

Differential schemes

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We start with the definition of differential scheme using the treatment of Hartshorne. Immediately we find that there are problems: the global section functor of an affine differential scheme does not recover the original ring. We give some examples to show what goes wrong. However, reduced differential schemes are more more tractable. We shall discuss what is known and not known.

All rings are commutative and unitary. We fix a set of commuting derivations $\Delta = \{\delta_1, \dots, \delta_m\}$. We use the prefix Δ instead of the word “differential”. R is a Δ -ring (arbitrary characteristic), K is a Δ -field. If S is a subset of R then $\{S\}$ is the smallest radical Δ -ideal of R that contains S . If $\mathbb{Q} \subset R$ (i.e. R is a Ritt algebra) then $\{S\} = \sqrt{[S]}$.

Definition

$X = \text{Diffspec } R$ is the set of prime Δ -ideals of R .

If \mathfrak{a} is a Δ -ideal then

$$V(\mathfrak{a}) = \{\mathfrak{p} \in X \mid \mathfrak{a} \subset \mathfrak{p}\}.$$

These sets are the closed sets in the *Kolchin* topology.

If $a \in R$ then

$$D(a) = \{\mathfrak{p} \in X \mid a \notin \mathfrak{p}\}.$$

These sets are the basic opens.

The structure sheaf is defined exactly as in Hartshorne.

Theorem

If $s \in \mathcal{O}_X(U)$, then, for $\mathfrak{p} \in U$,

$$s(\mathfrak{p}) = \begin{cases} \frac{a_1}{b_1} & \text{for } \mathfrak{p} \in D(b_1) \\ \vdots & \vdots \\ \frac{a_n}{b_n} & \text{for } \mathfrak{p} \in D(b_n) \end{cases}$$

and $D(b_1) \cup \dots \cup D(b_n) = U$ (equivalently $1 \in \llbracket b_1, \dots, b_n \rrbracket$).

Theorem

The stalk $\mathcal{O}_{X,\mathfrak{p}}$ is Δ -isomorphic to the local ring $R_{\mathfrak{p}}$.

Buium defines a Δ -scheme as a scheme whose sheaf consists of Δ -rings. Umemura calls this “a scheme with derivations”.

Carra Ferro changes the definition of the sheaf. If $X = \text{Diffspec } R$ and $Y = \text{Spec } R$ then $\mathcal{O}_X(U)$ is defined to be $\mathcal{O}_Y(V)$ where V is the largest open subset of Y with $V \cap X = U$.

Hrushovski has a paper in the Archives about difference schemes.

Definition

$\widehat{R} = \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X)$ is the ring of global sections of X .

Theorem

There is a canonical mapping

$$\iota: R \rightarrow \widehat{R}, \quad \iota(a)(p) = \frac{a}{1} \in R_p.$$

However ι is neither injective nor surjective in general.

Let

$$R = \mathbb{Q}[x]\{y\}/[xy],$$

where $x' = 1$ and y is a Δ -indeterminate. We claim that $\iota(y) = 0$.
For any $\mathfrak{p} \in \text{Diffspec } R$,

$$\iota(y)(\mathfrak{p}) = \frac{y}{1} \in R_{\mathfrak{p}}.$$

But $x \notin \mathfrak{p}$ (since $x' = 1$) so

$$\iota(y)(\mathfrak{p}) = \frac{y}{1} = \frac{xy}{x} = 0$$

The crucial property is that y is what is called a Δ -zero

$$1 \in [\text{Ann}(y)].$$

Definition

A Δ -ring R is AAD (Annihilators Are Differential) if for every $r \in R$, $\text{Ann}(r)$ is a Δ -ideal.

Any ring with trivial derivations is AAD. Thus the rings of classical Δ -algebraic geometry are AAD. The corresponding condition for difference rings is called “well-mixed”.

Theorem

If R is AAD then ι is injective.

Theorem

If R is reduced then R is AAD.

One would like the “hat” operation $R \rightarrow \widehat{R}$ to be a closure operation.

So I proved that $\iota_{\widehat{R}}: \widehat{R} \rightarrow \widehat{\widehat{R}}$ is a Δ -isomorphism.

And then Franck Benoist constructed a counterexample!

Surjectivity example

Even the simplest examples fail.

Let $R = \mathbb{Q}[x]$, where $x' = 1$. R has a unique prime Δ -ideal, namely (0) , so

$$\text{Diffspec } R = \{(0)\}.$$

It follows from the definition that

$$\widehat{R} = R_{(0)} = \mathbb{Q}(x),$$

Therefore

$$R = \mathbb{Q}[x] \rightarrow \widehat{R} = \mathbb{Q}(x)$$

is not surjective.

But, in this example, \widehat{R} is a ring (actually a field) of quotients of R .

Let $s \in \widehat{R}$. For any $\mathfrak{p} \in X = \text{Diffspec } R$ there is an open neighborhood U of \mathfrak{p} and $a, b \in R$, with $b \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{b}$ for $\mathfrak{q} \in U$. We can rewrite this as

$$\iota(b)s(\mathfrak{q}) = \iota(a).$$

So \widehat{R} is some sort of a ring of quotients. In fact, it is the ring of quotients for a certain Gabriel topology - this was first noted by Alexander Buium.

But this is too local; it is only valid for $\mathfrak{q} \in U$. The term of the next definition is due to Frank Benoit.

Definition

We say that $\iota: R \rightarrow \widehat{R}$ is *almost surjective* if for every $s \in \widehat{R}$ and every $\mathfrak{p} \in X$ there exist $a, b \in R$, $b \notin \mathfrak{p}$ such that

$$\iota(b)s(\mathfrak{q}) = \iota(a)$$

for every $\mathfrak{q} \in X$.

Almost surjective

Without the condition that ι be almost surjective, we know very little about \widehat{R} (and therefore about X).

Theorem

If R is reduced (even AAD) then ι is almost surjective.

Theorem

If ι is almost surjective then

$$X = \text{Dffspec } R \approx \widehat{X} = \text{Dffspec } \widehat{R}.$$

Suppose that $X = \text{Diffspec } R$ is reduced (the stalks are all reduced). Is R reduced? No. The example where ι is not injective is a counterexample.

Theorem

If X is reduced then \widehat{R} is reduced and $X \approx \widehat{X} = \text{Diffspec } \widehat{R}$.

Thus we can always choose a reduced ring S with $X \approx \text{Diffspec } S$. However we have to replace R with \widehat{R} , a much more complicated ring.

Let K be a Δ -field and $X = \text{Diffspec } R$. Then X is a Δ -scheme over $\text{Diffspec } K$ if there is a morphism $X \rightarrow \text{Diffspec } K$. Does not imply that R is a K -algebra. No.

Example

Let $K = \mathbb{Q}(x)$ and $R = \mathbb{Q}[x]$. Then $X = \text{Diffspec } R$ is a Δ -scheme over $Y = \text{Diffspec } K$ (in fact they are isomorphic). However R is not a K -algebra.

This affects the existence of products. If $X = \text{Diffspec } R$ and $Z = \text{Diffspec } S$ are Δ -schemes over $Y = \text{Diffspec } K$ then

$$X \times_Y Z \stackrel{?}{=} \text{Diffspec } R \otimes_K S.$$

But the right-hand side need not make sense!

The solution is easy:

Definition

By a Δ - K -scheme is meant a Δ -scheme X that can be covered by affine Δ -schemes $\text{Diffspec } R$ where R is a Δ - K -algebra.

Not much work as been done in this direction. In particular the various uses of products has not been studied.

Suppose that $Y \subset X = \text{Diffspec } R$ is a closed subscheme. Does there exist an ideal $\mathfrak{a} \subset R$ such that $Y \approx \text{Diffspec}(R/\mathfrak{a})$? I don't know.

Theorem

If $Y \subset X$ are both reduced then there is a radical Δ -ideal $\mathfrak{a} \subset R$ such that $Y \approx \text{Diffspec}(R/\mathfrak{a})$.

This theorem is probably true with “reduced” replaced by “AAD”, but I do not have a proof.

Definition

A Δ -ring R is *Rittian* if every strictly increasing chain of radical Δ -ideals is finite.

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Theorem

If R is finitely Δ -generated over K then R is Rittian.

Theorem

R is Rittian if and only if $X = \text{Diffspec } R$ is a Noetherian topological space.

This is actually better than the algebraic version!

Suppose that R is reduced and Rittian.

Theorem

R has a finite number of minimal prime ideals and they are Δ -ideals.

Theorem

The complete ring of quotients of R is a finite product of Δ -fields.

Theorem

R has a finite number of minimal idempotents.

Using these we get the usual theorems about connected and irreducible components of X .

If R is reduced then the canonical mapping $\iota: R \rightarrow \widehat{R}$ is injective and we identify R with a subring of \widehat{R} .

Theorem

If R is a domain then so is \widehat{R} and $\text{qf}(R) = \text{qf}(\widehat{R})$.

Hence the field of rational functions of an irreducible variety is what you expect.

If R is not a domain, the ring of rational functions classically is $Q(R)$, the complete of fractions of R , i.e. $R\Sigma^{-1}$ where Σ is the multiplicative set of elements that are not zero divisors in R .

Recall that a global section has the form

$$s(\mathfrak{p}) = \begin{cases} \frac{a_1}{b_1} & \text{for } \mathfrak{p} \in D(b_1) \\ \vdots & \vdots \\ \frac{a_n}{b_n} & \text{for } \mathfrak{p} \in D(b_n) \end{cases}$$

It may happen that every b_i is a zero divisor.

Theorem

If R is a reduced Rittian Ritt algebra then there is an injective homomorphism

$$\widehat{R} \rightarrow Q(R).$$

If we identify \widehat{R} with its image then \widehat{R} is the subring of $Q(R)$ consisting all everywhere defined functions (in the sense of Cassidy)

One of the more vexing problems for me is to understand the notion of non-singular or regular points of a Δ -scheme. We need a good notion so that we can study tangent spaces.

Joe Johnson has a definition however it seems to depend on the choice of affine neighborhood: there exists an affine open containing the point that has nice properties.

I don't know if there is an example where different affine neighborhoods give different results.

In algebraic geometry one has tools such as Nakayama's Lemma, Krull Intersection Theorem, Krull Dimension. The first two are false for Δ -rings.

Suppose that R is a Δ - K -algebra that is finitely Δ -generated over K . We choose a set of generators

$$R = K\{x_1, \dots, x_n\}.$$

For $s \in \mathbb{N}$, let be R_s the ring (not Δ -ring) generated by θx_i where θ is a “higher derivation”, $\theta = \delta_1^{d_1} \cdots \delta_m^{d_m}$, of order bounded by s .

Theorem

There is a numerical polynomial Φ_{Krull} such that

$$\Phi_{\text{Krull}}(s) = \dim R_s \quad (\text{Krull dimension})$$

for $s \gg 0$.

If we let $X_s = \text{Spec } R_s$ then $\Phi_{\text{Krull}}(s)$ is the topological dimension of X_s . Unfortunately the schemes X_s depend on the choice of Δ -generators of R .

We also have numerical polynomials for the vector space dimension of m/m^2 , etc. However I am unable to fit them together into a coherent theory.

My primary interest is in Galois theory. There is a well-established theory for linear homogeneous Δ -equations (The Picard-Vessiot Theory). Kolchin had established a theory for some (very few) non-homogeneous equations (The Galois Theory of Strongly Normal Extensions). However he used his own development of algebraic geometry, which has not been widely embraced. I wanted to put the theory in the Grothendieck language.

We suppose that L is a strongly normal extension of K . We then set

$$X = \text{Diffspec}(L \otimes_K L).$$

Then the C -rational points of X are in bijective correspondence to the Galois group of L over K .

To get an algebraic group we first define X^Δ to be the local ringed space X (same topological space) together with the sheaf

$$\mathcal{O}_{X^\Delta}(U) = \mathcal{O}_X(U)^\Delta$$

whose rings are the rings of constants of the structure sheaf on X .

Theorem

X^Δ is a group scheme of finite type of K .

Thank you

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