

Extensions by Antiderivatives and Iterated Logarithms

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Differential Field: A field \mathbf{F} with a map $' : \mathbf{F} \rightarrow \mathbf{F}$ satisfying

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▶ $(ab)' = a'b + ab'$

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A *differential field extension* \mathbf{E} of \mathbf{F} is a differential field such that $\mathbf{E} \supset \mathbf{F}$ and the restriction of the derivation of \mathbf{E} to \mathbf{F} coincides with the derivation of \mathbf{F} .

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All fields considered in this talk are of characteristic zero.

NNC Extensions: A differential field extension $\mathbf{E} \supset \mathbf{F}$ is a *No New Constant* (NNC) extension if the constants of \mathbf{E} are the same as the constants of \mathbf{F} .

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Antiderivatives: Let $\mathbf{E} \supset \mathbf{F}$ be a NNC extension. An element $u \in \mathbf{E}$ is an *antiderivative* if $u' \in \mathbf{F}$. A differential field extension $\mathbf{E} \supset \mathbf{F}$ is an *extension by antiderivatives* of \mathbf{F} if for $i = 1, 2, \dots, n$ there exists $u_i \in \mathbf{E}$ such that $u_i' \in \mathbf{F}$ and $\mathbf{E} = \mathbf{F}(u_1, u_2, \dots, u_n)$.

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Exponentials of Integrals: Let $\mathbf{E} \supset \mathbf{F}$ be a NNC extension. If $\epsilon \in \mathbf{E}$ and $\frac{\epsilon'}{\epsilon} \in \mathbf{F}$ then we call ϵ an *exponential of an integral* of an element (namely, $\frac{\epsilon'}{\epsilon}$) of \mathbf{F} , and if $\mathbf{E} = \mathbf{F}(\epsilon_1 \cdots, \epsilon_m)$ for some exponentials of integrals $\epsilon_1 \cdots, \epsilon_m \in \mathbf{E}$ of \mathbf{F} then we will call \mathbf{E} an *extension of \mathbf{F} by exponentials of integrals*.

Kolchin Ostrowski Theorem

Theorem

Let $\mathbf{E} \supset \mathbf{F}$ be a NNC differential field extension and let $x_1, \dots, x_n, e_1, \dots, e_m \in \mathbf{E}$ be such that $x'_i \in \mathbf{F}$ and $\frac{e'_j}{e_j} \in \mathbf{F}$. If $x_1, \dots, x_n, e_1, \dots, e_m$ are algebraically dependent over \mathbf{F} then there exist $(c_1, \dots, c_n) \in \mathbf{C}^n \setminus \{0\}$ such that $\sum_{i=1}^n c_i x_i \in \mathbf{F}$ or there exists $(r_1, \dots, r_m) \in \mathbb{Z}^m \setminus \{0\}$ such that $\prod_{j=1}^m e_j^{r_j} \in \mathbf{F}$.

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Application

Theorem

[VRS] Let $\mathbf{E} \subset \mathbf{F}$ be a NNC extension and let $\mathbf{E} = \mathbf{F}(x_1, \dots, x_n, e_1, \dots, e_m)$, where $x \in \mathbf{E}$, $x'_i \in \mathbf{F}$, $\frac{e'_i}{e_i} \in \mathbf{F}$ and $x_1, \dots, x_n, e_1, \dots, e_m$ are algebraically independent over \mathbf{F} . Let $u \in \mathbf{E}$ and suppose that $u = \frac{P}{Q}$, where $P, Q \in \mathbf{F}[x_1, \dots, x_n, e_1, \dots, e_m]$ and $(P, Q) = 1$. Then there are finite sets $\mathcal{U} \subset \text{span}_{\mathbf{F}}\{x_i \mid 1 \leq i \leq n\}$ and $\mathcal{V} \subset \{\prod_{i=1}^m e_i^{n_i} \mid 1 \leq i \leq m, n_i \in \mathbb{Z}\}$ such that

$$\mathbf{F}\langle u \rangle = \mathbf{F}(\mathcal{U}, \mathcal{V}).$$

Moreover these forms can be explicitly computed from the polynomials P and Q .

Let \mathbf{F} be a differential field with an algebraically closed field of Constants \mathbf{C} and let \mathbf{F}_∞ be a complete Picard-Vessiot closure of \mathbf{F} (every homogeneous linear differential equation over \mathbf{F}_∞ has a full set of solutions in \mathbf{F}_∞ and it has \mathbf{C} as its field of constants and \mathbf{F}_∞ is minimal with respect to these properties). All the differential fields under consideration are subfields of \mathbf{F}_∞ .

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Antiderivative Tower: A differential field extension \mathbf{E} of \mathbf{F} is called a *tower of extension by antiderivatives* if there are differential fields \mathbf{E}_i , $0 \leq i \leq n$ such that

$$\mathbf{E} := \mathbf{E}_n \supseteq \mathbf{E}_{n-1} \supseteq \cdots \supseteq \mathbf{E}_1 \supseteq \mathbf{E}_0 := \mathbf{F}$$

and \mathbf{E}_i is an extension by antiderivatives of \mathbf{E}_{i-1} for each $1 \leq i \leq n$.

Theorem

[VRS] Let $\mathbf{F}_\infty \supset \mathbf{M} \supseteq \mathbf{F}$ be differential fields and let

$$\mathbf{E} := \mathbf{E}_n \supset \mathbf{E}_{n-1} \supset \cdots \supset \mathbf{E}_1 \supset \mathbf{E}_0 := \mathbf{F}$$

be a tower of extensions by antiderivatives. Then $u \in \mathbf{E}$ is algebraic over \mathbf{M} only if $u \in \mathbf{M}$.

Generating Algebraically Independent Antiderivatives

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Theorem

[VRS] Let $\mathbf{E} \supseteq \mathbf{F}$ be a NNC extension. If there is an $x \in \mathbf{E} \setminus \mathbf{F}$ such that $x' \in \mathbf{F}$ then for any $n \in \mathbb{N}$ and distinct $\alpha_1, \dots, \alpha_n \in \mathbf{C}$, the elements $\eta_{\alpha_i} \in \mathbf{F}_\infty$ such that $\eta'_{\alpha_i} = \frac{1}{x+\alpha_i}$ are algebraically independent over $\mathbf{F}(x)$. Moreover, the differential field $\mathbf{F}(\eta_\alpha, x)$, where $\eta'_\alpha = \frac{1}{x+\alpha}$ and $\alpha \in \mathbf{C}$ is not imbeddable in any Picard-Vessiot extension of \mathbf{F} .

Tower of Extensions by J-I-E Antiderivatives

Let $\eta_{11}, \dots, \eta_{1n_1}$ be algebraically independent antiderivatives of \mathbf{F} and for $i = 1, 2, \dots, k$, let $\mathbf{E}_i := \mathbf{E}_{i-1}(\eta_{i1}, \eta_{i2}, \dots, \eta_{in_i})$, where $\mathbf{E}_0 := \mathbf{F}$ and for $i \geq 2$ the elements $\eta_{i1}, \eta_{i2}, \dots, \eta_{in_i}$ are antiderivatives of \mathbf{E}_{i-1} subject to the following conditions:

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- C0. $\eta'_{ij} = \frac{A_{ij}}{C_{ij}B_{ij}}$, where for each $2 \leq i \leq k$ and for all $1 \leq j \leq n_i$, $A_{ij}, B_{ij}, C_{ij} \in \mathbf{E}_{i-2}[\eta_{i-11}, \dots, \eta_{i-1n_{i-1}}]$ are polynomials such that $(A_{ij}, B_{ij}) = (B_{ij}, C_{ij}) = (A_{ij}, C_{ij}) = 1$.

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- C1. C_{ij} is an irreducible polynomial for each i, j . For every i , $C_{is} \nmid C_{it}$ (that is, C_{is} and C_{it} are non associates) if $s \neq t$ and $C_{is} \nmid B_{it}$ for any $1 \leq s, t \leq n_i$.

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- C2. For every $1 \leq j \leq n_i$ there is an element $\eta_{C_{ij}} \in \{\eta_{i-11}, \dots, \eta_{i-1n_{i-1}}\}$ such that the partial derivative $\frac{\partial C_{ij}}{\partial \eta_{C_{ij}}} \neq 0$ and $\frac{\partial A_{ij}}{\partial \eta_{C_{ij}}} = \frac{\partial B_{ij}}{\partial \eta_{C_{ij}}} = 0$.

Definition

We call

$$\mathbf{E} := \mathbf{E}_k \supset \mathbf{E}_{k-1} \supset \cdots \supset \mathbf{E}_2 \supset \mathbf{E}_1 \supset \mathbf{E}_0 := \mathbf{F}$$

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Let $l_i := \{\eta_{ij} | 1 \leq j \leq n_i\}$ and let $\Lambda_t := \text{Span}_{\mathbf{C}} \cup_{i=1}^t l_i$ for $1 \leq t \leq k$.

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Let $I_j := \{\eta_{ij} \mid 1 \leq j \leq n_i\}$ and let $\Lambda_t := \text{Span}_{\mathbf{C}} \cup_{i=1}^t I_i$ for $1 \leq t \leq k$.

Theorem

[VRS] Let $\mathbf{E}_k \supset \mathbf{K} \supset \mathbf{F}$ be an intermediate differential field. If $\cup_{i=1}^k I_i$ is algebraically dependent over \mathbf{K} then there is a nonzero $\mathfrak{s} \in \mathbf{K} \cap \Lambda_k$.

Theorem

[VRS] For every differential subfield \mathbf{K} of $\mathbf{E} := \mathbf{E}_k$, the field generated by \mathbf{F} and $S_k := \mathbf{K} \cap \Lambda_k$ equals the differential field \mathbf{K} . That is

$$\mathbf{K} = \mathbf{F}(S_k).$$

Moreover \mathbf{K} itself is a tower of extensions by antiderivatives, namely

$$\mathbf{K} = \mathbf{F}(S_k) \supset \mathbf{F}(S_{k-1}) \supset \mathbf{F}(S_{k-2}) \supset \cdots \supset \mathbf{F}(S_1) \supset \mathbf{F},$$

where $S_i := \mathbf{K} \cap \Lambda_i$ and if t is the largest integer such that $S_t \neq \emptyset$ then $S_i \neq \emptyset$ for each $i \leq t$.

Non-elementary J-I-E Antiderivatives

¹Elena Anne Marchisotto, Gholam-Ali Zakeri, *An Invitation to Integration in Finite Terms*, Math.Assoc.Amer (4) 25 (Sep., 1994), 295-308.

Non-elementary J-I-E Antiderivatives

If $a_i \in \mathbf{C}$ are distinct constants for $i = 1, \dots, n$ then the elements $\eta_j := \int \frac{\ln(x)}{x-a_j}$ are J-I-E antiderivatives of the differential field $\mathbf{C}(x, \ln(x))$ with $\eta'_j := \frac{A_j}{C_j B_j}$ where $A_j := \ln(x)$, $B_j := 1$ and $C_j := x - a_j$.

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These η_i 's are non-elementary functions¹.

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These η_i 's are non-elementary functions¹.

- ▶ The η_i 's are algebraically independent over $\mathbf{C}(x, \ln(x))$

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Extension by all iterated logarithms is an example of a normal sub-tower of \mathbf{C}_∞ . Consider the differential field $(\mathbf{C}(x), ')$ with $x' = 1$. We denote by $\ln(x + c)$, a designated solution of

$$Y' = \frac{1}{x + c},$$

where $c \in \mathbf{C}$ and call $\ln(x + c)$ an iterated logarithm of level 1.

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where $c \in \mathbf{C}$ and call $\ln(x + c)$ an iterated logarithm of level 1.

For $(c_1, c_2) \in \mathbf{C}^2$, we call the designated solution $\ln(\ln(x + c_1) + c_2)$ of the differential equation

$$Y' = \frac{1}{(x + c_1)(\ln(x + c_1) + c_2)},$$

an iterated logarithm of level 2.

Iteratively we define $\ln(\ln(\cdots \ln(\ln(x + c_1) + c_2) \cdots + c_{n-1}) + c_n)$
and call it an iterated logarithm of level n .

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- ▶ The finitely differentially generated subfields of $\mathbf{C}(\cup_{i=0}^n \Lambda_i)$ are of the form $\mathbf{C}(S)$, where $S \subset \text{span}_{\mathbf{C}} \cup_{i=0}^n \Lambda_i$ is a finite set.

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- ▶ $\mathbf{C}(\cup_{i=0}^n \Lambda_i)$ is a normal subfield of \mathbf{C}_{∞} .
- ▶ For $u \in \mathbf{C}(x, \ln x, \ln(x+1)) \setminus \mathbf{C}(x)$ with derivation $\frac{d}{dx}$,

$$\mathbf{C}\langle u \rangle = \mathbf{C}(x, S),$$

where $S \subset \text{span}_{\mathbf{C}}\{\ln x, \ln(x+1)\}$ is a finite nonempty subset.