

Effective differential Nullstellensatz

Alexey Ovchinnikov

University of Illinois at Chicago

Joint work with:

Oleg Golubitsky (University of Western Ontario)

Marina Kondratieva (Moscow State University)

Agnes Szanto (North Carolina State University)

The Problem of Consistency

Given a system of polynomial PDE, e.g.:

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 0 \end{cases}$$

Question: Is it consistent, i.e., does it have solutions?

(We look for solutions in differential extensions of the coefficient field ...)

The Problem of Consistency

Given a system of polynomial PDE, e.g.:

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 0 \end{cases}$$

Question: Is it consistent, i.e., does it have solutions?

(We look for solutions in differential extensions of the coefficient field ...)

Answer: YES

The Problem of Consistency

Given a system of polynomial PDE, e.g.:

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 1 \end{cases}$$

Question: Is it consistent, i.e., does it have solutions?

(We look for solutions in differential extensions of the coefficient field ...)

Answer: NO

Corresponding algebraic system

$$\left\{ \begin{array}{l} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 0 \end{array} \right. \longleftrightarrow \left\{ \begin{array}{l} z_1 + z_2 = 0 \\ z_3 - z_4 = 0 \\ (z_5 + z_6)^2 + (z_7 + z_8)^2 = 0 \end{array} \right.$$

PDE system is consistent

\implies

Algebraic system is consistent

Corresponding algebraic system

$$\left\{ \begin{array}{l} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 1 \end{array} \right. \longleftrightarrow \left\{ \begin{array}{l} z_1 + z_2 = 0 \\ z_3 - z_4 = 0 \\ (z_5 + z_6)^2 + (z_7 + z_8)^2 = 1 \end{array} \right.$$

PDE system is *inconsistent*

Algebraic system is consistent

The converse is not always true.

Differential Nullstellensatz

Notation $F^{(\leq k)}$

set of all partial derivatives of elements of F of order $\leq k$

Theorem Polynomial PDE system $F = 0$ has **no solutions**



$\exists k \geq 0$ such that $1 \in \langle F^{(\leq k)} \rangle$.

Example

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 1 \end{cases} \quad k = 1$$

The Problem

Given non-negative integers m, n, h, d .

Find $k(m, n, h, d)$ such that:

Polynomial PDE system $F = 0$
in m independent variables,
 n dependent variables
of order h
and degree d
has no solutions



$$1 \in \langle F(\leq k(m, n, h, d)) \rangle$$

Main Result

[Seidenberg, 1956] Proposed to analyse the differential elimination algorithm to obtain the bound.

Theorem [GKOS '08]

$$k(m, n, h, d) = A(m + 8, \max(n, h, d)).$$

Here $A(m, n)$ is the Ackermann function.

Why Ackermann Function?

Definition. A sequence of non-negative integer n -tuples τ_1, τ_2, \dots is called **dicksonian**, if for all $i < j$, $\tau_j - \tau_i$ has at least one negative coordinate.

Alternative definition: a sequence of monomials u_1, u_2, \dots such that $u_i \not\prec u_j$ for $i < j$.

Lemma [Dickson] Every dicksonian sequence terminates.

Lemma [G. Socias, 1991] Every dicksonian sequence of n -tuples, in which the maximal coordinate at each step increases by 1, has length at most

$$A(n, m - 1) - 1,$$

where m is the maximal coordinate of the first tuple.

Why Dicksonian Sequences?

Polynomial *completion* algorithms such as

```
Algorithm Buchberger ( $F, \leq$ )
  repeat
     $R := \text{NormalForm}(\text{SPoly}(F), F, \leq) \setminus \{0\}$ 
     $F := F \cup R$ 
  until  $R = \emptyset$ 
  return  $F$ 
end
```

produce dicksonian sequences of leading monomials.

Why Dicksonian Sequences?

Differential-algebraic completion algorithms, when applied to polynomial PDE systems, produce sequences of *powers of leading partial derivatives* of the form

$$\left(\frac{\partial^h u_j}{\partial x_1^{i_1} \dots \partial x_m^{i_m}} \right)^d$$

such that the corresponding $(m + n)$ -tuples

$$(i_1, \dots, i_m, 0, \dots, d, \dots, 0)$$

form dicksonian sequences.

How Fast Do Tuples Grow?

Polynomial case: at each iteration, degree doubles (at most).

Differential case: at each iteration,

- order h doubles
- degree d becomes $(4d)^{\binom{2h+m}{m}+1}$ [GKOS '08].

We have a dicksonian sequence of $(m + n)$ -tuples, in which the coordinates of the i -th tuple are bounded by a certain function $f(i)$.

How Long Are these Sequences?

Function $f(i)$ is not growing too fast: $\exists \delta$ such that $\forall i$

$$f(i + 1) - f(i) \leq A(\delta, f(i) - 1).$$

Lemma [GKOS'08] Length of such sequence does not exceed

$$\lceil f^{-1}(A[m + n + \delta, f(1) - 1]) \rceil$$

and the coordinates of the last tuple do not exceed

$$A(m + n + \delta, f(1) - 1).$$

Complexity of Differential Completion

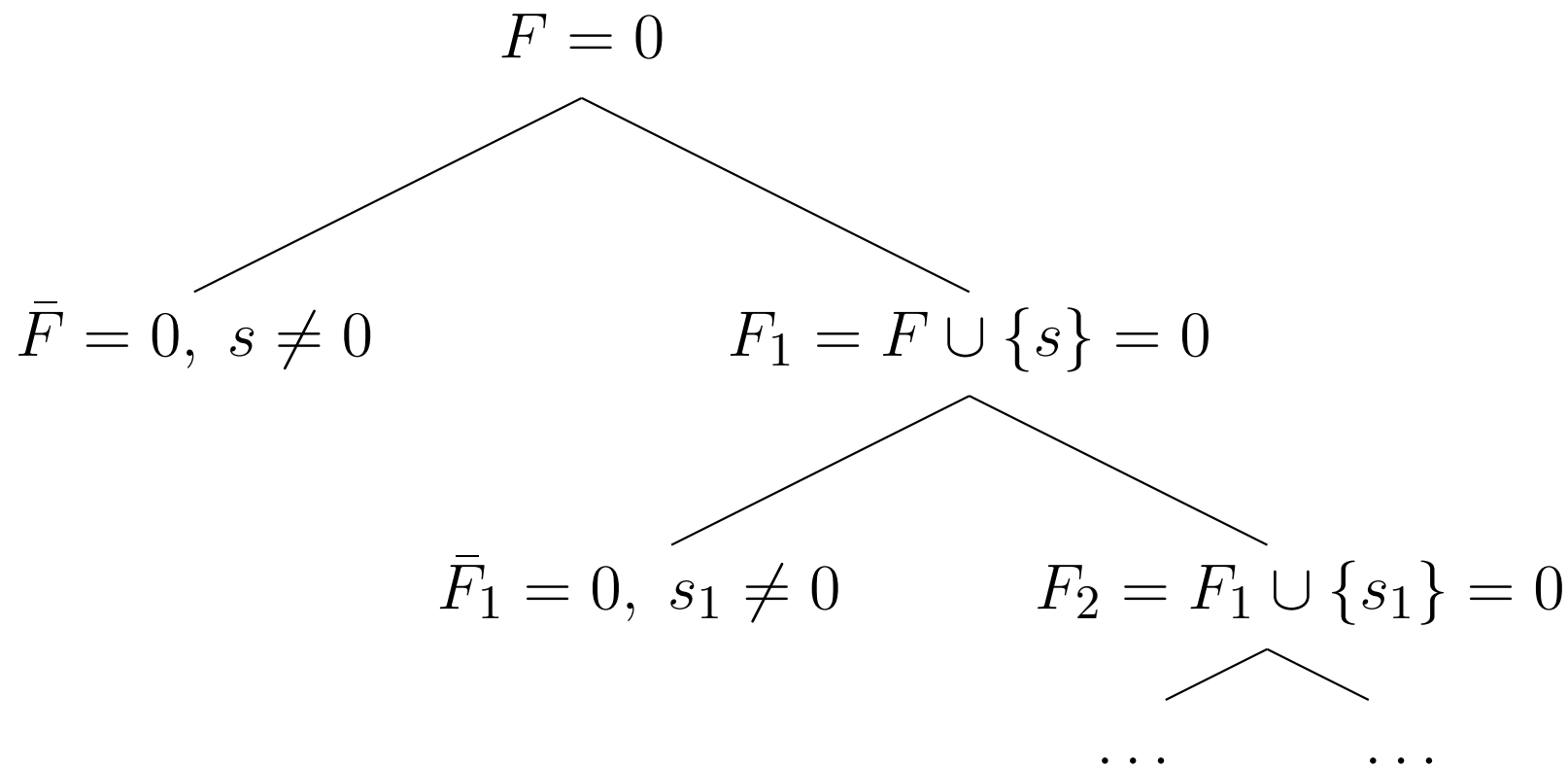
The number of iterations is bounded by:

$$A(m + n + 3, \max(9, 2^h, d) - 1)$$

and the orders and degrees of the output, as well as the number of differentiations required to produce it, by:

$$k_{\text{completion}} = A(m + n + 4, \max(3, h, d) - 1).$$

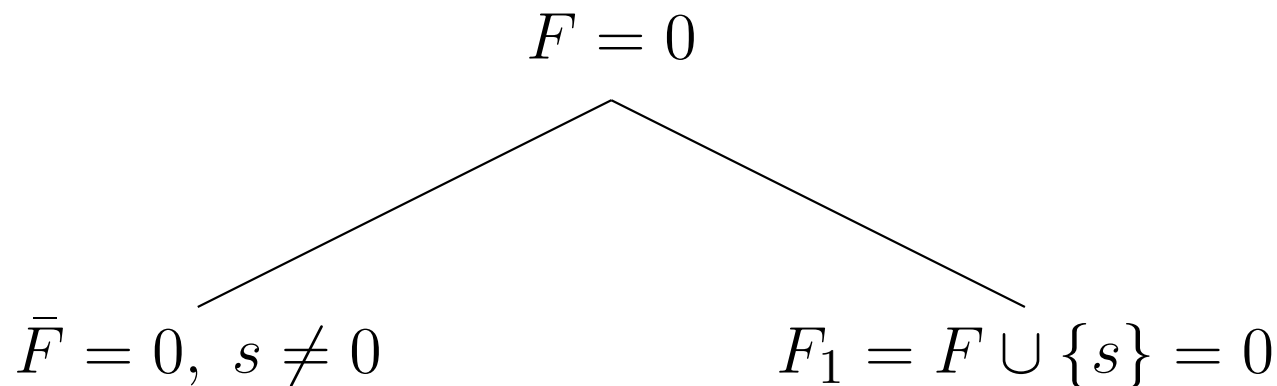
Splitting



Walking down the tree produces a dicksonian sequence

Obtaining Expression for 1

Let $F = 0$ be an inconsistent system.



$$\bar{F} \subset \langle F^{(\leq k_{\text{completion}})} \rangle$$
$$s \in \sqrt{\langle \bar{F} \rangle}$$

Suppose that we “recursively”
obtained bound k_1 for F_1 .
Then $1 \in \langle F^{(\leq k_1)} \rangle + \langle s^{(\leq k_1)} \rangle$.

And sufficiently large
powers of derivatives of s
can be expressed
in terms of derivatives of \bar{F} .

Expressing Powers of Derivatives

- $s \in \sqrt{\langle \bar{F} \rangle}$
- [Kollar, 1988] A bound for q such that

$$s^q \in \langle \bar{F} \rangle$$

in terms of number of variables and degrees of \bar{F} .

- [Ritt] If $s^q \in \langle G \rangle$, then

$$\left(\frac{\partial s}{\partial x_i} \right)^{2q-1} \in \langle G^{(\leq q)} \rangle.$$

Proof for $q = 2$:

$$\begin{aligned} s^2 \in \langle G \rangle &\Rightarrow ss' \in \langle G, G' \rangle \Rightarrow \\ (s')^2 + ss'' \in \langle G, G', G'' \rangle &\Rightarrow (s')^3 \in \langle G, G', G'' \rangle. \end{aligned}$$