Isomonodromic Garnier system and geometry of higher Painlevé VI equations

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0. The six Painlevé equations

(too) Many PsOV

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0. The six Painlevé equations. Painlevé and Gambier

While the solutions of a linear ODE ($t \in \mathbb{C}$):

$$\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + ... + P_{n-1}(t)\frac{dy}{dt} + P_n(t)w = 0$$

with meromorphic coefficients have fixed singularities (occurring only where the coefficients have singularities and independent of the n constants of integration), a non-linear ODE may exhibit both fixed and moveable singularities:

$$\frac{dy}{dt} + y^2 = 0$$

has general solution

$$y(t; t_0) = \frac{1}{t - t_0},$$

where t_0 is the constant of integration and also the location of the singularity. In general moveable meromorphic singularities cannot be avoided.

"Painlevé property": essential singularities and branch points, algebraic or logarithmic, are fixed[†]. Painlevé classified (with one omission remedied by his student Gambier) such second-order equations of the form:

$$\frac{d^2y}{dt^2} = R\left(t, y, \frac{dy}{dt}\right)$$

with R a rational function in $y, dy/dt$ and locally meromorphic in t (first-order had been classified by Fuchs). They found 50 canonical equations of which only six non-autonomous types could not be reduced to an equation already solved, in particular define new functions, the "Painlevé transcendents". Suitable limits of P_{VI} reduce to each of the other five (classical). The only known family of solutions of P_{VI} came from the hypergeometric equation, until the Inverse Scattering Transform was devised [FA].

[†] Kowalevski in her work on the top required that the equations of motion have no moveable critical points: this links the "Painlevé test" of PDE theory with the one for integrable (finite-dimensional) Hamiltonian systems; in both cases the test is implemented by finding "balances", the exponents of a local expansion near the singularities.

$\mathrm{P_I}$ $$\frac{d^2y}{dt^2} = 6y^2 + t$$

$\mathrm{P_{II}}$ $$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

$\mathrm{P_{III}}$ $$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$\mathrm{P_{IV}}$ $$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

$\mathrm{P_V}$ $$\begin{aligned}\frac{d^2y}{dt^2} &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} \\ &\quad + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}\end{aligned}$$

$\mathrm{P_{VI}}$ $$\begin{aligned}\frac{d^2y}{dt^2} =& \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 \\ & - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ & + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)\end{aligned}$$

1. Integrable hierarchies: similarity equations and the Painlevé-property conjecture

$$u_t + 6uu_x + u_{xxx} = 0 \qquad \text{KdV}$$

admits both traveling wave solutions and "self-similar (similarity)" solutions

$$(*)\, u(x,t) = U(x-ct), \quad (**)\, u(x,t) = \frac{1}{(3t)^{2/3}} f\left(\frac{x}{(3t)^{1/3}}\right),$$

$$(*) U'' + 3U^2 - cU = K, \qquad (**)\ f''' + 6ff' = zf' + 2f$$

Each of these ODE's is an "exact reduction" of the PDE, obtained by suitably restricting the set of solutions (no systematic way to find all)[‡]. Both have the Painlevé property. By integrating (*) once, we find an equation for $\wp(x-ct)$; set $f(s) = v_s - v^2$, where $s := x/(3t)^{1/3}$, and $v(s)$ solves $\mathrm{P_{II}}$ (with s in place of t).

[‡] For a group-theoretic approach cf. [ES,SE,O].

KdV is invariant under the group of transformations:

$$\tilde{x} = x + 6t\beta,\ \tilde{t} = t + \frac{\beta}{\alpha}, \tilde{u} = u + \beta.$$

Regarding α as fixed, invariance and KdV imply:

$$u = \alpha t + U(z),\ z = x - 3\alpha t^2,\quad U''' + 6UU' + \alpha = 0,$$

upon integrating, $\mathrm{P_I}$.

The Kadomtsev-Petviashvili (KP) equation:

$$(u_t - 6uu_x + u_{xxx})_x + 3\sigma u_{yy} = 0,\ s = \frac{x}{(3t)1/3} + \frac{y^2}{(3t)^{-4/3}}$$

gives an ODE for $F(s) = u(x, y, t)$ which, integrated once, is $\mathrm{P_{II}}$ for F^2. In [TNK], KP is reduced to KdV first, then to $\mathrm{P_I}$ or $\mathrm{P_{II}}$.

Painlevé Conjecture [ARS] *A nonlinear* PDE *is solvable by an inverse scattering transform only if every nonlinear* ODE *obtained by exact reduction is of Painlevé type, possibly after a transformation of variables.*

Inverse scattering transform (IST):

$$K(x,y) = F(x,y) + \int_x^\infty K(x,x)N(x;z,y)dz, \quad y \geq x,$$

N in terms of F, $d/dxK(x,x)$ a solution. IST comes from the linear problem, reconstruct potential from asymptotics of eigenfunctions, analogous to linear Fourier transform for an evolution equation:

$$u_t = F(u, u_x, u_{xx}, ...), \quad u(x,t) \sim Ae^{\imath kx - \imath\omega t}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)e^{\imath kx - \imath\omega t}dk.$$

Example: The linear problem

$$\begin{cases} N(x;z,y) = F(z,y), \\ (\partial_x^2 - \partial_y^2)F(x,y) = 0, \\ (\partial_t + (\partial_x + \partial_y)^3)F(x,y) = 0 \end{cases}$$

implies

$$u_t + 6uu_x + u_{xxx} = 0 \quad \text{if } u = 2\frac{d}{dx}K(x,x).$$

2. Integrable hierarchies: Lax pair and isomonodromy equations: the τ function

An important link between isomonodromy problems for ODEs and integrable equations of mathematical physics was first found by the "Kyoto school" (1978). We explore the link by an illustration: P_{VI}.

We specify the general construction to 2×2 matrices in $\mathfrak{sl}(2)$ whose entries are rational functions of λ:

$$M_\lambda(\lambda) = \sum_{k=1}^{K} \left(\frac{A_1^{(k)}}{\lambda - \lambda_k} + \dots + \frac{A_{n_k+1}^{(k)}}{(\lambda - \lambda_k)^{n_k+1}} \right) - A_0^{(\infty)} - \dots - A_{n_\infty - 1}^{(\infty)} \lambda^{n_\infty - 1}$$

(set $M_\lambda(\lambda) = \sum_k M_\lambda^{(k)}(\lambda)$ where $M_\lambda^{(k)}(\lambda)$ is the polar part of M at λ_k, including ∞), with one singularity at $\lambda = \infty$ and three regular singularities at $\lambda_0 = 0, \lambda_1 = 1, \lambda_t = t$.

Goal: Study the isomonodromy-deformation problem with respect to the parameter t.

The link: View the solution of $\partial_\lambda \Psi = M_\lambda \Psi$ as a "wave function":

$$\Psi(\lambda) = g(\lambda) e^{\sum_i \xi_i(\lambda) t_i}, \; i = (k, n, \alpha),$$

$g(\lambda)$ a regular matrix, $\xi_i(\lambda) = \sum_\alpha e_{\alpha\alpha}/(\lambda - \lambda_k)^n$, with

$$\partial_{t_i} \Psi(\lambda) = M_i(\lambda) \Psi(\lambda), \; M_i(\lambda) = \big(g(\lambda) \xi_i(\lambda) g^{-1}(\lambda) \big)_-$$

(the notation $()_-$ means taking the polar part at λ_k).

This gives a hierarchy of commuting flows, satisfying the zero-curvature equations:

$$\partial_{t_i} M_j - \partial_{t_j} M_i - [M_i, M_j] = 0.$$

Now the $\Psi(\lambda)$ have essential singularities at the $\lambda = \lambda_k$ and at $\lambda = \infty$ and in general have non-trivial monodromy around the singularities (are defined on a Riemann surface),

$$\Psi(\lambda) \sim g^{(k)}(\lambda) e^{\xi^{(k)}(\lambda)},$$

where

$$g^{(k)}(\lambda) = g_0^{(k)} + g_1^{(k)}(\lambda - \lambda_k) + \ldots$$

is regular at λ_k, and

$$\xi^{(k)} = B_0^{(k)} \log(\lambda - \lambda_k) + \sum_{\alpha;n=1}^{n_k} \frac{t_{(k,n,\alpha)}}{(\lambda - \lambda_k)^n} e_{\alpha\alpha}$$

$$\xi^{(\infty)} = B_0^{(\infty)} \log\left(\frac{1}{\lambda}\right) + \sum_{\alpha;n=1}^{n_\infty} t_{(\infty,n,\alpha)} \lambda^n e_{\alpha\alpha}$$

Goal: write evolution equations that describe the deformations of $M_\lambda(\lambda)$ under which the monodromy of

$\Psi(\lambda)$ around a singularity of $M(\lambda)$ is fixed. The isomonodromic deformation parameters will include the $t_{(k,n,\alpha)}$, and we have an enriched family of commuting flows:

$$\partial_{\lambda_k}\Psi(\lambda) = M_{\lambda_k}\Psi(\lambda),\ M_{\lambda_k} = \Big((g^{(k)}\partial_{\lambda_k}\xi_i^{(k)}g^{(k)-1}(\lambda)\Big)_-$$

In our illustration we choose the singularity data:

$$\xi^{(k)}(\lambda) = \begin{bmatrix}\theta_k & 0\\ 0 & 0\end{bmatrix}\log(\lambda-\lambda_k),\ k=0,1,t,$$

we diagonalize M at ∞ by global gauge transformation by a matrix constant in λ:

$$M(\lambda) = \frac{1}{\lambda}\begin{bmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{bmatrix} + O(\lambda^{-2}),\ \lambda\to\infty.$$

The equation $\partial_\lambda\Psi = M\Psi$ has a regular singularity at ∞ and the Fuchs condition reads: $\kappa_1+\kappa_2 = \theta_0+\theta_1+\theta_t$. Set:

$$M(\lambda) = \begin{bmatrix}m_{11}(\lambda) & m_{12}(\lambda)\\ m_{21}(\lambda) & m_{22}(\lambda)\end{bmatrix} = \frac{1}{\lambda(\lambda-1)(\lambda-t)}A(\lambda),$$

$A_{12}(\lambda) = \gamma(\lambda - y)$. The dynamical variables are now $y, \gamma, A_{11}(y)$ (from the condition that the determinant is zero at the singularities, we solve for the other parameters in terms of these). We use the equation of motion: $\partial_t M_\lambda = \partial_\lambda M_t + [M_t, M_\lambda]$ and afterwards set $\lambda = y$:

$$\begin{bmatrix} \dot m_{11}(\lambda) & \dot m_{12}(\lambda) \\ \dot m_{21}(\lambda) & \dot m_{22}(\lambda) \end{bmatrix}_{\lambda=y} = \frac{1}{t(t-1)(y-t)^2} \cdot \left\{ A(t) - \frac{1}{y(y-1)}[A(t), A(y)] \right\}$$

which by elimination give P_{VI} for y, with appropriate values for the parameters: $\alpha = 1/2(\kappa_1 - \kappa_2 + 1)^2$, $\beta = -(1/2)\theta_0^2$, $\gamma = 1/2\theta_1^2$, $\delta = 1/2(1-\theta_t)^2$.

The τ function.

The deformation equations imply that the following 1-form is closed:

$$Y = -\sum_k \mathrm{Res}_{\lambda=\lambda_k} \mathrm{Tr}(g^{(k)-1} \partial_\lambda g^{(k)} d\xi^{(k)}) d\lambda,$$

summed over all the singularities including ∞.

Example. In the case of the Schlesinger equations

$$Y = \frac{1}{2} \sum_{k \neq l} \mathrm{Tr}(A_k A_l) \frac{d\lambda_k - d\lambda_l}{\lambda_k - \lambda_l}.$$

Definition The tau-function: $Y = d \log \tau$.

The tau-function satisfies the Hirota equations: Define the bilinear operator

$$D_i^n f \cdot g = \left(\frac{\partial}{\partial y_i} \right)^n f(x+y) g(x-y)|_{y=0},$$

then for the polynomial operator

$$(D_1^4 + 3D_2^2 - 4D_1 D_3) \tau \cdot \tau = 0$$

implies the KP equation:

$$3u_{t_2 t_2} + (-4u_{t_3} - 6uu_{t_1} + u_{t_1 t_1 t_1})_{t_1} = 0, \ \ u = \frac{\partial^2}{\partial t_1^2} \log \tau.$$

The proof consists in showing that the tau-function obeys the Schlesinger transformations.

Project. (with G.N. Benes, BU GRS 2011)

There appears to be no reduction from an integrable PDE to P_{VI} (except for the alternative form in [AvdL]). Krichever [Kr] sets up a direct problem to produce self-similar PDEs from a Baker vector $(\psi_1, \psi_2, ..., \psi_n)$ with Stokes sectors given by angles ϕ_s :

$$\mathrm{Re}\left[\lambda^m \left(\exp\left(\frac{2\pi\imath}{n}k\right) - \exp\left(\frac{2\pi\imath}{n}l\right)\right)\right] = 0,\ 1 \le k, l \le n$$

G_s is the $n \times n$ Stokes matrix,

$$\psi^{+s}(r) = \psi^{-s}(r)G_s,\ \psi^{\pm s}(r) = \psi(re^{\imath\phi_s \pm \imath \lim(\delta\to 0)}).$$

If $\psi = v(x, t, \lambda)\psi_0$ with v a piecewise meromorphic function on the extended λ plane and $\psi_0 = \exp(\lambda Q x + \lambda^m Q^m t)$, $Q_{kl} = \exp[(2\pi\imath/n)k]\delta_{kl}$, then

$$L_n\psi = \lambda^n\psi,\ (\partial_t - L_m)\psi = 0,\ [L_n, \partial_t - L_m] = 0$$

and ψ can be rescaled $\psi(x, t, \lambda) = \psi(\beta x, \beta^m t, \beta^{-1}\lambda)$, to give self-similar solutions to the KP equation.

Thus, from the Lax pair for the PDE, we can find the appropriate reduced ODE. This should give us the general ODE associated to KP by reduction.

Combined with work by Flaschka and Newell [FN] that gives the Painlevé equations by linear Lax-pair conditions of type $[L, B] = L$, this may yield a PDE for P_{VI}.

3. Integrable hierarchies: the Penrose transform and Painlevé VI

An anti-self-dual conformal structure on a 4-manifold is the solution of an equation which can be interpreted as an isomonodromic deformation of a connection on its twistor space. When restricted to a 4-parameter family of projective lines on each of which it has 4 poles, the isomonodromy equation is P_{VI}. This allows Hitchin [H1] to "work backwards to find explicit formulas for the metric", using an abelian subgroup of the monodromy to pull-back the connection from the line to an elliptic curve. He thus finds solutions to P_{VI} for a special value of the parameters,

$$\begin{aligned}\frac{d^2y}{dt^2} =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 \\ & -\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt} \\ & +\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\frac{1}{8}+\frac{t}{8y}+\frac{(t-1)}{8(y-1)^2}+\frac{3t(t-1)}{8(y-t)^2}\right)\end{aligned}$$

depending on two constants c_1, c_2 [*typo alert*]:

$$y(t) = \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3}\left(1 + \frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}\right) +$$

$$\frac{\vartheta_1'''(\nu)\vartheta_1(\nu) - 2\vartheta_1''(\nu)\vartheta_1'(\nu) + 4\pi\imath c_1(\vartheta_1''(\nu)\vartheta_?(\nu) - {\vartheta_1'}^2(\nu))}{2\pi^2\vartheta_4^4(0)\vartheta_1(\nu)(\vartheta_1'(\nu) + 2\pi c_1\vartheta_1(\nu))},$$

where $\nu = c_1\tau + c_2$ and $t = \vartheta_3^4(0)/\vartheta_4^4(0)$.

N.B. We referred to a non-classical 1-parameter family of solutions above [FA] (also given for a constraint on the parameters). This is another family of solutions, and depends on two parameters.

In [H2], Hitchin gives a more algebraic formula for special cases,e.g.

$$y = \frac{s^2(2s^2 + 5s + 2)}{(2s + 1)(s^2 + s + 1)},\ t = \frac{s^3(s + 2)}{2s + 1}$$

by projecting the lines in the Penrose space to conics in $\mathbb{P}^2$ and observing that solutions are given by conics in Poncelet position (3-inscribed in the example).

4. Garnier's isomonodromy equations

Back to the classics. [joint project with Victor Z. Enolskii (Institute of Metal Physics, Kiev) and Frank W. Nijhoff (University of Leeds)]

Part I: Genus 1 revisited. The Legendre-Picard-Fuchs operator.

Richard Fuchs [F] poses the isomonodromy problem for a second-order linear ODE, which has regular-singular points at $\{0, 1, t, \infty\}$ (according to his father's theory [(L)F]):

$$\frac{d^2y}{dx^2} = \left[\frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{\alpha}{x} + \frac{\beta}{(x-1)} + \frac{\gamma}{(x-t)} + \frac{3}{4(x-\lambda)^2} + \frac{\epsilon}{(x-\lambda)}\right] y, \tag{1}$$

where λ is an apparent singularity (namely, a singular point of the equation which is not a singularity of any of its solutions), a, b, c are independent of x and t, $\alpha, \beta, \gamma, \epsilon$ are functions of x independent of t.

The monodromy of (1) is independent of t when λ satisfies the Painlevé VI equation:

$$\frac{d^2\lambda}{dt^2} + \left[\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda - t}\right]\frac{d\lambda}{dt} -$$

$$\frac{1}{2}\left[\frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t}\right]\left(\frac{d\lambda}{dt}\right)^2 =$$

$$\frac{1}{2}\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\times$$

$$\left[k_\infty - k_0 + \frac{t}{\lambda^2} + k_1\frac{t-1}{(\lambda-1)^2} - (k_t - 1)\frac{t(t-1)}{(\lambda-t)^2}\right],$$

where k, a, b, c (equivalently, k_0, k_1, k_t, k_∞) are arbitrary, and

$$-4(k-1) + 4a + 4b + 4c = k_\infty,$$

$$4\left(a + \frac{1}{4}\right) = k_0,\ 4\left(b + \frac{1}{4}\right) = k_1,\ 4\left(c + \frac{1}{4}\right) = k_t,$$

which Fuchs turns into an equation for the Weierstrass $\wp$ function:

$$u = \int_0^\lambda \frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}},$$

$$\frac{d^2u}{dt^2} + \frac{2t-1}{t(t-1)}\frac{du}{dt} + \frac{u}{t(t-1)} = \frac{\sqrt{\lambda(\lambda-1)(\lambda-t)}}{2t^2(t-1)^2}\times$$

$$\left[k_\infty - k_0\frac{t}{\lambda^2} + k_1\frac{t-1}{(\lambda-1)^2} - k_t\frac{t(t-1)}{(\lambda-t)^2}\right]. \qquad (2)$$

Note: The left-hand-side is the Legendre operator, which annihilates all the periods; the right-hand-side is a doubly-periodic function of u.

Motivational Aside: Manin [M] recently interpreted the inhomogeneous Picard-Fuchs equation (2) as an indication that the mirror manifold of $\mathbb{P}^2$ is an elliptic fibration with labelled sections of order two, since the potential of the quantum cohomology of $\mathbb{P}^2$ is a particular solution of the Painlevé VI equation.

Part II: The Garnier system

Garnier [G1] in one of the three main parts of his dissertation extended Fuchs' question to a linear second-order equation with $n+3$ regular-singular points $\{0, 1, \infty, t_1, ..., t_n\}$ and n apparent singularities $\{\lambda_1, ..., \lambda_n\}$ (so $n = 1$ in the case above),

$$\frac{d^2y}{dx^2} = p(x)y$$

$$p(x) = \frac{c_{n+1}}{x^2} + \frac{c_{n+2}}{(x-1)^2} + \frac{c_{n+3}}{x(x-1)} +$$

$$\sum_{l=1}^{n} \left[\frac{c_l}{(x-t_l)^2} + \frac{\alpha_l}{x(x-1)(x-t_l)} \right] +$$

$$\sum_{j=1}^{n} \left[\frac{3}{4(x-\lambda_j)^2} + \frac{\beta_j}{x(x-1)(x-\lambda_j)} \right].$$

The monodromy group of the equation is independent of the parameters $\{t_1, ..., t_n\}$ when the following system of PDEs is satisfied:

$$(f_n) \qquad \frac{\varphi'(t_i)(t_i-\lambda_j)}{\psi(t_i)} \frac{\partial \lambda_j}{\partial t_i} - \frac{\varphi'(t_k)(t_k-\lambda_j)}{\psi(t_k)} \frac{\partial \lambda_j}{\partial t_k} =$$

$$\frac{t_i - t_k}{(\lambda_j - t_i)(\lambda_j - t_k)} \frac{\varphi(\lambda_j)}{\psi'(\lambda_j)},$$

$$(F_n) \qquad \frac{\partial^2 \lambda_j}{\partial t_i^2} =$$

$$\frac{1}{2}\left(\frac{\varphi'(\lambda_j)}{\varphi(\lambda_j)} - \frac{\psi''(\lambda_j)}{2\psi'(\lambda_j)}\right)\left(\frac{\partial\lambda_j}{\partial t_i}\right)^2 -$$

$$\left(\frac{\varphi''(t_i)}{2\varphi'(t_i)} - \frac{\psi'(t_i)}{\psi(t_i)}\right)\frac{\partial\lambda_j}{\partial t_i} +$$

$$\frac{1}{2}\sum_{\substack{l=1\\l\neq j}}^{n}\frac{\varphi(\lambda_j)\psi'(\lambda_l)(\lambda_l - t_i)^2}{\varphi(\lambda_l)\psi'(\lambda_j)(\lambda_j - t_i)^2(\lambda_j - \lambda_l)}\left(\frac{\partial\lambda_l}{\partial t_i}\right)^2 -$$

$$\sum_{\substack{l=1\\l\neq j}}^{n}\frac{\lambda_j - t_i}{(\lambda_l - t_i)(\lambda_l - \lambda_j)}\frac{\partial\lambda_j}{\partial t_i}\frac{\partial\lambda_l}{\partial t_i} +$$

$$2\frac{\psi^2(t_i)}{\varphi'^2(t_i)(\lambda_j - t_i)^2}\frac{\varphi(\lambda_j)}{\psi'(\lambda_j)}\times$$

$$\left[\sum_{k=1}^{n+3}\left(c_k + \frac{3}{4}\right) - 2 + \right.$$

$$\left.\sum_{\substack{k=1\\k\neq i}}^{n+2}\frac{\varphi'(t_k)}{\psi(t_k)}\frac{c_k + \frac{1}{4}}{\lambda_j - t_k} + \frac{\varphi'(t_i)}{\psi(t_i)}\frac{c_i}{\lambda_j - t_i}\right],$$

where

$$\varphi(x) \equiv x(x-1)\prod_{l=1}^{n}(x - t_l), \quad \psi(x) \equiv \prod_{j=1}^{n}(x - \lambda_j).$$

We study Garnier's system as a system of higher order ODEs, which was Garnier's original point of view, the lowest member of which is the Painlevé VI equation itself. We derive the ODEs as similarity solutions to a Schwarzian PDE [NHJ]:

$$\begin{gathered}
z_{sstt} = z_{sst}\left(\frac{z_{st}}{z_s} + \frac{z_{tt}}{z_t}\right) + z_{stt}\left(\frac{z_{st}}{z_t} + \frac{z_{ss}}{z_s}\right) - \\
z_{st}\left(\frac{z_{st}z_{ss}}{z_s^2} + \frac{z_{st}z_{tt}}{z_t^2} + \frac{z_{ss}z_{tt}}{z_s z_t}\right) + \\
\frac{1}{s-t}\left[\frac{s}{t}\left(z_{sst} - \frac{z_{st}z_{ss}}{z_s} - \frac{1}{2}\frac{z_{st}^2}{z_t}\right) - \right. \\
\left. \frac{t}{s}\left(z_{stt} - \frac{z_{st}z_{tt}}{z_t} - \frac{1}{2}\frac{z_{st}^2}{z_s}\right)\right] - \\
\frac{1}{(s-t)^2}\left[\nu^2\frac{s^2}{t^2}\frac{z_s}{z_t}\left(z_{st} - \frac{z_s z_{tt}}{z_t}\right) + \right. \\
\left. \mu^2\frac{t^2}{s^2}\frac{z_t}{z_s}\left(z_{st} - \frac{z_t z_{ss}}{z_s}\right)\right] - \\
\frac{1}{2}\frac{1}{(s-t)^3}\left[\nu^2\frac{s}{t}z_s\left(1 + \frac{(4t-3s)s}{t^2}\frac{z_s}{z_t}\right) - \right. \\
\left. \mu^2\frac{t}{s}z_t\left(1 + \frac{(4s-3t)t}{s^2}\frac{z_t}{z_s}\right)\right].
\end{gathered}$$

5. Garnier's Hamiltonian system

It is clear from Garnier's text that the system should be considered to constitute a coupled system of higher order ODEs, namely in any one of the critical points t_i. Thus, choosing a specific t_i, t_1 say, we have a system of n coupled second order ODEs which possesses the Painlevé property. Garnier shows that the system (F_n) is completely integrable in the Pfaffian sense. We establish various aspects of that system, including its Lagrangian structure which can be interpreted as a minimal action principle on the moduli space of hyperelliptic curves, and its connection to Picard-(L.)Fuchs equations of a class of hyperelliptic curves, leading to a reformulation of the Garnier equations in hyperelliptic form using a class of Abelian functions introduced by Klein and extended in [BEL], where characterizing differential equations are derived. We would like to further the analogy of the interrelation between the two problems, isomonodromy and variation of incomplete period integrals, and actually find a hyperelliptic curve of genus n to reexpress the equation in terms of n-variable abelian functions, and generalized hypergeometric equations to provide special solutions.

Note: In recent studies of the Garnier systems, motivated by isomonodromy problems connected to complete integrability, as well as modular arithmetic (cf., e.g., [IKSY]), the emphasis was placed on regarding it as an overdetermined system of PDEs rather than as a system of coupled higher-order ODEs.

Our first observation is that the system of Garnier ODEs for any chosen independent variable t_i (fixing i) has a natural Lagrange structure:

For any chosen label i the associated system F_n of ODEs for the functions $\lambda_j = \lambda_j(t_i)$, $j = 1, \ldots, n$ (i fixed) can be derived as the Lagrange equations from the following Lagrangian:

$$
\begin{gathered}
\mathcal{L}_i = \frac{1}{2} \sum_{j=1}^{n} (\lambda_j - t_i) \frac{\varphi'(t_i)\psi'\lambda_j)}{\varphi(\lambda_j)\psi(t_i)} \left(\frac{\partial \lambda_j}{\partial t_i} \right)^2 + \\
2 \Biggl[\left(\sum_{l=1}^{n+3} \left(c_l + \frac{3}{4} \right) - 2 \right) \frac{\psi(t_i)}{\varphi'(t_i)} - \\
\sum_{\substack{l=1 \\ l \neq i}}^{n} \left(c_l + \frac{1}{4} \right) \frac{\varphi'(t_l)\psi(t_i)}{\psi(t_l)\varphi'(t_i)(\lambda_j - t_i)} - \frac{c_i}{\lambda_j - t_i} \Biggr],
\end{gathered}
$$

where $\mathcal{L}_i = \mathcal{L}_i(\lambda_1, \ldots, \lambda_n; \dot{\lambda}_1, \ldots, \dot{\lambda}_n; t_i)$, with $\dot{\lambda}_j = \partial \lambda_j / \partial t_i$.

In the case $n=1$, setting $\lambda_1=\lambda$, $t_1=t$, we have the Lagrangian

$$\mathcal{L}(\lambda,\dot{\lambda},t)=\frac{1}{2}\frac{t(t-1)}{\lambda(\lambda-1)(t-\lambda)}\dot{\lambda}^2+$$

$$2\left[\alpha\frac{t-\lambda}{t(t-1)}+\frac{\beta}{(t-1)\lambda}+\frac{\gamma}{t(\lambda-1)}+\delta\lambda-t\right],$$

which is a Lagrangian for the PVI equation. An alternative Lagrangian for PVI, which however seems more complicated, has been given in [S].

The PDE aspect of the Garnier system, through the mixed equations, tell us that the Lagrangian different choices of the label i are all compatible in the following sense:

The Lagrangians obey the following closure relation

$$\frac{\partial \mathcal{L}_i}{\partial t_j} = \frac{\partial \mathcal{L}_j}{\partial t_i}, \quad \forall i, j = 1, \ldots, n.$$

As a consequence, the differential 1-form $\sum_i \mathcal{L}_i dt_i$ is closed, and this leads to the introduction of the following action integral:

$$\mathcal{S}[\lambda(\mathbf{t})] \equiv \int_\Gamma \sum_{i=1}^{n} \mathcal{L}_i dt_i,$$

(in which $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mathbf{t} = (t_1, \ldots, t_n)$), which can be interpreted as an action on the moduli space of hyperelliptic curves of genus n of the form

$$y^2 = \varphi(x)\psi(x).$$

Garnier proceeds to give an explicit integration in terms of genus-n hyperelliptic functions, in the special case when the λ_i are actually assumed to be algebraic functions [G2]; this Algebraically Completely Integrable Hamiltonian system is now called the "Garnier system". For $n = 1$ the system reduces to a single ODE which happens to be the Painlevé VI equation. For $n = 2$ Garnier gives in his paper [G1] the explicit form:

(a) $$\frac{\partial^2 w}{\partial t^2} =$$

$$\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}+\frac{1}{w-s}-\frac{1}{w-z}\right)\left(\frac{\partial w}{\partial t}\right)^2-$$

$$\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{t-s}-\frac{1}{t-w}-\frac{1}{t-z}\right)\frac{\partial w}{\partial t}+$$

$$\frac{1}{2}\frac{w(w-1)(w-s)(z-t)}{z(z-1)(z-s)(w-t)(z-w)}\left(\frac{\partial z}{\partial t}\right)^2-$$

$$\frac{w-t}{(z-t)(z-w)}\left(\frac{\partial w}{\partial t}\right)\left(\frac{\partial z}{\partial t}\right)+$$

$$\frac{2w(w-1)(w-t)(w-s)(z-t)^2}{t^2(t-1)^2(t-s)^2(w-z)}\times$$

$$\times\Big[\alpha+\beta+\gamma+\delta+\kappa+\frac{7}{4}-\frac{ts}{z}\frac{\alpha+\frac{1}{4}}{w^2}+$$

$$\frac{(t-1)(s-1)}{(z-1)}\frac{\beta+\frac{1}{4}}{(w-1)^2}+$$

$$\frac{t(t-1)(t-s)}{(z-t)}\frac{\gamma}{(w-t)^2}+$$

$$\frac{s(s-1)(s-t)}{(z-s)}\frac{\delta}{(w-s)^2}\Big]$$

together with coupled first order PDE's

$$\text{(b)} \qquad \frac{t(t-1)}{t-z}\frac{\partial w}{\partial t}+\frac{s(s-1)}{s-z}\frac{\partial w}{\partial s}=\frac{w(w-1)}{w-z}$$

$$\text{(c)} \qquad \frac{t(t-1)}{t-w}\frac{\partial z}{\partial t}+\frac{s(s-1)}{s-w}\frac{\partial z}{\partial s}=\frac{z(z-1)}{z-w}.$$

It should be pointed out that the system consisting of (a), (b), (c), amounts actually to a fourth order ODE in terms of $w = w(t)$ only, and as such can be considered to be the first higher order member of the Painlevé VI hierarchy.

6. Picard-Fuchs equations in integrable systems

We use the Kleinian sigma function, which replaces the Riemann theta function and in the case of hyperelliptic curves was used very explicitly by H.F. Baker to derive the algebraic relations of the function field of the Jacobi variety, analogous to the cubic equation for genus 1. The Kleinian σ function can be constructed by analogy with the $g = 1$ case for any (nonsingular) hyperelliptic curve

$$\mu^2 = 4\prod_{n=1}^{2g+1}(\lambda - e_i) = 4\lambda^{2g+1} - \sum_{i=0}^{2g} a_i\lambda^i.$$

As such, we only recall that σ depends on a g-dimensional complex vector $\vec{x}$, that the ζ function and $\wp$ function are defined as: $\zeta_i(\vec{x}) = \frac{\partial \log \sigma(\vec{x})}{\partial x_i}, \wp_{ij}(\vec{x}) = \frac{\partial^2 \log \sigma(\vec{x})}{\partial x_i \partial x_j}$, cf. [BEL]. Using σ, Enolskii and Richter [ER] compute the analog of the Legendre operator for the periods of a genus 2 curve; on the other hand, Dullin *et al.* [DRVW] express the Picard-Fuchs equations as a fourth-order differential equation; indeed, the corresponding 4 linear system will act on a basis of (four) differentials of the second kind integrated around the four standard generators of the first homology. As the authors point out, the equation results from differentiation along a given branchpoint (an 'autonomous' one-time limit) and the expression is simple when a basis of $2g$ differentials of the second kind is chosen suitably. Specifically for genus $g=2$:

$$2\pi J := \oint \frac{(\lambda-e_1)\cdots(\lambda-e_g)}{\mu}d\lambda$$

for the curve above,taking derivatives in the e_{g+1} variable, yields a fourth-order equation for J whose coefficients are polynomial in $e_1, \ldots e_{2g+1}$, [DRVW]:

$$4(e_3-e_1)(e_4-e_2)(e_3-e_4)J''''+(4e_3^2(7e_3-6e_4)-4e_3(5e_3-4e_4)A+$$

$$+(9e_3(4e_3-3e_4)-3(5e_3-3e_4)A+3B)J''+3(e_3-e_4)J'+\frac{3}{4}J=0,$$

$A,\ B,\ C$ the elementary symmetric functions of $e_1,\ e_2,\ e_3$ (of degrees 1, 2, 3 respectively).

What remains to be done is the calculation of the inhomogeneous Picard-Fuchs operators, in terms of the σ function, to obtain the right-hand side of Garnier's equation:

$$\frac{2w(w-1)(w-t))(w-s)(z-t)^2}{t^2(t-1)^2(t-u)^2(w-s)}$$

(times the partial-fraction expansion analogous to Fuchs' in genus 1). It seems plausible that in this ODE model, the autonomous limit of the Garnier system will lead to the hyperelliptic curve of genus n with branchpoints the roots of:

$$\lambda_1(\lambda_1-1)(\lambda_1-t_1)\cdots(\lambda_1-t_n)(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_n).$$

In addition, when we derive the analog of the Lagrange operator for the autonomous limit, in which Garnier obtains the Algebraically Completely Integrable Hamiltonian system known nowadays by “the Garnier system”, and shows that it can be integrated by quadratures involving (a second set of) hyperelliptic integrals of genus n:

$$\sum_{j=1}^{n}\int_{\lambda_j^0}^{\lambda_j}\frac{(\lambda_j-a_1)\cdots(\lambda_j-a_n)d\lambda_j}{(\lambda_j-a_i)\sqrt{(\lambda_j-a_1)\cdots(\lambda_j-a_n)P(x)}}$$

($i=1,\ldots n$, $P(x)$ an arbitrary polynomial of degree $n-1$) [G2], we can express the Picard-Fuchs equations in terms of the (algebraic) Hamiltonians, just as [DRVW] does for the Neumann system.

7. Generalized hypergeometric functions

If $\alpha = \beta = \gamma = \delta = 0$ the general solution of PVI can be expressed as

$$w(t; 0, 0, 0, 0) = \Lambda(C_1\omega_1 + C_2\omega_2 + y(t); t) \ ,$$

in which $y(t)$ is an arbitrary solution of the linear equation $\mathcal{D}_t y = 0$.

Classically solutions of PVI were generated from the Gauss hypergeometric functions for special parameter values. Similarly, Garnier points out that in special cases the symmetric functions of the λ's are a solution of Appell's hypergeometric equation in two variables.

Consider the Appell operators $\mathcal{L}(s, t)$ and $\mathcal{L}(t, s)$ [K,Y]:

$$\mathcal{L}(s, t) : s(s-1)\frac{\partial^2}{\partial s^2} + s(1-t)\frac{\partial^2}{\partial s \partial t} +$$

$$(1-2s)\frac{\partial}{\partial s} - \frac{t}{2}\frac{\partial}{\partial t} - \frac{1}{4}$$

They annihilate the periods [B,5.8.2–5.9]:

$$\oint_\gamma \frac{dz}{\sqrt{z(1-z)(1-sz)(1-st)}} =$$

$$2\int_0^1 \frac{dz}{\sqrt{z(1-z)(1-sz)(1-st)}} \equiv$$
$$F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1;s,t\right),$$

Appell's hypergeometric function.

Following the case $n = 1$ to transform the equations, and using [DRVW,ER] to derive higher-genus inhomogeneous Picard-Fuchs equations, we have formulas of this kind:

$$\mathcal{L}(s,t)\ \int_0^{\lambda(s,t)} \frac{dz}{\sqrt{z(1-z)(1-sz)(1-st)}} =$$

terms of Fuchs type:

$$\lambda_{ss}(s,t) + \frac{t}{s}\lambda_{st}(s,t) + \left[\frac{1}{\lambda} + \frac{1}{\lambda - s} + \ldots\right]\lambda_s(s,t)^2 + \ldots$$

$+$ inhomogeneous terms

$$a\frac{\sqrt{z(1-z)(1-sz)(1-tz)}}{(z-s)^2}+$$

$$b\frac{\sqrt{z(1-z)(1-sz)(1-tz)}}{(z-s)}$$

8. Lagrange operators on chains with boundaries

Specialize the curve to the form

$$y^2 = 4x(x-1)(x-s)(x-t)(x-\mu(s,t)) \equiv R(x) \quad (*)$$

Polynomials in $x, \mathcal{U}_j(x)$, $\mathcal{R}_j(x)$ defining canonical differentials are given as

$$\begin{aligned}
\mathcal{U}_1(x) =& 1,\ \mathcal{U}_2(x) = x^2 \\
\mathcal{R}_1(x) =& 12x^3 - 8(1+s+t+\mu(s,t))x^2 \\
& \quad + 4(s+t+\mu(s,t)+st+\mu(s,t)(t+s))x, \\
\mathcal{R}_2(x) =& 4x^2.
\end{aligned}$$

We shall also consider

$$w^2 = 4z(z-1)(z-s)(z-t)(z-\lambda(s,t)) = \widetilde{R}(z) \quad (**)$$

Introduce the 4-vector defined on the curve (*)

$$\mathcal{K}(s,t) = \left(\int_\infty^{\lambda(s,t)} d\mathbf{u}, \int_\infty^{\lambda(s,t)} d\mathbf{r} \right)^T.$$

Theorem. The vector $\mathcal{K}(s,t)$ defined on the curve $y^2 = 4x(x-1)(x-s)(x-t)(x-\mu(s,t))$ satisfies the partial differential equations

$$\begin{aligned}
\frac{\partial}{\partial s}\mathcal{K} &= \mathcal{M}(s,t)\mathcal{K} + \mathcal{P}(s,t), \\
\frac{\partial}{\partial t}\mathcal{K} &= \mathcal{N}(s,t)\mathcal{K} + \mathcal{Q}(s,t),
\end{aligned}$$

where vectors $\mathcal{P}(s,t) = (\mathcal{P}_1(s,t),\ldots,\mathcal{P}_4(s,t))^T$ and $\mathcal{Q}(s,t) = (\mathcal{Q}_1(s,t),\ldots,\mathcal{Q}_4(s,t))^T$ are given as

$$\begin{aligned}\mathcal{P}(s,t) =& y(\lambda(s,t))\left[\frac{1}{\lambda(s,t)-s}\mathcal{V}(s)\right.\\ &\left.+\frac{1}{\lambda(s,t)-\mu(s,t)}\frac{\partial\mu(s,t)}{\partial s}\mathcal{V}(\mu(s,t))\right]\\ \mathcal{Q}(s,t) =& y(\lambda(s,t))\left[\frac{1}{\lambda(s,t)-t}\mathcal{V}(t)\right.\\ &\left.+\frac{1}{\lambda(s,t)-\mu(s,t)}\frac{\partial\mu(s,t)}{\partial t}\mathcal{V}(\mu(s,t))\right],\end{aligned}$$

and

$$\mathcal{V}(s) = \begin{pmatrix}\mathcal{U}_1(s)\\ \mathcal{U}_2(s)\\ \mathcal{R}_1(s)\\ \mathcal{R}_2(s)\end{pmatrix},\ \mathcal{V}(t) = \begin{pmatrix}\mathcal{U}_1(t)\\ \mathcal{U}_2(t)\\ \mathcal{R}_1(t)\\ \mathcal{R}_2(t)\end{pmatrix}$$

The matrices $\mathcal{M}(s,t)$ and $\mathcal{N}(s,t)$ depend on both variables, (s,t) and are built in terms of blocks α, β, γ:

$$\mathcal{M}(s,t) = \begin{pmatrix} \alpha(s) & \beta(s) \\ \gamma(s) & -\alpha(s)^T \end{pmatrix}$$
$$\frac{\partial \mu(s,t)}{\partial s} \begin{pmatrix} \alpha(\mu(s,t)) & \beta(\mu(s,t)) \\ \gamma(\mu(s,t)) & -\alpha(\mu(s,t))^T \end{pmatrix},$$
$$\mathcal{N}(s,t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & -\alpha(t)^T \end{pmatrix}$$
$$+ \frac{\partial \mu(s,t)}{\partial t} \begin{pmatrix} \alpha(\mu(s,t)) & \beta(\mu(s,t)) \\ \gamma(\mu(s,t)) & -\alpha(\mu(s,t))^T \end{pmatrix}.$$

We also introduce a 4-vector defined on the curve (**)

$$\widetilde{\mathcal{V}}(s,t) = \left(\int_{\infty}^{\mu(s,t)} d\mathbf{u}, \int_{\infty}^{\mu(s,t)} d\mathbf{r} \right)^T$$

and prove analogous relations for this vector.

By iteration one can obtain a PDE up to order four.

$$\begin{aligned}
\frac{\partial^4}{\partial s^4}\mathcal{K} &= (\mathcal{M}_{sss} + 3\mathcal{M}_s^2 + 4\mathcal{M}\mathcal{M}_{ss} + 6\mathcal{M}^2\mathcal{M}_s + \mathcal{M}^4)\mathcal{K} \\
&\quad + \frac{\partial^3}{\partial s^3}\mathcal{P}, \\
\frac{\partial^3}{\partial s^3}\mathcal{K} &= (\mathcal{M}_{ss} + 3\mathcal{M}\mathcal{M}_s + \mathcal{M}^3)\mathcal{K} + \frac{\partial^2}{\partial s^2}\mathcal{P}, \\
\frac{\partial^2}{\partial s^2}\mathcal{K} &= (\mathcal{M}_s + \mathcal{M}^2)\mathcal{K} + \frac{\partial}{\partial s}\mathcal{P}, \\
\frac{\partial}{\partial s}\mathcal{K} &= \mathcal{M}\mathcal{K} + \mathcal{P},
\end{aligned}$$

where subscripts mean partial derivatives. One can deduce

$$W \begin{pmatrix} \int_\infty^{\lambda(s,t)} du_2, \\ \int_\infty^{\lambda(s,t)} dr_1, \\ \int_\infty^{\lambda(s,t)} dr_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial^3}{\partial s^3} \int_\infty^{\lambda(s,t)} du_1 - \frac{\partial^2}{\partial s^2}\mathcal{P}_1 \\ \frac{\partial^2}{\partial s^2} \int_\infty^{\lambda(s,t)} du_1 - \frac{\partial}{\partial s}\mathcal{P}_1 \\ \frac{\partial}{\partial s} \int_\infty^{\lambda(s,t)} du_1 - \mathcal{P}_1 \end{pmatrix}$$

where the 3 × 3 matrix W is given as product of 3 × 9 and 3 × 3 matrices

$$W = \begin{pmatrix} \mathcal{M}_{ss} + 3\mathcal{M}\mathcal{M}_s + \mathcal{M}^3 \\ \mathcal{M}_s + \mathcal{M}^2 \\ \mathcal{M} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The first component of the vector equation represents a fourth order differential equation with respect to the variable s of

$$\mathcal{X}_1 = \int_{\infty}^{\lambda(s,t)} \frac{dx}{\sqrt{x(x-1)(x-s)(x-t)(x-\mu(s,t))}}.$$

In the same way one can obtain equations for other integrals.

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