Representations of Rota-Baxter Algebras

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1. Rota-Baxter Algebra

$\mathbf{k}$— a fixed field. All algebras are associative unital $\mathbf{k}$-algebras.

Definition 1. A Rota-Baxter algebra of weight $\lambda \in \mathbf{k}$ is a $\mathbf{k}$-algebra A with a $\mathbf{k}$-linear map $Q : A \to A$ satisfing

$$Q(a)Q(b) = Q(Q(a)b + aQ(b) + \lambda Q(ab))$$

for all $a, b \in A$. Q is called the Rota-Baxter operator. We will denote by (A, Q) a Rota-Baxter algebra, RB-algebra for short.

2. Matrix representations

Let (A, Q) be a Rota-Baxter algebra (of weight λ). For any positive integer n, we obtained a Rota-Baxter operator $\mathcal{Q}$ on the algebra of $n \times n$ matrices $M_n(A)$ on A by defining $\mathcal{Q}$ entry-wise:

$$\mathcal{Q}([a_{ij}]) = \left[Q(a_{ij})\right].$$

This is easy to check that $\mathcal{Q}$ is a Rota-Baxter operator on $M_n(A)$. Such a Rota-Baxter algebra is called a **matrix Rota-Baxter algebra.**

Let (R, P) be a Rota-Baxter algebra (of weight λ). A **matrix representation** (with coefficients in A) of (R, P) is a homomorphism

$$f : (R, P) \to (M_n(A), \mathcal{Q})$$

of Rota-Baxter algebras, i.e., f is an k-algebra homomorphism and

$$\mathcal{Q}(f(r)) = f(P(r)) \quad (1)$$

for all $r \in R$. Such representations have appeared in QFT renormalization

Remark 1. For any $\mathbf{k}$-algebra A, the identity operator on A is a Rota-Baxter operator of weight $\lambda = -1$. In this case, any $\mathbf{k}$-algebra homomorphism $R \to M_n(A)$ is a matrix representation of the Rota-Baxter algebra (R, Id) of weight -1. Hence representations of Rota-Baxter algebras enriches the ordinary representations of algebras.

We will see that matrix representations are special cases of more general module theories of Rota-Baxter algebras. The matrix representations are just modules of (R, P) over a free (A, Q)-modules

3. Rota-Baxter modules

Recall the well-known differential case:

Let (A, d) be a differential algebra of weight λ:

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y), \ \forall x, y \in A.$$

A differential module for (A, d) of weight λ is a pair (M, δ) with M being an A-module and $\delta : M \to M$ is a $\mathbf{k}$-linear map such that

$$\delta(ax) = d(a)x + a\delta(x) + \lambda d(a)\delta(x), \forall a \in A, x \in M.$$

Algebraically, let $k[d]$ = polynomial algebra with variable d. For any $\lambda \in k$, make $k[d]$ into a commutative bialgebra, denoted by $k_\lambda[d]$, with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d + \lambda d \otimes d$ and counit $\epsilon(d) = 0$. If $\lambda = 0$ then antipode $S(d) = -d$ makes $k[d]$ a Hopf algebra.

A is a differential algebra is equivalent to A is a $k[d]$-module algebra, i.e., the multiplication map $A \otimes A \to A$ is a $k[d]$-module homomorphism.

Form the smash product algebra $A_\lambda[d] = A\#k_\lambda[d]$ with the relation

$$(1\#d)(a\#1) = d(a)\#1 + a\#d + \lambda d(a)\#d.$$

Theorem 1. *An A-module M is a differential module if and only if M is a module for the smash product algebra $A\#k_\lambda[d]$. In particular, the category of all differential modules for a differential algebra (A, d) is an abelian category with enough projectives.*

An interesting question is to determine all irreducible and projective indecomposable objects and homological properties can be studied. A lot of work are done!

Definition 2. Let (R, P) be a Rota-Baxter algebra. An **Rota-Baxter** (R, P)**-module** is a pair (M, p) with M being an R-module and $p : M \to M$ a $\mathbf{k}$-linear map such that

$$P(a)p(x) = p(ap(x) + P(a)x + \lambda ax),\ \forall a \in R,\ x \in M.$$

Given two (R, P)-modules (M, p) and (M', p'), a homomorphism is an R-module homomorphism $\phi : M \to M'$ such that

$$\phi \circ p = p' \circ \phi.$$

As in the case of usual module theory, any Rota-Baxter (left) ideal I of (R, P) (meaning an ideal I of R such that $P(I) \subseteq I$) is a Rota-Baxter (R, P)-module under the restriction $P : I \to I$ and the quotient R/I is also an (R, P)-module.

4. Ring of Rota-Baxter operators

Similar to the ring of differential operators, we construct the ring of Rota-Baxter operators acting on a Rota-Baxter algebra. Then the category of Rota-Baxter modules is equivalent to the category of modules over the ring of Rota-Baxter operators.

Definition 3. Let (R, P) be a Rota-Baxter algebra of weight λ. The **ring of Rota-Baxter operators on** (R, P), denoted by $R_{RB}\langle Q\rangle$, is defined to the the quotient of the free product of the $\mathbf{k}$-algebras R and $\mathbf{k}[Q]$ modulo the relation

$$QfQ - P(f)Q + QP(f) + \lambda Qf, \quad \forall f \in R. \quad (2)$$

More precisely, let $\mathbf{k}\langle R, Q\rangle$ be the free product of the $\mathbf{k}$-algebras R and $\mathbf{k}[Q]$, where Q is a variable. Let $I_{R,Q}$ be the ideal of $\mathbf{k}\langle R, Q\rangle$ generated by the element in (2). Then

$$R_{RB}\langle Q\rangle = \mathbf{k}\langle R, Q\rangle / I_{R,Q}.$$

Theorem 2. *Let (R, P) be a Rota-Baxter algebra.*

- *For a Rota-Baxter module (M, p), define*

  $$R_{RB}\langle Q\rangle \otimes M \to M \tag{3}$$

  sending $Q \otimes m$ to $p(m)$. Then M is a $R_{RB}\langle Q\rangle$-module.

- *Conversely, for a $R_{RB}\langle Q\rangle$-module M, define*

  $$p : M \to M, \quad m \mapsto Qm, \quad m \in M.$$

  Then (M, p) is a Rota-Baxter module.

- *Any R-module module homomorphism $\phi : M \to M'$ is a homomorphism of (R, P)-modules if any only if it is an R_{RB}-module homomorphism.*

Because of the theorem, the study of Rota-Baxter modules becomes the study of $R_{RB}\langle Q\rangle$-modules in the usual sense. In particular, the category (R,P)-mod is an abelian category with enough projective objects.

5. The structure of $R_{RB}\langle Q\rangle$

In order to study $R_{RB}\langle Q\rangle$-modules, it is necessarily to get precise information on the algebra $R_{RB}\langle Q\rangle$. We now give a concrete construction of this algebra motivated by the following observation. An element in $R_{RB}\langle Q\rangle$ can be formally written as finite sums of

$$f_1 Q \cdots Q f_{k-1} Q f_k,\ \ f_i \in R,$$

modulo the relation

$$QfQ = P(f)Q - QP(f) - \lambda Qf.$$

Theorem 3. *$R_{RB}\langle Q\rangle = R \oplus RQR$ as R-R-bimodule.*

Question 1. How are the representation theory of R and (R,P) related?

6. Induced representations

The forgetful functor

$$(R, P)\text{-mod} \to R\text{-mod}$$

is exact and admit a left adjoint functor V : R-mod $\to$ (R, P)-mod. For each R-module M, define

$$V(M) = R_{RB}\langle Q\rangle \otimes_R M$$

Question 2. If R is finite representation type, is (R, P)-always of finite representation type?

7. Examples

Let $R = \mathbf{k}$, $\lambda \in \mathbf{k}$. Then $(\mathbf{k}, -\lambda)$ is a Rota-Baxter algebra of weight λ. Each finite dimensional (R, P)-module is a necessarily a finite dimensional vector space $\mathbf{k}^n$. A linear map $p : \mathbf{k}^n \to \mathbf{k}^n$ defines an (R, P)-module structure if and only if

$$Q(Q + \lambda) = 0.$$

If $\lambda \neq 0$, then Q is a diagonalizable over $\mathbf{k}$ with eigenvalues 0 and $-\lambda$.

Thus: The $(\mathbf{k}, P)$-module category is semisimple with exactly two irreducible representations $(\mathbf{k}, 0)$ and $(\mathbf{k}, -\lambda)$.

If $\lambda = 0$, then the equation for Q becomes $Q^2 = 0$.

Thus: The $(\mathbf{k}, P)$-module category is not semisimple with exactly one irreducible representation $(\mathbf{k}, 0)$, but there are exactly two indecomposable representations $(\mathbf{k}, 0)$ and $(\mathbf{k}^2, J_2)$ where J_2 is a Jordan block of size 2 with eigenvalue 0.