Representations of Rota-Baxter Algebras

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1. Rota-Baxter Algebra

$k$— a fixed field. All algebras are associative unital $k$-algebras.

**Definition 1.** A **Rota-Baxter algebra** of weight $\lambda \in k$ is a $k$-algebra $A$ with a $k$-linear map $Q : A \rightarrow A$ satisfying

$$Q(a)Q(b) = Q(Q(a)b + aQ(b) + \lambda Q(ab))$$

for all $a, b \in A$. $Q$ is called the Rota-Baxter operator. We will denote by $(A, Q)$ a Rota-Baxter algebra, **RB-algebra** for short.

2. Matrix representations

Let $(A, Q)$ be a Rota-Baxter algebra (of weight $\lambda$). For any positive integer $n$, we obtained a Rota-Baxter operator $Q$ on the algebra of $n \times n$ matrices $M_n(A)$ on $A$ by defining $Q$ entry-wise:

$$Q([a_{ij}]) = [Q(a_{ij})].$$

This is easy to check that $Q$ is a Rota-Baxter operator on $M_n(A)$. Such a Rota-Baxter algebra is called a **matrix Rota-Baxter algebra**.
Let \((R, P)\) be a Rota-Baxter algebra (of weight \(\lambda\)). A **matrix representation** (with coefficients in \(A\)) of \((R, P)\) is a homomorphism

\[
f : (R, P) \to (M_n(A), Q)
\]

of Rota-Baxter algebras, i.e., \(f\) is an \(k\)-algebra homomorphism and

\[
Q(f(r)) = f(P(r))
\]

for all \(r \in R\). Such representations have appeared in QFT renormalization.

**Remark 1.** For any \(k\)-algebra \(A\), the identity operator on \(A\) is a Rota-Baxter operator of weight \(\lambda = -1\). In this case, any \(k\)-algebra homomorphism \(R \to M_n(A)\) is a matrix representation of the Rota-Baxter algebra \((R, \text{Id})\) of weight \(-1\). Hence representations of Rota-Baxter algebras enriches the ordinary representations of algebras.

We will see that matrix representations are special cases of more general module theories of Rota-Baxter algebras. The matrix repre-
sentations are just modules of \((R, P)\) over a free \((A, Q)\)-modules

3. Rota-Baxter modules
Recall the well-known differential case:

Let \((A, d)\) be a differential algebra of weight \(\lambda\):

\[
d(xy) = d(x)y + xd(y) + \lambda d(x)d(y), \quad \forall x, y \in A.
\]

A differential module for \((A, d)\) of weight \(\lambda\) is a pair \((M, \delta)\) with \(M\) being an \(A\)-module and \(\delta : M \to M\) is a \(k\)-linear map such that

\[
\delta(ax) = d(a)x + a\delta(x) + \lambda d(a)\delta(x), \quad \forall a \in A, x \in M.
\]
Algebraically, let $k[d] = \text{polynomial algebra}$ with variable $d$. For any $\lambda \in k$, make $k[d]$ into a commutative bialgebra, denoted by $k_\lambda[d]$, with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d + \lambda d \otimes d$ and counit $\epsilon(d) = 0$. If $\lambda = 0$ then antipode $S(d) = -d$ makes $k[d]$ a Hopf algebra.

A is a differential algebra is equivalent to $A$ is a $k[d]$-module algebra, i.e., the multiplication map $A \otimes A \to A$ is a $k[d]$-module homomorphism.

Form the smash product algebra $A_\lambda[d] = A \# k_\lambda[d]$ with the relation

$$(1 \# d)(a \# 1) = d(a) \# 1 + a \# d + \lambda d(a) \# d.$$  

**Theorem 1.** An $A$-module $M$ is a differential module if and only if $M$ is a module for the smash product algebra $A \# k_\lambda[d]$. In particular, the category of all differential modules for a differential algebra $(A,d)$ is an abelian category with enough projectives.
An interesting question is to determine all irreducible and projective indecomposable objects and homological properties can be studied. A lot of work are done!

**Definition 2.** Let \((R, P)\) be a Rota-Baxter algebra. An **Rota-Baxter** \((R, P)\)-module is a pair \((M, p)\) with \(M\) being an \(R\)-module and \(p : M \to M\) a \(k\)-linear map such that

\[
P(a)p(x) = p(ap(x) + P(a)x + \lambda ax), \quad \forall a \in R, \ x \in M.
\]

Given two \((R, P)\)-modules \((M, p)\) and \((M', p')\), a homomorphism is an \(R\)-module homomorphism \(\phi : M \to M'\) such that

\[
\phi \circ p = p' \circ \phi.
\]

As in the case of usual module theory, any Rota-Baxter (left) ideal \(I\) of \((R, P)\) (meaning an ideal \(I\) of \(R\) such that \(P(I) \subseteq I\)) is a Rota-Baxter \((R, P)\)-module under the restriction \(P : I \to I\) and the quotient \(R/I\) is also an \((R, P)\)-module.
4. Ring of Rota-Baxter operators

Similar to the ring of differential operators, we construct the ring of Rota-Baxter operators acting on a Rota-Baxter algebra. Then the category of Rota-Baxter modules is equivalent to the category of modules over the ring of Rota-Baxter operators.

**Definition 3.** Let \((R, P)\) be a Rota-Baxter algebra of weight \(\lambda\). The ring of Rota-Baxter operators on \((R, P)\), denoted by \(R_{RB}\langle Q\rangle\), is defined to be the quotient of the free product of the \(k\)-algebras \(R\) and \(k[Q]\) modulo the relation

\[ QfQ - P(f)Q + QP(f) + \lambda Qf, \quad \forall f \in R. \]  

More precisely, let \(k\langle R, Q\rangle\) be the free product of the \(k\)-algebras \(R\) and \(k[Q]\), where \(Q\) is a variable. Let \(I_{R,Q}\) be the ideal of \(k\langle R, Q\rangle\) generated by the element in (2). Then

\[ R_{RB}\langle Q\rangle = k\langle R, Q\rangle/I_{R,Q}. \]
Theorem 2. Let $(R, P)$ be a Rota-Baxter algebra.

- For a Rota-Baxter module $(M, p)$, define
  \[ R_{RB}\langle Q \rangle \otimes M \rightarrow M \]  
  sending $Q \otimes m$ to $p(m)$. Then $M$ is a $R_{RB}\langle Q \rangle$-module.

- Conversely, for a $R_{RB}\langle Q \rangle$-module $M$, define
  \[ p : M \rightarrow M, \quad m \mapsto Qm, \quad m \in M. \]
  Then $(M, p)$ is a Rota-Baxter module.

- Any $R$-module module homomorphism $\phi : M \rightarrow M'$ is a homomorphism of $(R, P)$-modules if and only if it is an $R_{RB}$-module homomorphism.
Because of the theorem, the study of Rota-Baxter modules becomes the study of $R_{RB}\langle Q\rangle$-modules in the usual sense. In particular, the category $(R, P)$-mod is an abelian category with enough projective objects.

5. The structure of $R_{RB}\langle Q\rangle$

In order to study $R_{RB}\langle Q\rangle$-modules, it is necessarily to get precise information on the algebra $R_{RB}\langle Q\rangle$. We now give a concrete construction of this algebra motivated by the following observation. An element in $R_{RB}\langle Q\rangle$ can be formally written as finite sums of

$$f_1Q \cdots Qf_{k-1}Qf_k, \; f_i \in R,$$

modulo the relation

$$QfQ = P(f)Q - QP(f) - \lambda Qf.$$

**Theorem 3.** $R_{RB}\langle Q\rangle = R \oplus RQR$ as $R$-$R$-bimodule.

**Question 1.** How are the representation theory of $R$ and $(R, P)$ related?
6. Induced representations

The forgetful functor

\[ (R, P)\text{-mod} \to R\text{-mod} \]

is exact and admit a left adjoint functor \( V : R\text{-mod} \to (R, P)\text{-mod} \). For each \( R \)-module \( M \), define

\[ V(M) = R_{RB}\langle Q \rangle \otimes_R M \]

**Question 2.** If \( R \) is finite representation type, is \((R, P)\)-always of finite representation type?

7. Examples

Let \( R = k, \lambda \in k \). Then \((k, -\lambda)\) is a Rota-Baxter algebra of weight \( \lambda \). Each finite dimensional \((R, P)\)-module is a necessarily a finite dimensional vector space \( k^n \). A linear map \( p : k^n \to k^n \) defines an \((R, P)\)-module structure if and only if

\[ Q(Q + \lambda) = 0. \]

If \( \lambda \neq 0 \), then \( Q \) is a diagonalizable over \( k \) with eigenvalues 0 and \(-\lambda\).
Thus: The \((k, P)\)-module category is semisimple with exactly two irreducible representations \((k, 0)\) and \((k, -\lambda)\).

If \(\lambda = 0\), then the equation for \(Q\) becomes \(Q^2 = 0\).

Thus: The \((k, P)\)-module category is not semisimple with exactly one irreducible representation \((k, 0)\), but there are exactly two indecomposable representations \((k, 0)\) and \((k^2, J_2)\) where \(J_2\) is a Jordan block of size 2 with eigenvalue 0.