

# Differential Algebra and Related Topics

## Workshop Abstracts

November 2–3, 2000, Rutgers University at Newark

## Differential Algebra and Symbolic Integration

Manuel Bronstein

**Abstract:** An *elementary function* of a variable  $x$  is a function that can be obtained from the rational functions in  $x$  by repeatedly adjoining a finite number of nested logarithms, exponentials, and algebraic numbers or functions. This tutorial describes how differential algebra provides an algorithmic solution to the *problem of integration in finite terms*: to decide in a finite number of steps whether a given elementary function has an elementary antiderivative, and to compute it explicitly if it exists.

## History and Basics of Differential Algebra

Phyllis Cassidy

**Abstract:** In this talk, we will discuss the evolution of differential algebraic geometry. Algebraic geometry has been the prime mover in the formulation of the Ritt-Kolchin study of algebraic differential equations. Ritt's theory of characteristic sets of prime differential ideals, inspired by the work of E. Noether and van der Waerden, enabled him to pin down such concepts as the general solution and singular solutions of a single differential equation, and the consistency and completeness of a system of differential equations, which had been only roughly sketched in the classical theory. Kolchin, influenced not only by his teacher but also by Weil and Chevalley, deepened and modernized Ritt's "differential algebra" by developing "differential algebraic geometry" and a theory of Lie-like groups called "differential algebraic groups." This "new geometry," a natural context for defining "Bäcklund transformations," and for clarifying the central notion of "dependence of the solution set of a system of differential equations on arbitrary functions and constants" (Kolchin's "dimension polynomial"), is the setting for recent work of Buium on algebraic curves, abelian varieties, and their moduli spaces, defined over differential fields (for example, function fields). This approach to Diophantine geometry through differential algebraic geometry, a phenomenon unanticipated by Ritt and Kolchin, goes beyond, and even reverses the traditional interplay between algebraic geometry and differential algebra.

## Applications of Differential Galois Theory in Hamiltonian Mechanics

Richard Churchill

**Abstract:** The existence of conservation laws independent of energy is a desirable property

for a classical Hamiltonian system. In this talk I outline recently developed methods for proving the non-existence of such laws by means of differential Galois theory.

## Differential Algebra and Schemes

Henri Gillet

**Abstract:** This will be an expository talk, in which I will describe how one can formulate and prove results in differential algebra from a scheme theoretic point of view. I.e. viewing various constructions in differential algebra as functors on appropriate categories of rings. This "Grothendieck style algebraic geometry" approach is in contrast to the "Weil style algebraic geometry" of the more classical texts in the subject. Hopefully I will also be able to say something about how one can use Artin approximation to prove a result of Buium.

## Baxter Algebra and Differential Algebra

Li Guo

**Abstract:** A Baxter algebra (of weight  $\lambda$ ) is a commutative algebra  $A$  with a linear operator  $P$  such that

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy), \quad x, y \in A.$$

When  $\lambda = 0$ , we get an algebra with an integral operator, an algebraic structure that is "dual" to a differential algebra.

One of the first major goals in the study of Baxter algebra is to understand free Baxter algebras. In the first part of the talk we will recall earlier work of Rota and Cartier on this subject and will explain recent work on explicit constructions of free Baxter algebras. We will consider some examples and show some variations of free Baxter algebras, and indicate connections of free Baxter algebras with shuffle products and Hurwitz series in differential algebra.

In the second part of the talk we use these explicit constructions to relate Baxter algebras and Hopf algebras. We then give an application of Baxter algebra and differential algebra to the umbral calculus in combinatorics.

## Differential Schemes

Jerald Kovacic

**Abstract:** The language of schemes, which has proven to be of value to algebraic geometry, has not yet been widely accepted into differential algebraic geometry. One reason may be that there are "challenges"; some hoped-for properties of the ring of global sections are absent.

We examine a class of differential rings (schemes, modules, sheaves) called AAD (Annihilators Are Differential). Any differential ring has an AAD-radical which gives an AAD quotient. We then show that the theory of AAD differential schemes can be developed with relative ease.

This is an introductory talk, focusing on concepts, rather than drilling down to the details.

## Inverse Differential Galois Theory

Andy R. Magid

**Abstract:** Let  $F$  be a differential field with field of constants  $C$ , which we assume to be algebraically closed and of characteristic 0. Let  $G$  be an algebraic group over  $C$ . The *Inverse Galois Problem* or *IGP* for  $G$  over  $F$  asks whether there is a differential Galois extension  $E \supset F$  whose differential Galois group  $G(E/F)$  is isomorphic to  $G$ . More generally, if  $E \supset F$  is a differential Galois extension and  $\phi : G \rightarrow G(E/F)$  is an epimorphism of algebraic groups over  $C$ , the *Lifting Problem* or *LP* for  $G$  and  $E$  over  $F$  asks whether there is a differential Galois extension  $K \supset F$  containing  $E$  with  $G(K/F)$  isomorphic to  $G$  such that the natural map  $G(K/F) \rightarrow G(E/F)$  becomes isomorphic to  $\phi$ . (*IGP* is *LP* for the case  $E = F$ .) The easiest contexts in which to understand the problems is when  $G$  is connected algebraic and, for the lifting problem, when the kernel of  $\phi$  is commutative unipotent or a torus, and this tutorial will focus on those.

## Applications of Differential Algebra to Symmetries and Vice Versa

Elizabeth Mansfield

**Abstract:** Sophus Lie's method for calculating Lie symmetries of differential equations is a perfect application for computer algebra. The method generates an overdetermined system of partial differential equations which is often, but not always, amenable to symbolic integrators. In this talk we show how a simple differential elimination algorithm can be used to help solve those systems that integrators struggle on, particularly if the differential equation being analysed contains parameters or arbitrary functions. In fact, many integrators now contain differential eliminations as part of their heuristics.

Overdetermined systems which are invariant under the action of a Lie group may suffer from expression swell when processed by differential elimination algorithms. In the second half of the talk I will show how the recent Moving Frames method of Fels and Olver can be used to rewrite the system in terms of a generating set of differential invariants and differential invariant operators. One can then perform invariantized differential elimination processes. There are many theoretical challenges in analysing such processes and we discuss these.

## Model Theory and Differential Algebra

Thomas Scanlon

**Abstract:** Differential algebra has played a central role in model theory over the past three decades. I will discuss the theory of differentially closed fields as developed within model theory concentrating on differential Galois theory, Zariski geometries, Manin kernels, and the fine structure of finite rank differential varieties. I will also discuss the role of Hardy fields and Pfaffian equations in the model-theoretic analysis of certain real-valued smooth functions (such as the exponential function and the Riemann zeta function).

## Galois Theory of Difference Equations

Michael F. Singer

**Abstract:** One of the aims of classical Galois theory of polynomial equations is to give criteria and algorithms that allow one to solve these equations in terms of radicals. A similar theory has been developed for linear differential equations and this allows one to decide if a linear differential equation can be solved in finite terms, that is in terms of exponentials, integrals and algebraic functions. Recent joint work with Marius van der Put (*Galois Theory of Difference Equations*, LNM 1666, Springer, 1997) developed the Galois theory of difference equations. This theory was applied, in joint work with Peter Hendriks (*Solving Difference Equations in Finite Terms*, Journal of Symbolic Computation, Vol. 27, No. 3, 1999, pp. 239-259), to develop criteria and an algorithm to solve difference equations in “finite terms”. The notion of finite terms for difference equations corresponds to solving in terms of indefinite sums, indefinite products, the usual operations of arithmetic and the operation of interlacing, that is, taking two sequences  $a_0, a_1, a_2, \dots$  and  $b_0, b_1, b_2, \dots$  and forming the sequence  $a_0, b_0, a_1, b_1, a_2, b_2, \dots$ .

A simple example is the equation

$$a_{n+2} - (n+3)a_{n+1} + (n+2)a_n = 0$$

all of whose solutions are linear combinations of the constant sequence  $\{u_n = 1\}$  and the sequence  $\{v_n = \sum_{k=1}^n k!\}$ .

A less obvious example is given by

$$a_{n+2} + \frac{(5n^2 + 8n + 4)}{(n+2)(n^5 + 2n^4 + n^3 - 1)}a_{n+1} - \frac{n^2(n^5 + 7n^4 + 19n^3 + 25n^2 + 16n + 3)}{(n+2)(n^5 + 2n^4 + n^3 - 1)}a_n = 0$$

Applying the algorithm, one finds that all solutions of this equation are linear combinations of two sequences  $\{u_n = \frac{1}{n}c_n + nc_{n+1}\}$  and  $\{v_n = \frac{1}{n}d_n + nd_{n+1}\}$  where  $c_n = \prod_{k=1}^{\frac{n}{2}-1} 2k$  if  $n$  is even and  $c_n = 0$  if  $n$  is odd and  $d_n = \prod_{k=1}^{\frac{n-1}{2}} (2k-1)$  if  $n$  is odd and  $d_n = 0$  if  $n$  is even.

This talk will present an elementary introduction to this Galois theory, motivate and give

a formal definition of solving a difference equation in finite terms and give an exposition of this algorithm.

## The Ritt-Kolchin Theory on Differential Polynomials

William Sit

**Abstract:** This will be an introductory talk that will cover the basic concepts in the Ritt-Kolchin theory on differential polynomials. A finite set of differential polynomials represents a system differential equations and the Ritt- Kolchin theory uses algebraic methods to study the properties of the solution set. A major computational problem is: given a finite system of algebraic differential equations, decompose it into its irreducible components. Ritt and Kolchin succeeded in solving

a major part of this problem, at least in principle, but the complete solution still eludes us today. I shall sketch the Ritt-Kolchin algorithm. Another key contribution of algebraic methods is the precise notion of (differential) dimension which measures the arbitrariness of the set of solutions. I shall give an exposition of the known results. If time permits, I shall also talk about the special case when the differential polynomials are linear.

## Differential Galois theory

Marius van der Put

**Abstract:** After introducing this subject and some elementary examples I will concentrate on the Galois theory of complex analytic differential equations. In particular J.-P. Ramis' work on the inverse problem of differential Galois theory will be explained. (See also *Recent work on differential Galois theory* - Séminaire Bourbaki, exp 849,, volume 97/98, Astérisque 252, 1998).