

# **$\mathrm{PGL}_n$ as a differential Galois group**

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## **A quick review:**

We consider Picard-Vessiot extensions  $M/K$  over a differential field  $K$  of characteristic 0, with algebraically closed field of constants  $\mathcal{C}$ .

The differential Galois group  $G(\mathcal{C})$  is considered as a subgroup of  $GL_n(\mathcal{C})$  for some  $n$ .

**The Kolchin Structure Theorem:**

$M$  is the function field over  $K$  of an irreducible  $G$ -torsor.

Moreover, if the  $n \times n$  matrix  $Y$  is a generic point for the torsor, then the derivation on  $M$  is given by

$$Y' = YA$$

for some  $n \times n$  matrix  $A$  over  $K$ .

**Result:** The  $G$ -torsors over  $K$  correspond bijectively to the elements of  $H^1(K, G)$ , i.e., equivalence classes of continuous *crossed homomorphisms*  $f: \text{Gal}(K) \rightarrow G(\bar{K})$ .

Here, a crossed homomorphism is a map

$$f: \text{Gal}(K) \rightarrow G(\bar{K})$$

satisfying

$$f_{\sigma\tau} = f_\sigma \sigma(f_\tau),$$

and two crossed homomorphisms  $f$  and  $g$  are equivalent if

$$f_\sigma = e g_\sigma \sigma(e^{-1})$$

for some  $e \in G(\bar{K})$ .

Given a crossed homomorphism  $f$ , there exists (by **Speiser's Theorem**) a matrix  $P$  in  $\mathrm{GL}_n(\bar{K})$  with

$$f_\sigma = P \sigma(P)^{-1}.$$

If  $X$  is a generic point for  $G$ , then  $Y = XP$  will be a generic point for the torsor corresponding to  $f$ .

Therefore, we want to determine  $H^1(K, G)$ .

If  $G$  is the automorphism group (functorially) of a finite-dimensional vector space  $V$ , then  $H^1(K, G)$  also classifies vector spaces over  $K$  that become isomorphic to  $V$  under scalar extension to  $\bar{K}$ .

**Examples:**  $H^1(K, O_n)$  classifies regular  $n$ -dimensional quadratic spaces.

$H^1(K, \mathrm{PGL}_n)$  classifies central simple algebras of degree  $n$ .

**New idea:** Given an inclusion  $H \subseteq G$ , we have an induced map

$$\iota: H^1(K, H) \rightarrow H^1(K, G).$$

If this map is onto for all  $K$ , we speak of a *surjectivity*.

Obviously, if we have a surjectivity, determining  $H^1(K, G)$  can be done by determining  $H^1(K, H)$ .

**Example:**  $\mu_2^n \subseteq O_n$  is a surjectivity, since all regular quadratic forms can be diagonalised.

**A better example:**  $\text{Spin}_7$  maps onto  $\text{SO}_7$ .  
 $\text{SO}_7$  contains the exceptional Lie group  $G_2$ .  
The pre-image of  $G_2$  in  $\text{Spin}_7$  splits:

$$\mu_2 \times G_2 \subseteq \text{Spin}_7.$$

This is a surjectivity.

Since  $G_2$  is the automorphism group for the octonions,  $H^1(K, G_2)$  classifies octonion algebras over  $K$ . They are given by three parameters  $a = i^2$ ,  $b = j^2$  and  $c = \ell^2$ . Thus,  $H^1(K, G_2)$  is easily described.

By extension, so is  $H^1(K, \text{Spin}_7)$ .

If we let  $A =$

$$\begin{pmatrix} \frac{a'}{2a} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ -\frac{a_{12}a}{b} & \frac{b'}{2b} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ -\frac{a_{13}}{b} & -\frac{a_{23}}{a} & \frac{a'b+ab'}{2ab} & -\frac{a_{16}a-a_{25}b}{ab} & \frac{a_{24}b-a_{17}}{b} & -\frac{a_{27}+a_{14}a}{a} & -a_{15} - a_{26} \\ -\frac{a_{14}a}{c} & -\frac{a_{24}b}{a} & -\frac{a_{16}a-a_{25}b}{2ab} & \frac{c'}{2c} & a_{45} & a_{46} & a_{47} \\ \frac{a_{15}}{c} & \frac{a_{25}b}{ac} & -\frac{a_{24}b-a_{17}}{c} & \frac{a_{45}}{a} & \frac{a'c+ac'}{2ac} & \frac{a_{12}a-a_{47}}{a} & a_{13} - a_{46} \\ \frac{a_{16}a}{bc} & \frac{a_{26}}{ac} & \frac{a_{27}+a_{14}a}{c} & \frac{a_{46}}{b} & -\frac{a_{12}a-a_{47}}{b} & \frac{b'c+bc'}{2bc} & a_{45} + a_{23} \\ -\frac{a_{17}}{bc} & -\frac{a_{27}}{ac} & \frac{a_{15}+aa_{26}}{c} & -\frac{a_{47}}{ab} & \frac{a_{13}-a_{46}}{b} & \frac{a_{45}+a_{23}}{a} & \frac{a'bc+ab'c+abc'}{2abc} \end{pmatrix}$$

we get an equation

$$Y' = YA$$

for  $G_2$ .

If  $H$  is a finite group, then  $H^1(K, H)$  classifies Galois algebras over  $K$  with group  $H$ .

Thus, if we have a surjectivity with finite  $H$ , the question of finding  $G$ -torsors has been reduced to a question in classical Galois theory.

**Theorem** [Chernousov, Gille & Reichstein]

Given  $G$ , a finite subgroup  $H$  inducing a surjectivity can be found as follows:

- (1) Replace  $G$  by a Levi subgroup  $G_{\text{red}}$ .
- (2) Let  $T$  be a maximal torus in  $G_{\text{red}}$ , and let  $N$  be its normaliser.
- (3) Let  $n = [N : T]$ , and let  $\mu$  be the  $n$ -torsion of  $T$ . Then there exists a finite subgroup  $H$  of  $N$ , such that

$$1 \rightarrow \mu \rightarrow H \rightarrow N/T \rightarrow 1$$

is exact.

This  $H$  will work.

**Example:** For  $\mathrm{PGL}_n$ , we get

$$H = A \rtimes S_n,$$

where  $A$  is the  $n!$ -torsion of the diagonal, and  $S_n$  is the group of permutation matrices.

A Galois algebra with group  $A \rtimes S_n$  is the splitting ring of a polynomial

$$x^{n n!} + a_{n-1} x^{(n-1) n!} + \dots + a_1 x^{n!} + a_0^{n!}.$$

In principle, we therefore have a description of  $\mathrm{PGL}_n$ -torsors.