Extensions by Antiderivatives and Iterated Logarithms

V. Ravi Srinivasan

December 3, 2008
Differential Field: A field $F$ with a map $' : F \to F$ satisfying

- $(a + b)' = a' + b'$
- $(ab)' = a'b + ab'$

is called a *differential field* and the map $'$ is called a *derivation* on $F$. 
**Differential Field:** A field $F$ with a map $': F \rightarrow F$ satisfying

- $(a + b)' = a' + b'$
- $(ab)' = a'b + ab'$

is called a *differential field* and the map $'$ is called a *derivation* on $F$.

A *differential field extension $E$ of $F$* is a differential field such that $E \supset F$ and the restriction of the derivation of $E$ to $F$ coincides with the derivation of $F$. 
**Differential Field:** A field $F$ with a map $': F \to F$ satisfying

- $(a + b)' = a' + b'$
- $(ab)' = a'b + ab'$

is called a *differential field* and the map $'$ is called a *derivation* on $F$.

A *differential field extension* $E$ of $F$ is a differential field such that $E \supset F$ and the restriction of the derivation of $E$ to $F$ coincides with the derivation of $F$.

**Field of Constants:** Let $(F, ')$ be a differential field. The differential field $C := \{ c \in F | c' = 0 \}$ is called the field of constants.
**Differential Field:** A field $F$ with a map $': F \to F$ satisfying

- $(a + b)' = a' + b'$
- $(ab)' = a'b + ab'$

is called a *differential field* and the map $'$ is called a *derivation* on $F$.

A *differential field extension* $E$ of $F$ is a differential field such that $E \supset F$ and the restriction of the derivation of $E$ to $F$ coincides with the derivation of $F$.

**Field of Constants:** Let $(F, ')$ be a differential field. The differential field $C := \{c \in F | c' = 0\}$ is called the field of constants.

All fields considered in this talk are of characteristic zero.
NNC Extensions: A differential field extension $E \supset F$ is a *No New Constant* (NNC) extension if the constants of $E$ are the same as the constants of $F$. 
**NNC Extensions:** A differential field extension $E \supset F$ is a *No New Constant* (NNC) extension if the constants of $E$ are the same as the constants of $F$.

**Antiderivatives:** Let $E \supset F$ be a NNC extension. An element $u \in E$ is an antiderivative if $u' \in F$. A differential field extension $E \supset F$ is an extension by antiderivatives of $F$ if for $i = 1, 2, \cdots, n$ there exists $u_i \in E$ such that $u_i' \in F$ and $E = F(u_1, u_2, \cdots, u_n)$. 
**NNC Extensions:** A differential field extension $E \supset F$ is a *No New Constant* (NNC) extension if the constants of $E$ are the same as the constants of $F$.

**Antiderivatives:** Let $E \supset F$ be a NNC extension. An element $u \in E$ is an *antiderivative* if $u' \in F$. A differential field extension $E \supset F$ is an *extension by antiderivatives* of $F$ if for $i = 1, 2, \cdots, n$ there exists $u_i \in E$ such that $u_i' \in F$ and $E = F(u_1, u_2, \cdots, u_n)$.

**Exponentials of Integrals:** Let $E \supset F$ be a NNC extension. If $e \in E$ and $\frac{e'}{e} \in F$ then we call $e$ an *exponential of an integral* of an element (namely, $\frac{e'}{e}$) of $F$, and if $E = F(e_1 \cdots, e_m)$ for some exponentials of integrals $e_1 \cdots, e_m \in E$ of $F$ then we will call $E$ an *extension of $F$ by exponentials of integrals*. 
Kolchin Ostrowski Theorem

Theorem

Let $E \supset F$ be a NNC differential field extension and let $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m \in E$ be such that $\xi'_i \in F$ and $\frac{\eta'_j}{\eta_j} \in F$. If $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m$ are algebraically dependent over $F$ then there exist $(c_1, \cdots, c_n) \in \mathbb{C}^n \setminus \{0\}$ such that $\sum_{i=1}^n c_i \xi_i \in F$ or there exists $(r_1, \cdots, r_m) \in \mathbb{Z}^m \setminus \{0\}$ such that $\prod_{j=1}^m \eta_j^{r_j} \in F$. 
Theorem

Let $E \supset F$ be a NNC differential field extension and let $\xi_1, \cdots, \xi_n, e_1, \cdots, e_m \in E$ be such that $\xi'_i \in F$ and $\frac{e'_j}{e_j} \in F$. If $\xi_1, \cdots, \xi_n, e_1, \cdots, e_m$ are algebraically dependent over $F$ then there exist $(c_1, \cdots, c_n) \in \mathbb{C}^n \setminus \{0\}$ such that $\sum_{i=1}^{n} c_i \xi_i \in F$ or there exists $(r_1, \cdots, r_m) \in \mathbb{Z}^n \setminus \{0\}$ such that $\prod_{j=1}^{m} e_j^{r_j} \in F$.

Application

Theorem

[VRS] Let \( E \subseteq F \) be a NNC extension and let \( E = F(\xi_1, \ldots, \xi_n, \epsilon_1 \cdot \ldots \cdot, \epsilon_m) \), where \( \xi \in E, \xi'_i \in F, \frac{\epsilon'_i}{\epsilon_i} \in F \) and \( \xi_1, \ldots, \xi_n, \epsilon_1 \cdot \ldots \cdot, \epsilon_m \) are algebraically independent over \( F \). Let \( u \in E \) and suppose that \( u = \frac{P}{Q} \), where \( P, Q \in F[\xi_1, \ldots, \xi_n, \epsilon_1 \cdot \ldots \cdot, \epsilon_m] \) and \( (P, Q) = 1 \). Then there are finite sets \( U \subseteq \text{span}_F \{\xi_i| 1 \leq i \leq n\} \) and \( V \subseteq \{\prod_{i=1}^m \epsilon_i^{n_i}| 1 \leq i \leq m, n_i \in \mathbb{Z}\} \) such that

\[
F\langle u \rangle = F(U, V).
\]

Moreover these forms can be explicitly computed from the polynomials \( P \) and \( Q \).
Let $F$ be a differential field with an algebraically closed field of Constants $C$ and let $F_\infty$ be a complete Picard-Vessiot closure of $F$ (every homogeneous linear differential equation over $F_\infty$ has a full set of solutions in $F_\infty$ and it has $C$ as its field of constants and $F_\infty$ is minimal with respect to these properties). All the differential fields under consideration are subfields of $F_\infty$. 
Let $\mathbb{F}$ be a differential field with an algebraically closed field of Constants $\mathbb{C}$ and let $\mathbb{F}_\infty$ be a complete Picard-Vessiot closure of $\mathbb{F}$ (every homogeneous linear differential equation over $\mathbb{F}_\infty$ has a full set of solutions in $\mathbb{F}_\infty$ and it has $\mathbb{C}$ as its field of constants and $\mathbb{F}_\infty$ is minimal with respect to these properties). All the differential fields under consideration are subfields of $\mathbb{F}_\infty$.

**Antiderivative Tower:** A differential field extension $\mathbb{E}$ of $\mathbb{F}$ is called a *tower of extension by antiderivatives* if there are differential fields $\mathbb{E}_i$, $0 \leq i \leq n$ such that

$$\mathbb{E} := \mathbb{E}_n \supseteq \mathbb{E}_{n-1} \supseteq \cdots \supseteq \mathbb{E}_1 \supseteq \mathbb{E}_0 := \mathbb{F}$$

and $\mathbb{E}_i$ is an extension by antiderivatives of $\mathbb{E}_{i-1}$ for each $1 \leq i \leq n$. 
Theorem

[TVS] Let $F_\infty \supset M \supseteq F$ be differential fields and let

$$E := E_n \supset E_{n-1} \supset \cdots \supset E_1 \supset E_0 := F$$

be a tower of extensions by antiderivatives. Then $u \in E$ is algebraic over $M$ only if $u \in M$. 
Generating Algebraically Independent Antiderivatives

Theorem

Let $E \supseteq F$ be a NNC extension. If there is an $x \in E \setminus F$ such that $x' \in F$ then for any $n \in \mathbb{N}$ and distinct $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, the elements $y_{\alpha_i} \in F_\infty$ such that $y_{\alpha_i}' = 1 + x + \alpha_i$ are algebraically independent over $F(x)$. Moreover, the differential field $F(y_{\alpha_i}, x)$, where $y_{\alpha_i}' = 1 + x + \alpha_i$ and $\alpha_i \in \mathbb{C}$ is not imbeddable in any Picard-Vessiot extension of $F$. 

V. Ravi Srinivasan
Generating Algebraically Independent Antiderivatives

**Theorem**

[VRS] Let \( E \supseteq F \) be a NNC extension. If there is an \( \xi \in E \setminus F \) such that \( \xi' \in F \) then for any \( n \in \mathbb{N} \) and distinct \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), the elements \( \eta_{\alpha_i} \in F_{\infty} \) such that \( \eta_{\alpha_i}' = \frac{1}{\xi+\alpha_i} \) are algebraically independent over \( F(\xi) \). Moreover, the differential field \( F(\eta_\alpha, \xi) \), where \( \eta_\alpha' = \frac{1}{\xi+\alpha} \) and \( \alpha \in \mathbb{C} \) is not imbeddable in any Picard-Vessiot extension of \( F \).
Tower of Extensions by J-I-E Antiderivatives

Let $\eta_{11}, \cdots, \eta_{1n_1}$ be algebraically independent antiderivatives of $F$ and for $i = 1, 2, \cdots, k$, let $E_i := E_{i-1}(\eta_{i1}, \eta_{i2}, \cdots, \eta_{in_i})$, where $E_0 := F$ and for $i \geq 2$ the elements $\eta_{i1}, \eta_{i2}, \cdots, \eta_{in_i}$ are antiderivatives of $E_{i-1}$ subject to the following conditions:
Tower of Extensions by J-I-E Antiderivatives

Let $\eta_{11}, \cdots, \eta_{1n_1}$ be algebraically independent antiderivatives of $F$ and for $i = 1, 2, \cdots, k$, let $E_i := E_{i-1}(\eta_{i1}, \eta_{i2}, \cdots, \eta_{in_i})$, where $E_0 := F$ and for $i \geq 2$ the elements $\eta_{i1}, \eta_{i2}, \cdots, \eta_{in_i}$ are antiderivatives of $E_{i-1}$ subject to the following conditions:

C0. $\eta'_{ij} = \frac{A_{ij}}{C_{ij}B_{ij}}$, where for each $2 \leq i \leq k$ and for all $1 \leq j \leq n_i$, $A_{ij}, B_{ij}, C_{ij} \in E_{i-2}[\eta_{i-11}, \cdots, \eta_{i-1n_{i-1}}]$ are polynomials such that $(A_{ij}, B_{ij}) = (B_{ij}, C_{ij}) = (A_{ij}, C_{ij}) = 1$. 

C1. $C_{ij}$ is an irreducible polynomial for each $i, j$. For every $i$, $C_{it} \nmid C_{it}$ (that is, $C_{it}$ and $C_{it}$ are non associates) if $s \neq t$ and $C_{it} \nmid B_{it}$ for any $1 \leq s, t \leq n_i$.

C2. For every $1 \leq j \leq n_i$ there is an element $y_{C_{ij}} \in \{y_{i-11}, \cdots, y_{i-1n_{i-1}}\}$ such that the partial derivative $\frac{\partial C_{ij}}{\partial y_{C_{ij}}} \neq 0$ and $\frac{\partial A_{ij}}{\partial y_{C_{ij}}} = \frac{\partial B_{ij}}{\partial y_{C_{ij}}} = 0$. 

V. Ravi Srinivasan
Tower of Extensions by J-I-E Antiderivatives

Let $\eta_{11}, \cdots, \eta_{1n_1}$ be algebraically independent antiderivatives of $F$ and for $i = 1, 2, \cdots, k$, let $E_i := E_{i-1}(\eta_{i1}, \eta_{i2}, \cdots, \eta_{in_i})$, where $E_0 := F$ and for $i \geq 2$ the elements $\eta_{i1}, \eta_{i2}, \cdots, \eta_{in_i}$ are antiderivatives of $E_{i-1}$ subject to the following conditions:

C0. $\eta'_{ij} = \frac{A_{ij}}{C_{ij}B_{ij}}$, where for each $2 \leq i \leq k$ and for all $1 \leq j \leq n_i$, $A_{ij}, B_{ij}, C_{ij} \in E_{i-2}[\eta_{i-11}, \cdots, \eta_{i-1n_{i-1}}]$ are polynomials such that $(A_{ij}, B_{ij}) = (B_{ij}, C_{ij}) = (A_{ij}, C_{ij}) = 1$.

C1. $C_{ij}$ is an irreducible polynomial for each $i, j$. For every $i$, $C_{is} \nmid C_{it}$ (that is, $C_{is}$ and $C_{it}$ are non associates) if $s \neq t$ and $C_{is} \nmid B_{it}$ for any $1 \leq s, t \leq n_i$. 

V. Ravi Srinivasan
Tower of Extensions by J-I-E Antiderivatives

Let $\eta_{11}, \cdots, \eta_{1n_1}$ be algebraically independent antiderivatives of $F$ and for $i = 1, 2, \cdots, k$, let $E_i := E_{i-1}(\eta_{i1}, \eta_{i2}, \cdots, \eta_{in_i})$, where $E_0 := F$ and for $i \geq 2$ the elements $\eta_{i1}, \eta_{i2}, \cdots, \eta_{in_i}$ are antiderivatives of $E_{i-1}$ subject to the following conditions:

C0. $\eta'_{ij} = \frac{A_{ij}}{C_{ij}B_{ij}}$, where for each $2 \leq i \leq k$ and for all $1 \leq j \leq n_i$, $A_{ij}, B_{ij}, C_{ij} \in E_{i-2}[\eta_{i-11}, \cdots, \eta_{i-1n_{i-1}}]$ are polynomials such that $(A_{ij}, B_{ij}) = (B_{ij}, C_{ij}) = (A_{ij}, C_{ij}) = 1$.

C1. $C_{ij}$ is an irreducible polynomial for each $i, j$. For every $i$, $C_{is} \nmid C_{it}$ (that is, $C_{is}$ and $C_{it}$ are non associates) if $s \neq t$ and $C_{is} \nmid B_{it}$ for any $1 \leq s, t \leq n_i$.

C2. For every $1 \leq j \leq n_i$ there is an element $\eta_{C_{ij}} \in \{\eta_{i-11}, \cdots, \eta_{i-1n_{i-1}}\}$ such that the partial derivative $\frac{\partial \eta_{C_{ij}}}{\partial \eta_{C_{ij}}} \neq 0$ and $\frac{\partial A_{ij}}{\partial \eta_{C_{ij}}} = \frac{\partial B_{ij}}{\partial \eta_{C_{ij}}} = 0$. 
Definition

We call

\[ E := E_k \supset E_{k-1} \supset \cdots \supset E_2 \supset E_1 \supset E_0 := F \]

a tower of extensions by J-I-E antiderivatives. Note that \( E_1 \) is an ordinary antiderivative extension of \( F \).
Definition
We call
\[ E := E_k \supset E_{k-1} \supset \cdots \supset E_2 \supset E_1 \supset E_0 := F \]
a tower of extensions by J-I-E antiderivatives. Note that \( E_1 \) is an ordinary antiderivative extension of \( F \).

Let \( l_i := \{ \eta_{ij} | 1 \leq j \leq n_i \} \) and let \( \Lambda_t := \text{Span}_C \cup_{i=1}^t l_i \) for \( 1 \leq t \leq k \).
**Definition**

We call

\[ E := E_k \supset E_{k-1} \supset \cdots \supset E_2 \supset E_1 \supset E_0 := F \]

a tower of extensions by J-I-E antiderivatives. Note that \( E_1 \) is an ordinary antiderivative extension of \( F \).

Let \( l_i := \{ y_{ij} | 1 \leq j \leq n_i \} \) and let \( \Lambda_t := \text{Span}_C \bigcup_{i=1}^{t} l_i \) for \( 1 \leq t \leq k \).

**Theorem**

[VRS] Let \( E_k \supset K \supset F \) be an intermediate differential field. If \( \bigcup_{i=1}^{k} l_i \) is algebraically dependent over \( K \) then there is a nonzero \( s \in K \cap \Lambda_k \).
Theorem

[VRS] For every differential subfield \(K\) of \(E := E_k\), the field generated by \(F\) and \(S_k := K \cap \Lambda_k\) equals the differential field \(K\). That is

\[K = F(S_k)\].

Moreover \(K\) itself is a tower of extensions by antiderivatives, namely

\[K = F(S_k) \supset F(S_{k-1}) \supset F(S_{k-2}) \supset \cdots \supset F(S_1) \supset F,\]

where \(S_i := K \cap \Lambda_i\) and if \(t\) is the largest integer such that \(S_t \neq \emptyset\) then \(S_i \neq \emptyset\) for each \(i \leq t\).
Non-elementary J-I-E Antiderivatives

Non-elementary J-I-E Antiderivatives

If $a_i \in \mathbb{C}$ are distinct constants for $i = 1, \cdots, n$ then the elements

$$\eta_i := \int \frac{\ln(x)}{x-a_i}$$

are J-I-E antiderivatives of the differential field $\mathbb{C}(x, \ln(x))$ with $

\eta_i' := \frac{A_i}{C_iB_i}$

where $A_i := \ln(x)$, $B_i := 1$ and $C_i := x - a_i$.

---

Non-elementary J-I-E Antiderivatives

If $a_i \in \mathbb{C}$ are distinct constants for $i = 1, \cdots, n$ then the elements $\eta_i := \int \frac{\ln(x)}{x-a_i}$ are J-I-E antiderivatives of the differential field $\mathbb{C}(x, \ln(x))$ with $\eta_i' := \frac{A_i}{C_i B_i}$ where $A_i := \ln(x)$, $B_i := 1$ and $C_i := x - a_i$.

These $\eta_i$’s are non-elementary functions\(^1\).

---

Non-elementary J-I-E Antiderivatives

If \( a_i \in C \) are distinct constants for \( i = 1, \cdots, n \) then the elements
\[
\eta_i := \int \frac{\ln(x)}{x - a_i} \, dx
\]
are J-I-E antiderivatives of the differential field \( C(x, \ln(x)) \) with
\[
\eta'_i := \frac{A_i}{C_i B_i}
\]
where \( A_i := \ln(x) \), \( B_i := 1 \) and
\( C_i := x - a_i \).

These \( \eta_i \)'s are non-elementary functions\(^1\).

- The \( \eta_i \)'s are algebraically independent over \( C(x, \ln(x)) \)

---
Non-elementary J-I-E Antiderivatives

If \( a_i \in \mathbb{C} \) are distinct constants for \( i = 1, \cdots, n \) then the elements \( \eta_i := \int \frac{\ln(x)}{x-a_i} \) are J-I-E antiderivatives of the differential field \( \mathbb{C}(x, \ln(x)) \) with \( \eta'_i := \frac{A_i}{C_i B_i} \) where \( A_i := \ln(x) \), \( B_i := 1 \) and \( C_i := x - a_i \).

These \( \eta_i \)'s are non-elementary functions\(^1\).

- The \( \eta_i \)'s are algebraically independent over \( \mathbb{C}(x, \ln(x)) \)
- Any differential field \( K, \mathbb{C}(x, \ln(x), \eta_i | 1 \leq i \leq n) \supseteq K \supseteq \mathbb{C} \) is of the form \( \mathbb{C}(S) \), where \( S \subseteq \text{span}_\mathbb{C}\{x, \ln(x), \eta_i | 1 \leq i \leq n\} \) is a finite set.

---

Non-elementary J-I-E Antiderivatives

If $a_i \in \mathbb{C}$ are distinct constants for $i = 1, \cdots, n$ then the elements $\eta_i := \int \frac{\ln(x)}{x-a_i}$ are J-I-E antiderivatives of the differential field $\mathbb{C}(x, \ln(x))$ with $\eta'_i := \frac{A_i}{C_iB_i}$ where $A_i := \ln(x)$, $B_i := 1$ and $C_i := x - a_i$.

These $\eta_i$’s are non-elementary functions$^1$.

- The $\eta_i$’s are algebraically independent over $\mathbb{C}(x, \ln(x))$
- Any differential field $K$, $\mathbb{C}(x, \ln(x), \eta_i|1 \leq i \leq n) \supseteq K \supseteq \mathbb{C}$ is of the form $\mathbb{C}(S)$, where $S \subset \text{span}_{\mathbb{C}}\{x, \ln(x), \eta_i|1 \leq i \leq n\}$ is a finite set.
- Moreover $\mathbb{C}(S)$ itself is a tower of (Picard-Vessiot) extensions by antiderivatives.

---

Iterated Logarithms:
$C_\infty :=$ the complete Picard-Vessiot Closure of $C$. 
Iterated Logarithms:
$\mathcal{C}_\infty :=$ the complete Picard-Vessiot Closure of $\mathbb{C}$.

Extension by all iterated logarithms is an example of a normal sub-tower of $\mathcal{C}_\infty$. Consider the differential field $(\mathcal{C}(x),')$ with $x' = 1$. We denote by $\ln(x + c)$, a designated solution of

$$Y' = \frac{1}{x + c},$$

where $c \in \mathcal{C}$ and call $\ln(x + c)$ an iterated logarithm of level 1.
Iterated Logarithms:

$C_\infty :=$ the complete Picard-Vessiot Closure of $C$.

Extension by all iterated logarithms is an example of a normal sub-tower of $C_\infty$. Consider the differential field $(C(x),')$ with $x' = 1$. We denote by $\ln(x + c)$, a designated solution of

$$Y' = \frac{1}{x + c},$$

where $c \in C$ and call $\ln(x + c)$ an iterated logarithm of level 1.

For $(c_1, c_2) \in C^2$, we call the designated solution $\ln(\ln(x + c_1) + c_2)$ of the differential equation

$$Y' = \frac{1}{(x + c_1)(\ln(x + c_1) + c_2)},$$

an iterated logarithm of level 2.
Iteratively we define $\ln(\ln(\cdots \ln(\ln(x + c_1) + c_2) \cdots + c_{n-1}) + c_n)$ and call it an iterated logarithm of level $n$. 
\[ \Lambda_n := \text{the set of all } n - \text{th level iterated logarithms and } \Lambda_0 := \{x\}. \]
Λₙ := the set of all n−th level iterated logarithms and Λ₀ := {x}.

▶ Λₙ is a set of J-I-E antiderivatives of the differential field 

\[ C(\bigcup_{i=0}^{n-1} \Lambda_i). \]
\( \Lambda_n := \) the set of all \( n \) - th level iterated logarithms and \( \Lambda_0 := \{ x \} \).

- \( \Lambda_n \) is a set of J-I-E antiderivatives of the differential field \( \mathbb{C}(\bigcup_{i=0}^{n-1} \Lambda_i) \).
- The finitely differentially generated subfields of \( \mathbb{C}(\bigcup_{i=0}^{n} \Lambda_i) \) are of the form \( \mathbb{C}(S) \), where \( S \subset \text{span}_\mathbb{C} \bigcup_{i=0}^{n} \Lambda_i \) is a finite set.
\( \Lambda_n := \) the set of all \( n \) – \( th \) level iterated logarithms and \( \Lambda_0 := \{x\} \).

- \( \Lambda_n \) is a set of J-I-E antiderivatives of the differential field \( C(\bigcup_{i=0}^{n-1} \Lambda_i) \).
- The finitely differentially generated subfields of \( C(\bigcup_{i=0}^{n} \Lambda_i) \) are of the form \( C(S) \), where \( S \subset \text{span}_C \bigcup_{i=0}^{n} \Lambda_i \) is a finite set.
- \( C(\bigcup_{i=0}^{n} \Lambda_i) \) is a normal subfield of \( C_\infty \).
\( \Lambda_n := \text{the set of all } n - \text{th level iterated logarithms and } \Lambda_0 := \{x\}. \)

- \( \Lambda_n \) is a set of J-I-E antiderivatives of the differential field \( \mathbb{C}(\bigcup_{i=0}^{n-1}\Lambda_i) \).
- The finitely differentially generated subfields of \( \mathbb{C}(\bigcup_{i=0}^{n}\Lambda_i) \) are of the form \( \mathbb{C}(S) \), where \( S \subset \text{span}\mathbb{C} \bigcup_{i=0}^{n}\Lambda_i \) is a finite set.
- \( \mathbb{C}(\bigcup_{i=0}^{n}\Lambda_i) \) is a normal subfield of \( \mathbb{C}_\infty \).
- For \( u \in \mathbb{C}(x, \ln x, \ln(x + 1)) \setminus \mathbb{C}(x) \) with derivation \( \frac{d}{dx} \),
  \[
  \mathbb{C}\langle u \rangle = \mathbb{C}(x, S),
  \]
  where \( S \subset \text{span}\mathbb{C}\{\ln x, \ln(x + 1)\} \) is a finite nonempty subset.