Dimension Polynomials of Intermediate Fields of a Finitely Generated Difference-Differential Field Extension

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Third International Workshop on Differential Algebra and Related Topics

November 13-16, 2008

• Let K be a difference-differential field ($Char\, K = 0$) with basic sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of derivation operators and automorphisms of K, respectively (any two mappings of the set $\Delta \bigcup \sigma$ commute).

• Let Λ be the free commutative semigroup of all elements of the form

$\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ $(k_i \in \mathbf{N},\ l_j \in \mathbf{Z})$.

• The order of such an element is defined as

$ord\, \lambda = \sum_{i=1}^{m} k_i + \sum_{j=1}^{n} |l_j|$, and

$\Lambda(r) = \{\lambda \in \Lambda \,|\, ord\, \lambda \le r\}$ $(r \in \mathbf{N})$.

• Let $L = K\langle\eta_1, \dots, \eta_s\rangle$ be a difference-differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$.

As a field, $L = K(\{\lambda\eta_j | \lambda \in \Lambda, 1 \leq j \leq s\})$.

• The following is a unified version of E. Kolchin's theorem on differential dimension polynomial and the speaker's theorem on the dimension polynomial of a difference field extension.

Theorem 1. With the above notation, there exists a polynomial $\phi_{\eta|K}(t) \in \mathbf{Q}[t]$ such that

(i) $\phi_{\eta|K}(r) = trdeg_K K(\{\lambda\eta_j | \lambda \in \Lambda(r), 1 \leq j \leq s\})$ for all sufficiently large $r \in \mathbf{Z}$;

(ii) $\deg \phi_{\eta|K} \leq m+n$ and $\phi_{\eta|K}(t)$ can be written as

$$\phi_{\eta|K}(t) = \sum_{i=0}^{m+n} a_i \binom{t+i}{i}$$

where $a_0, \dots, a_{m+n} \in \mathbf{Z}$ and $2^n | a_{m+n}$.

(iii) $d = \deg \phi_{\eta|K}$, a_{m+n} and a_d do not depend on the set of difference-differential generators η of L/K ($a_d \neq a_{m+n}$ iff $d < m+n$).

Moreover, $\frac{a_{m+n}}{2^n}$ is equal to the difference-differential transcendence degree of L over K (denoted by Δ-σ-$trdeg_K L$), that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the family $\{\lambda\xi_i | \lambda \in \Lambda, 1 \leq i \leq k\}$ is algebraically independent over K.

The next result is an essential generalization of Theorem 1; it shows the existence of a dimension polynomial associated with any subextension of a finitely generated difference-differential field extension.

Theorem 2. With the above notation, let F be an intermediate difference-differential field of the extension L/K and for any $r \in \mathbf{N}$, let $F_r = F \bigcap K(\{\lambda\eta_j | \lambda \in \Lambda(r), 1 \leq j \leq s\})$.

Then there exists a polynomial $\phi_{K,F,\eta}(t) \in \mathbf{Q}[t]$ such that

(i) $\phi_{K,F,\eta}(r) = trdeg_K F_r$ for all sufficiently large $r \in \mathbf{Z}$;

(ii) $\deg \phi_{K,F,\eta} \leq m + n$ and $\phi_{K,F,\eta}(t)$ can be written as $\phi_{K,F,\eta}(t) = \sum_{i=0}^{m+n} b_i \binom{t+i}{i}$ where $b_0, \ldots, b_{m+n} \in \mathbf{Z}$ and $2^n | b_{m+n}$.

(iii) $d = \deg \phi_{K,F,\eta}(t)$, b_{m+n} and b_d do not depend on the set of difference-differential generators η of the extension L/K. Furthermore, $\dfrac{b_{m+n}}{2^n} = \Delta\text{-}\sigma\text{-}trdeg_K F$.

The proof of this theorem is based on some properties of difference-differential modules we are going to introduce.

As before, let K be a difference-differential field with the same basic sets Δ and σ (in what follows we often use prefix Δ-σ- instead of the adjective "difference-differential") and let Λ be the commutative semigroup defined above.

Let $\mathcal{D}$ denote the set of all finite sums of the form $\sum_{\lambda\in\Lambda} a_\lambda \lambda$ where $a_\lambda \in K$ (such a sum is called a Δ-σ-*operator* over K; two Δ-σ-operators are equal iff their corresponding coefficients are equal).

The set $\mathcal{D}$ can be treated as a ring with respect to its natural structure of a left K-module and the relationships $\delta a = a\delta + \delta(a)$, $\alpha a = \alpha(a)\alpha$, $\alpha^{-1}a = \alpha^{-1}(a)\alpha^{-1} \quad (\delta \in \Delta$, $\alpha \in \sigma^* = \{\alpha_1, \dots, \alpha_n, \alpha_1^{-1}, \dots, \alpha_n^{-1}\}$, $a \in K)$ extended by distributivity.

By a difference-differential module over K (also called a Δ-σ-K-module) we mean a left $\mathcal{D}$-module M, that is, a vector K-space where elements of $\Delta \cup \sigma^*$ act as additive mutually commuting operators such that

$\delta(ax) = a\delta(x) + \delta(a)x, \quad \alpha(ax) = \alpha(a)\alpha(x),$ and $\alpha^{-1}(\alpha(x)) = x$ for any $\delta \in \Delta$, $\alpha \in \sigma^*$, $x \in M$, $a \in K$.

We say that M is a finitely generated Δ-σ-K-module if M is finitely generated as a left $\mathcal{D}$-module.

By a filtration of a Δ-σ-K-module M we mean an exhaustive and separated filtration of M as a $\mathcal{D}$-module, that is, an ascending chain $(M_r)_{r\in\mathbf{Z}}$ of vector K-subspaces of M such that $\mathcal{D}_r M_s \subseteq M_{r+s}$ for all $r, s \in \mathbf{Z}$, $M_r = 0$ for all sufficiently small $r \in \mathbf{Z}$, and $\bigcup_{r\in\mathbf{Z}} M_r = M$. Such a filtration is called *excellent* if every M_r $(r \in \mathbf{Z})$ is finitely generated over K and there exists $r_0 \in \mathbf{Z}$ such that $M_r = \mathcal{D}_{r-r_0} M_{r_0}$ for any $r \geq r_0$.

The proofs of the following two results can be found in [KLMP, Chapter 6].

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[KLMP] Kondrateva, M. V.; Levin, A. B.; Mikhalev, A. V.; Pankratev, E. V. Differential and Difference Dimension Polynomials. *Kluwer Acad.Publ.*, 1999

Theorem 3 (KLMP, Theorem 6.7.3). With the above notation, let M be a Δ-σ-K-module with an excellent filtration $(M_r)_{r\in\mathbf{Z}}$. Then there is a polynomial $\psi(t) \in \mathbf{Q}[t]$ such that:

(i) $\psi(r) = dim_K M_r$ for all sufficiently large $r \in \mathbf{Z}$.

(ii) $\deg \psi \le m + n$ and $\psi(t)$ can be written as

$$\psi(t) = \sum_{i=0}^{m+n} a_i \binom{t+i}{i}$$

where $a_0, \ldots, a_{m+n} \in \mathbf{Z}$ and $2^n | a_{m+n}$.

(iii) $d = \deg \psi$, a_{m+n} and a_d do not depend on the excellent filtration of M.

Furthermore, $\dfrac{a_{m+n}}{2^n}$ equals the difference-differential dimension of M over K (denoted by Δ-σ-$dim_K M$), that is, to the maximal number of elements $x_1, \dots, x_k \in M$ such that the family $\{\lambda x_i | \lambda \in \Lambda, 1 \leq i \leq k\}$ is linearly independent over K.

Theorem 4 (KLMP, Theorem 6.7.10). Let $\mu : N \to M$ be an injective homomorphism of filtered Δ-σ-K-modules M and N with filtrations $(M_r)_{r\in\mathbf{Z}}$ and $(N_r)_{r\in\mathbf{Z}}$, respectively.

(It means that μ is a homomorphism of $\mathcal{D}$-modules and $\mu(N_r) \subseteq M_r$ for any $r \in \mathbf{Z}$.)

If the filtration of M is excellent, then the filtration of N is also excellent.

Sketch of the proof of Theorem 2.

Let $L = K\langle\eta_1, \dots, \eta_s\rangle$ and let $\Omega_{L|K}$ be the associated module of Kähler differentials. Then $\Omega_{L|K}$ can be treated as a Δ-σ-L-module where the action of the elements of $\Delta\bigcup\sigma$ is defined in such a way that $\delta(d\zeta) = d\delta(\zeta)$ and $\alpha(d\zeta) = d\alpha(\zeta)$ for any $\zeta \in L$, $\delta \in \Delta$, $\alpha \in \sigma^*$.

Let $M = \Omega_{L|K}$ and for any $r \in \mathbf{N}$ let M_r denote the vector L-space generated by all elements $d\zeta$ where $\zeta \in K(\bigcup_{i=1}^{s}\Lambda(r)\eta_i)$.

It is easy to check that $(M_r)_{r\in\mathbf{Z}}$ ($M_r = 0$ if $r < 0$) is an excellent filtration of the Δ-σ-L-module M.

Let F be any intermediate differential-differential field of L/K and for any $r \in \mathbf{N}$, let

$$F_r = F \bigcap K(\{\lambda\eta_j | \lambda \in \Lambda(r), 1 \leq j \leq s\})$$

Let $\mathcal{D}_L$ denote the ring of Δ-σ-operators over L and let N be the $\mathcal{D}_L$-submodule of M generated by all elements of the form $d\zeta$ where $\zeta \in F$. Furthermore, for any $r \in \mathbf{N}$, let N_r be the vector L-space generated by all elements $d\zeta$ where $\zeta \in F_r$, and let $N_r = 0$ if $r < 0$.

Then $(N_r)_{r\in\mathbf{Z}}$ is a filtration of the Δ-σ-L-module N, and the embedding $N \rightarrow M$ becomes a homomorphism of filtered Δ-σ-L-modules.

Since the filtration $(M_r)_{r\in\mathbf{Z}}$ is excellent, one can apply Theorem 4 and obtain that the filtration $(N_r)_{r\in\mathbf{Z}}$ is also excellent. Therefore, there exists a polynomial $\phi_{K,F,\eta}(t) \in \mathbf{Q}[t]$ such that $\phi_{K,F,\eta}(r) = dim_K N_r$ for all sufficiently large $r \in \mathbf{Z}$.

Since a family $(\zeta_i)_{i\in I}$ of elements of F_r ($r \in \mathbf{Z}$) is algebraically independent over K iff the family $(d\zeta_i)_{i\in I}$ is linearly independent over L, $dim_K N_r = trdeg_K F_r$ for all $r \in \mathbf{N}$.

Applying Theorem 3 we obtain the result of Theorem 2.

• Theorem 2 shows that the Einstein's strength of any system of algebraic difference-differential equations with an action of any group commuting with basic operators is a polynomial function. We shall specify this statement in the case of a system of differential equations. In this case our theorem sounds as follows.

Theorem 2′. Let K be a differential field with a basic set of derivations $\Delta = \{\delta_1, \dots, \delta_m\}$, let Θ denote a the free commutative multiplicative semigroup generated by $\delta_1, \dots, \delta_m$, and for any $r \in \mathbf{N}$, let $\Theta(r)$ denote the set $\{\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \,|\, ord\,\theta = \sum_{i=1}^{m} k_i \leq r\}$.

Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$, and let F be an intermediate differential field of L/K, $K \subseteq F \subseteq L$.

Furthermore, for any $r \in \mathbf{N}$, let

$$F_r = F \bigcap K(\{\theta\eta_j | \theta \in \Theta(r), 1 \leq j \leq s\}) \ .$$

Then there exists $\omega_{K,F,\eta}(t) \in \mathbf{Q}[t]$ such that

(i) $\omega_{K,F,\eta}(r) = trdeg_K F_r$ for all sufficiently large $r \in \mathbf{Z}$;

(ii) $\deg \omega_{K,F,\eta} \leq m$ and

$$\omega_{K,F,\eta}(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i} \quad \text{where } a_i \in \mathbf{Z}.$$

(iii) $d = \deg \omega_{K,F,\eta}(t)$, a_m and a_d do not depend on the set of generators η of the extension L/K. Furthermore, $a_m = \Delta\text{-}trdeg_K F$.

If $F = L$, one obtains the Kolchin's theorem on differential dimension polynomial $\omega_{\eta|K}(t)$ such that $\omega_{\eta|K}(r) = trdeg_K K(\bigcup_{j=1}^{s} \Theta(r)\eta_j)$ for all sufficiently large $r \in \mathbf{Z}$.

Theorem 1 allows one to assign numerical polynomials to certain systems of algebraic differential equations as follows.

• Let $R = K\{y_1, \ldots, y_s\}$ be the ring of differential polynomials.
($R = K[\{\theta y_j | \theta \in \Theta, 1 \leq j \leq s\}]$; the structure of a differential ring on R is defined by $\delta(\theta y_j) = (\delta\theta) y_j$ for any $\delta \in \Delta$.)

• By a system of algebraic differential equations over K we mean a system of the form

$$f_i(y_1, \ldots, y_s) = 0 \qquad (i \in I)$$

where $\{f_i\}_{i \in I} \subseteq R$; by a solution we mean an s-tuple with coordinates in some differential field extension of K that annuls all f_i.

• Let $\mathcal{P}$ be the differential ideal generated by $\{f_i | i \in I\}$ in R (as an ideal, $\mathcal{P}$ is generated by $\{\theta f_i | \theta \in \Theta, i \in I\}$). If it is prime, then $Q(R/\mathcal{P}) = K\langle \eta_1, \ldots, \eta_s \rangle$ where η_j is the image of y_j in $R/\mathcal{P}$. By Theorem 1, we obtain a numerical polynomial $\omega_{\eta|K}(t)$ called the differential dimension polynomial of the system.

• The concept of a differential dimension polynomial can be viewed as the algebraic version of A. Einstein's concept of strength of a system of partial differential equations governing a physical field. In his work "The Meaning of Relativity" [Princeton, 1953, pp. 133 - 165]

A. Einstein defined the strength as follows.

”... the system of equations is to be chosen so that the field quantities are determined as strongly as possible. In order to apply this principle, we propose a method which gives a measure of strength of an equation system. We expand the field variables, in the neighborhood of a point P, into a Taylor series (which presupposes the analytic character of the field); the coefficients of these series, which are the derivatives of the field variables at P, fall into sets according to the degree of differentiation.

In every such degree there appear, for the first time, a set of coefficients which would be free for arbitrary choice if it were not that the field must satisfy a system of differential equations. Through this system of differential equations (and its derivatives with respect to the coordinates) the number of coefficients is restricted, so that in each degree a smaller number of coefficients is left free for arbitrary choice. The set of numbers of "free" coefficients for all degrees of differentiation is then a measure of the "weakness" of the system of equations, and through this, also of its "strength".

• Theorem $2'$, which shows the existence of a dimension polynomial associated with an intermediate differential field of a finitely generated field extension, leads to the partial solution of a more general A. Einstein's problem: to evaluate the strength of a system of PDE whose solutions should be invariant with respect to the action of some group G.

If the basic derivations commute with G (that is, $\delta G = G\delta$ for any $\delta \in \Delta$), then the elements of the differential field $L = K\langle \eta_1, \ldots, \eta_s \rangle$, which are invariant with respect to the group action, form a differential subfield F of L, and the Einstein's strength of the system is the dimension polynomial $\chi(t) \in \mathbf{Q}[t]$ such that $\chi(r) = trdeg_K\left(F \bigcap K\left(\bigcup_{i=1}^{s} \Theta(r)\eta_i\right)\right)$ for all sufficiently large $r \in \mathbf{Z}$. (We assume that $g(a) = a$ for any $g \in G$, $a \in K$.)

The following considerations lead to other essential generalizations of the Kolchin theorem. We are going to prove the existence of multivariate numerical polynomials associated with partitions of the basic set of derivation operators; these polynomials represent the "generalized" strength of a system of algebraic differential equations, which is defined in the same way as the Einstein's concept of strength if one imposes separate restrictions on the orders of derivations with respect to each group of basic derivation operators.

Let K be a differential field ($Char\,K = 0$) whose basic set Δ is a union of p disjoint finite sets ($p \geq 1$): $\Delta = \Delta_1 \bigcup \cdots \bigcup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($i = 1, \dots, p$). Thus, we fix a partition of the set Δ.

For any $\theta = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} \in \Theta$, we define the order of the element θ with respect to Δ_i as follows:

$$ord_i\,\theta = \sum_{j=1}^{m_i} k_{ij} \quad (i = 1, \dots, p).$$

Furthermore, for any $r_1, \dots, r_p \in \mathbf{N}$, we set

$\Theta(r_1, \dots, r_p) = \{\theta \in \Theta | ord_i\,\theta \leq r_i$ for $i = 1, \dots, p\}$.

Theorem 5. (Levin, 2007). Let $L = K\langle\eta_1, \dots, \eta_s\rangle$ be a differential field extension generated by a set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\Phi_\eta(t_1, \dots, t_p)$ in p variables $t_1, \dots, t_p$ with rational coefficients such that
(i) $\Phi_\eta(r_1, \dots, r_p) =$

$$trdeg_K K(\bigcup_{j=1}^{s} \Theta(r_1, \dots, r_p)\eta_j)$$

for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{N}^p$
(i. e., there exist $s_1, \dots, s_p \in \mathbf{N}$ such that the last equality holds for all elements $(r_1, \dots, r_p) \in \mathbf{N}^p$ with $r_1 \geq s_1, \dots, r_p \geq s_p$);

(ii) $deg_{t_i}\Phi_\eta \leq m_i$ $(1 \leq i \leq p)$, so that $deg\,\Phi_\eta \leq m$ and the polynomial $\Phi_\eta(t_1, \dots, t_p)$ can be represented as

$$\Phi_\eta(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p}$$

where $a_{i_1 \dots i_p} \in \mathbf{Z}$ for all $i_1, \dots, i_p$.

The polynomial $\Phi_\eta(t_1, \dots, t_p)$ is called the *differential dimension polynomial* of the extension L/K associated with the set of differential generators η (and the given partition of the basic set Δ).

For any permutation $(j_1, \dots, j_p)$ of the set $\{1, \dots, p\}$, we define the lexicographic order $<_{j_1,\dots,j_p}$ on $\mathbf{N}^p$ as follows: $(r_1, \dots, r_p) <_{j_1,\dots,j_p} (s_1, \dots, s_p)$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbf{N}$, $1 \le k \le p-1$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$.

If $\Sigma \subseteq \mathbf{N}^p$, then Σ' denotes the set $\{e \in \Sigma | e$ is a maximal element of Σ with respect to one of the $p!$ lexicographic orders $<_{j_1,\dots,j_p}\}$.

Example 1.

Let $\Sigma = \{(3,0,2), (2,1,1), (0,1,4), (1,0,3), (1,1,6), (3,1,0), (1,2,0)\} \subseteq \mathbf{N}^3$.

Then $\Sigma' = \{(3,0,2), (3,1,0), (1,1,6), (1,2,0)\}$.

Theorem 6. Let K be a differential field whose basic set of derivations Δ is a union of p disjoint finite sets ($p \geq 1$): $\Delta = \Delta_1 \bigcup \cdots \bigcup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ $(i = 1, \dots, p)$. Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a Δ-field extension of K with the finite set of Δ-generators $\eta = \{\eta_1, \dots, \eta_s\}$ and

$\Phi_\eta(t_1, \dots, t_p) =$

$$\sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p}$$

the corresponding dimension polynomial. Let

$E_\eta = \{(i_1, \dots, i_p) \in \mathbf{N}^p | 0 \leq i_k \leq m_k$ for $k = 1, \dots, p$ and $a_{i_1 \dots i_p} \neq 0\}$.

Then the degree d of the polynomial Φ_η, the coefficient $a_{m_1 \dots m_p}$, elements $(j_1, \dots, j_p) \in E'_\eta$, the corresponding coefficients $a_{j_1 \dots j_p}$ and the coefficients of the terms of total degree d do not depend on the choice of the system of differential generators η of L/K. Furthermore, $a_{m_1,\dots,m_p}$ is equal to the differential transcendence degree of L over K.

In order to prove Theorem 5 and find a method of computation of differential dimension polynomials, we consider the ring $\mathcal{D}$ of differential operators over the differential field $L = K\langle\eta_1, \dots, \eta_s\rangle$ as a ring with p-dimensional filtration $\{\mathcal{D}_{r_1 \dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ where $\mathcal{D}_{r_1 \dots r_p}$ is the vector L-subspace of $\mathcal{D}$ generated by $\Theta(r_1, \dots, r_p)$ if all $r_i \geq 0$, and $\mathcal{D}_{r_1 \dots r_p} = 0$ if $(r_1, \dots, r_p) \in \mathbf{Z}^p \setminus \mathbf{N}^p$.

If M is a differential L-module (that is, a left $\mathcal{D}$-module), then a family $\{M_{r_1 \dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ of vector L-subspaces of M is called a p-dimensional filtration of M if

(i) For any fixed $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_p$ $M_{r_1 \dots r_i \dots r_p} \subseteq M_{r_1 \dots r_{i-1}, r_i+1, r_{i+1} \dots r_p}$ and $M_{r_1 \dots r_p} = 0$ for all sufficiently small $r_i \in \mathbf{Z}$;

(ii) $\bigcup \{M_{r_1 \dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\} = M$;

(iii) $\mathcal{D}_{r_1 \dots r_p} M_{s_1 \dots s_p} \subseteq M_{r_1+s_1, \dots, r_p+s_p}$ for any $(r_1, \dots, r_p), (s_1, \dots, s_p) \in \mathbf{Z}^p$.

- If every vector L-space $M_{r_1 \dots r_p}$ is finitely generated and there is $(h_1, \dots, h_p) \in \mathbf{Z}^p$ such that $\mathcal{D}_{r_1 \dots r_p} M_{h_1 \dots h_p} = M_{r_1+h_1, \dots, r_p+h_p}$ for any $(r_1, \dots, r_p) \in \mathbf{N}^p$, then the p-dimensional filtration is called *excellent*.

(If $M = \sum_{i=1}^{s} \mathcal{D} u_i$, then $\{\sum_{i=1}^{s} \mathcal{D}_{r_1 \dots r_p} u_i | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ is an excellent filtration.)

Recall that the module of Kähler differentials $\Omega_{L|K}$ $(L = K\langle \eta_1, \dots, \eta_s \rangle)$ can be treated as a differential L-module such that

$\delta(db) = d\delta(b)$ for every $\delta \in \Delta$, $b \in L$.

Clearly, $\Omega_{L|K} = \sum_{i=1}^{s} \mathcal{D} d\eta_i$.

Let $(\Omega_{L|K})_{r_1 \dots r_p}$ $(r_1, \dots, r_p \in \mathbf{N})$ be the vector L-subspace of $\Omega_{L|K}$ generated by all elements $d\eta$ with $\eta \in K(\{\theta\eta_j \,|\, \theta \in \Theta(r_1, \dots, r_p), 1 \leq j \leq s\})$ and let $(\Omega_{L|K})_{r_1 \dots r_p} = 0$ whenever at least one r_i is negative.

Then $\{(\Omega_{L|K})_{r_1 \dots r_p} \,|\, (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ is an excellent p-dimensional filtration of $\Omega_{L|K}$.

Furthermore,

$$trdeg_K K(\bigcup_{j=1}^{s} \Theta(r_1, \dots, r_p)\eta_j) =$$

$$dim_L(\Omega_{L|K})_{r_1 \dots r_p}$$

so the proof of Theorem 5 can be reduced to the proof of the following theorem for differential modules.

Theorem 7 (Levin, 2007). Suppose that $\{M_{r_1\dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ is an excellent p-dimensional filtration of a left $\mathcal{D}$-module M. Then there exists a polynomial $\phi(t_1, \dots, t_p) \in \mathbf{Q}[t_1, \dots, t_p]$ such that

(i) $\phi(r_1, \dots, r_p) = dim_L M_{r_1\dots r_p}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{Z}^p$.

(ii) $deg_{t_i}\phi \leq m_i$ $(1 \leq i \leq p)$, so that $deg\,\phi \leq m$ and

$$\phi = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1\dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p}$$

where $a_{i_1\dots i_p} \in \mathbf{Z}$ for all $i_1, \dots, i_p$.

(iii) Let $A = \{(i_1, \dots, i_p) \in \mathbf{N}^p | 0 \leq i_k \leq m_k$ for $k = 1, \dots, p$ and $a_{i_1 \dots i_p} \neq 0\}$. Then $d = deg\, \phi$, $a_{m_1 \dots m_p}$, elements $(j_1, \dots, j_p) \in A'$, the corresponding $a_{j_1 \dots j_p}$ and the coefficients of the terms of total degree d do not depend on the excellent filtration. Furthermore, $a_{m_1, \dots, m_p}$ is equal to the maximal number of elements of M linearly independent over $\mathcal{D}$.

The last theorem can be proven via the technique of Gröbner bases with respect to several orderings (see [L, 2007]),

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[L, 2007] Levin, A. B. Gröbner Bases with respect to Several Orderings and Multivariable Dimension Polynomials. *J. Symbolic Comput.*, 42 (2007), 561-578.

The following is an analog of Theorem 4 for multivariate filtrations:

Theorem 8. Let $\mu : N \to M$ be an injective homomorphism of multifiltered differential L-modules N and M with p-dimensional filtrations $\{N_{r_1 \dots r_p} \,|\, (r_1 \dots r_p) \in \mathbf{Z}^p\}$ and $\{M_{r_1 \dots r_p} \,|\, (r_1 \dots r_p) \in \mathbf{Z}^p\}$, respectively. (That is, μ is a homomorphism of differential modules and $\mu(N_{r_1 \dots r_p}) \subseteq M_{r_1 \dots r_p}$ for any $r_1 \dots r_p \in \mathbf{Z}$.) If the filtration of M is excellent, then the filtration of N is excellent as well.

Applying Theorem 8, one can use the arguments of the proof of Theorem 2 to obtain the following statement that shows the existence of multivariate dimension polynomials associated with an intermediate differential field of a finitely generated field extension. (The same arguments allow one to obtain a similar result in the difference-differential case as well.)

Theorem 9. With the above notation, let F be an intermediate differential field extension of the extension $L = K\langle\eta_1, \dots, \eta_s\rangle$ and for any $r_1, \dots, r_p \in \mathbf{N}$, let

$F_{r_1,\dots,r_p} = F \bigcap K(\bigcup_{i=1}^{s} \Theta(r_1, \dots, r_p)\eta_j)$.

Then there exists a polynomial $\Psi_\eta(t_1, \dots, t_p) \in \mathbf{Q}[t_1, \dots, t_p]$ such that

(i) $\Psi_\eta(r_1, \dots, r_p) = trdeg_K F_{r_1,\dots,r_p}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{N}^p$.

(ii) $deg_{t_i}\Psi_\eta \leq m_i$ $(1 \leq i \leq p)$, so that $deg\,\Psi_\eta \leq m$ and

$$\Psi_\eta = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1\dots i_p} \binom{t_1+i_1}{i_1} \dots \binom{t_p+i_p}{i_p}$$

where $a_{i_1\dots i_p} \in \mathbf{Z}$ for all $i_1, \dots, i_p$.

(iii) Let $A = \{(i_1, \ldots, i_p) \in \mathbf{N}^p | 0 \leq i_k \leq m_k$ for $k = 1, \ldots, p$ and $a_{i_1 \ldots i_p} \neq 0\}$. Then $d = deg\, \Psi_\eta$, $a_{m_1 \ldots m_p}$, elements $(j_1, \ldots, j_p) \in A'$, the corresponding coefficients $a_{j_1 \ldots j_p}$ and the coefficients of the terms of total degree d do not depend on the choice of the system of differential generators η of L/K. Furthermore, $a_{m_1, \ldots, m_p}$ is equal to the differential transcendence degree of L over K.

This result allows one to associate a family of multivariate dimension polynomials with a system of algebraic differential equations with an action of a group G commuting with basic derivations.

Thus, we obtain an algebraic description of A. Einstein's strength of such a system (in the sense explained after Theorem $2'$).