

TOPOLOGICAL GALOIS THEORY

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METHODS

1. ABEL AND LIOUVILLE (1833).
2. GALOIS AND PICARD- VESSIO (1910).
3. TOPOLOGICAL VERSION: ONE DIMENSIONAL CASE (1972), MULTI-DIMENSIONAL CASE (2003).

LIST OF BASIC ELEMENTARY FUNCTIONS IN ONE VARIABLE (or in n variables).

all constants, x (or x_1, \dots, x_n);
 exp, ln, $x \mapsto x^\alpha$;
 sin, cos, tan;
 arcsin, arccos, arctan.

LIST OF CLASSICAL OPERATIONS.

- 1) composition: $f, g \in L \Rightarrow f \circ g \in L$;
- 2) arithmetic operation $(+, -, \times, /)$: $f, g \in L \Rightarrow f \pm g, f \times g, f/g \in L$;
- 3) differentiation: $f \in L \Rightarrow f' \in L$ (or $f(x_1, \dots, x_n) \in L \Rightarrow \frac{\partial f}{\partial x_i} \in L$);
- 4) integration: $f \in L \Rightarrow \int_{x_0}^x f(t)dt \in L$ (or $f_1, \dots, f_n \in L$ and $d\alpha = 0$ where
 $\alpha = f_1 dx_1 + \dots + f_n dx_n \Rightarrow \int_{x_0}^x \alpha \in L$);
- 5) extension by exponent of integral: $f \in L$ and $y' = fy$ [$y = \exp \int_{x_0}^x f dt$] $\Rightarrow y \in L$
 (or $f_1, \dots, f_n \in L$ and $\frac{\partial y}{\partial x_i} = f_i$ [$y = \exp \int_{x_0}^x \alpha$ where $\alpha = f_1 dx_1 + \dots + f_n dx_n$]
 $\Rightarrow y \in L$;
- 6) algebraic extension: $f_1, \dots, f_n \in L$ and $y^n + f_1 y^{n-1} + \dots + f_n = 0 \Rightarrow y \in L$;
- 7) extension by exponent: $f \in L$ and $dy = ydf$ [$y = \exp f$] $\Rightarrow y \in L$;
- 8) extension by logarithm: $f \in L$ and $dy = \frac{df}{f}$ [$y = \ln f$] $\Rightarrow y \in L$;
- 9) meromorphic operation: if $F : \mathbb{C}^n \rightarrow \mathbb{C}$ is a meromorphic function, $f_1, \dots, f_n \in L$, and $y = F(f_1, \dots, f_n) \Rightarrow y \in L$.

The arithmetic operations and exponentiation are meromorphic operations.

CLASSICAL CLASSES OF FUNCTIONS

I. RADICALS.

Basic functions: rational functions.

Operations: arithmetic operations and extensions by radicals.

II. ELEMENTARY FUNCTIONS.

Basic functions: basic elementary functions.

Operations: composition, arithmetic operations, differentiation.

III. GENERALIZED ELEMENTARY FUNCTIONS.

The same as elementary functions + algebraic extensions.

IV. QUADRATURES.

Basic functions: basic elementary functions.

Operations: composition, arithmetic operations, differentiation and integration.

V. GENERALIZED QUADRATURES.

The same as quadratures + algebraic extensions.

LIOUVILLE THEORY

Theorem (of Liouville on quadratures). *Consider a class: {List of basic functions: all constant. List of allowed operations: arithmetic operations, integration, extension by exponent of integral}. Then this class coincide with the class of quadratures.*

IDEOLOGY:

Simple equation should have simple solution. Otherwise it is unsolvable in finite terms.

Theorem (Liouville). *If $a(x)$ is an algebraic function and $y(x) = \int_{x_0}^x f(t)dt$, then $y(x)$ is a generalized elementary function if and only if $y(x) = a_0(x) + \lambda_1 \ln a_1(x) + \dots + \lambda_n \ln a_n(x)$, where a_0, \dots, a_n are algebraic functions.*

Examples of unsolvable integrals. Indefinite integrals $\int \exp(t^2)dt$, $\int \frac{\exp t}{t}dt$, $\int \frac{dt}{\ln t}dt$, $\int \frac{\sin t}{t}dt$ are not generalized elementary functions.

Theorem (Liouville). *The equation $y'' + py' + qy = 0$, where p, q -rational functions is solvable by generalized quadratures if and only if it has a solution of the form $y_1(x) = \exp \int_{x_0}^x a(t)dt$ where $a(t)$ is an algebraic function.*

GALOIS AND PICARD–VESSIOT THEORY

THE ANALOGY BETWEEN LINEAR DIFFERENTIAL EQUATIONS AND ALGEBRAIC EQUATIONS.

- 1) Reduction of order as an analog of Bezout theorem.
- 2) Analog of the Vieta formula for differential operators.
- 3) An analog of the theorem on symmetric functions for differential operator: Every rational function of independent solutions y_1, \dots, y_n of a linear differential equation of order n and of derivatives of the solutions that is $GL(n)$ -invariant is in fact a rational function of the coefficients of the equation and of their derivatives.

Theorem (Picard–Vessiot, 1910). *A linear differential equation is solvable by quadratures if and only if its Galois group is solvable. It is solvable by generalized quadratures if and only if the connected component of the identity in its Galois group is solvable.*

Corollary. *A linear differential equation of order n is solvable by generalized quadratures if and only if: first it has a solution of the form $y_1 = \exp \int_{x_0}^x a(t)dt$ where $a(t)$ is an algebraic function, and, second, if the differential equation of order $(n - 1)$ obtained from the original equation by the reduction of order is solvable by generalized quadratures.*

This corollary was proved independently (in 1910) by Mordukhai-Boltovskii using Liouville's method. Picard–Vessio theory not only explains this corollary but allows (for an equation which coefficients are rational functions with integral coefficients) to determine explicitly is equation solvable by generalized quadratures or not.

A TOPOLOGICAL APPROACH.

Theorem. *The Galois group of an equation $y^n + r_1y^{n-1} + \dots + r_n = 0$ over the field P of rational functions, $r_i \in P$ is isomorphic to the monodromy group of the (multivalued) algebraic function defined by the same equation.*

Corollary. *If the monodromy group of an algebraic function is unsolvable then the function is not representable by radicals.*

PROGRAM:

I. Find a wide class of functions which is closed under natural operations (compositions, arithmetic operations, ...) and such that for all functions in the class the monodromy group is well defined.

II. Use the monodromy group within this class instead of the Galois group.

CLASS OF \mathcal{S} -FUNCTIONS. A multivalued analytic function of one complex variable is called \mathcal{S} -function if the set of its singular points is at most countable.

Theorem. *The class of \mathcal{S} -functions is closed under composition, arithmetic operations, differentiation, integration, meromorphic operations, solving algebraic equations, solving linear differential equations.*

Corollary. *A function having an uncountable number of singular points can not be expressed by generalized quadratures*

Example. Consider $f = \ln \sum_{i=1}^n \lambda_i \ln(x - a_i)$. If $n \geq 3$, λ_i - generic and $a_i \neq a_j$ if $i \neq j$ then: 1) the monodromy group of f contains continuum elements, 2) the set of singular points of f is everywhere dense on the complex line.

Theorem. *The class of \mathcal{S} -functions whose monodromy group is solvable is under integration, differentiation, compositions and meromorphic (in particular arithmetic) operations.*

Corollary. *The monodromy group of a function representable by quadratures is solvable. Moreover if the monodromy group of a function is unsolvable it cannot be represented via meromorphic functions using integration, differentiation, composition and meromorphic operations.*

Theorem. *If the monodromy group of an algebraic function is solvable one can represent it via radicals. But if it is unsolvable one cannot represent it by a formula which involves meromorphic functions and elementary functions and uses integration, composition and meromorphic operations.*

Theorem. *If the monodromy group of a Fuchs-type linear differential equation is solvable then this equation is solvable by quadratures. But if it is unsolvable one can not represent its solutions by a formula which involves integration, composition and meromorphic operations and uses meromorphic and elementary functions.*

Corollary. *Consider a system $y' = \sum \frac{A_i}{x - a_i} y$, where y is n -vector and A_i are $n \times n$ matrices with constants entries. Assume that the matrices A_i have sufficiently small entries. Then the system is solvable if and only if all the matrices A_i are triangular in some basis*

Corollary. *Let G be a polygon bounded by arcs of circles on the complex line. Let $f_G : B_1 \rightarrow G$ be a Riemann map from a unit ball onto G . One can classify all polygons G such that the function f_G is representable by quadratures.*

MULTIDIMENSIONAL CASE

One can construct a class of functions in many complex variables which contains all meromorphic functions and which is closed under compositions, integrations, meromorphic operations, etc. One can describe how the monodromy groups of the functions from the class change under these operations.

Theorem. *If the monodromy group of an algebraic function in many variables is solvable then the function is representable via radical. Otherwise one cannot represent it by a formula which involves meromorphic and elementary functions and uses integrations, differentiations, compositions and meromorphic operations.*

Theorem. *If the monodromy group of a regular holonomic system of equation is solvable then this system is solvable by quadratures. Otherwise one cannot represent a solution by a formula which involves meromorphic and elementary functions and used integrations, differentiations, compositions and meromorphic operations.*

AN ALGEBRAIC VERSION OF THE 13th HILBERT PROBLEM

Let x be an algebraic function of n variables, defined by an algebraic equation $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$.

PROBLEM. *Find the smallest integral $k(n)$ such that the function x could be represented by algebraic functions of k variable, using compositions and arithmetic operations. What are the conditions for representing a function of n variables by algebraic functions of a smaller number of variables?*

NOTE. The problem of the composition was formulated by Hilbert for classes of continuous functions, not for algebraic functions. It was solved by Kolmogorov and Arnold. Kolmogorov represented continuous function of n variables as the composition of function of a single variable with the help of a sole addition.

CLASSICAL RESULTS:

- 1) if $n = 2, 3, 4$ then $k(n) = 1$ (equations of degree < 5 are solvable by radicals);

- 2) if $n = 5$ then $k(n) = 1$;
- 3) if $n > 5$ then $k(n) \leq (n - 4)$;
- 4) if $n > 8$ then $k(n) \leq (n - 5)$;
- 5) for any m there exists N such that if $n > N$ then $k(n) \leq (N - m)$.

Theorem. *An integral algebraic function y of two variables (a, b) , defined by equation $y^5 + ay + b = 0$, cannot be expressed in terms of integral algebraic functions of a single variable by means of compositions, additions and multiplications.*

The reason is the following. If a function can be express in terms of functions of one variable then about each point a local monodromy group of the function is solvable. But the local monodromy group about the origin of the function $y(a, b)$ is the group $S(5)$ of all permutations of five elements, which is not solvable.

NOTE. The function $y(a, b)$ can be expressed by means of division in terms of functions $\sqrt[4]{u}$ and $g(u)$ of one variable u , where $g(u)$ is defined by equation $g^5 + g + u = 0$. It is easily to see that $y(a, b) = g(b/\sqrt[4]{a^5})\sqrt[4]{a}$.

References

1. Solvability of equations in finete terms, “Uspehi Matem. Nauk”, Vol. 59, No. 4(358), 2004, 69–146; translation in Russian Math. Surveys 59 no. 4 (2004), 661–736.
2. On the continuability of multivalued analytic functions to an analytic subset, “Funkts. analiz”, V. 35, N.1, 62–73, 2001; translation in Funct. Anal. Appl. 35 (2001), no. 1, 51–59 (2001).
3. On the monodromy of a multivalued function on it’s ramification set, “Funkts. analiz”, Vol. 37, No 2, 2003, 65–74. translation in Funct. Anal. Appl. 37 (2003), no. 2, 65–74 (2003).
4. On the nonrepresentability of functions of several variables by quadratures “Funkts. analiz”, Vol. 37, No 4, 2003, 74–85; translation in Funct. Anal. Appl. 37 (2003), no. 4, 302–310 (2003).
5. Galois theory, coverings and Riemann surfaces, MCCME, Moscow, 2007, 90 pp.
6. Topological Galois theory. Solvability and non-solvability of equations in finite terms. Monograph. MCCME. 296 pp. 2008.