty & \dots & 0 & 1 & -\infty \\ -\infty & -\infty & -\infty & \dots & -\infty & 0 & -\infty \end{pmatrix}$$

$$J(\mathcal{E}) = 0 = \text{ord}(\Delta)$$