Pre-Lie algebras and Lie and Jordan analogues of Loday algebras (I)

Chengming Bai

Chern Institute of Mathematics, Nankai University

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Introduction to pre-Lie algebras
• What is a pre-Lie algebra?
• Examples of pre-Lie algebras
• Pre-Lie algebras and classical Yang-Baxter equation
• Pre-Lie algebras and vertex (operator) algebras
What is a pre-Lie algebra?

Definition

A pre-Lie algebra $A$ is a vector space with a binary operation $(x, y) \rightarrow xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \quad \forall x, y, z \in A. \quad (1)$$

- Other names:
  1. left-symmetric algebra
  2. Koszul-Vinberg algebra
  3. quasi-associative algebra
  4. right-symmetric algebra
  5. …
What is a pre-Lie algebra?

Proposition

Let $A$ be a pre-Lie algebra.

1. The commutator

$$[x, y] = xy - yx, \quad \forall x, y \in A$$

(2)

defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of $A$.

2. For any $x, y \in A$, let $L(x)$ denote the left multiplication operator. Then Eq. (1) is just

$$[L(x), L(y)] = L([x, y]), \quad \forall x, y \in A,$$

(3)

which means that $L : \mathfrak{g}(A) \rightarrow \text{gl}(\mathfrak{g}(A))$ with $x \rightarrow L(x)$ gives a representation of the Lie algebra $\mathfrak{g}(A)$.
What is a pre-Lie algebra?

- The “left-symmetry” of associators.
  That is, for any \( x, y, z \in A \), the associator

\[
(x, y, z) = (xy)z - x(yz)
\]

satisfies

\[
(x, y, z) = (y, x, z)
\]

which is equivalent to Eq. (1).
What is a pre-Lie algebra?

- A geometric interpretation of “left-symmetry”:
  Let $G$ be a Lie group with a left-invariant affine structure: there is a flat torsion-free left-invariant affine connection $\nabla$ on $G$, namely, for all left-invariant vector fields $X, Y, Z \in \mathfrak{g} = T(G)$,

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z = 0,
\]  
\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0.
\]

This means both the curvature $R(X, Y)$ and torsion $T(X, Y)$ are zero for the connection $\nabla$. If we define

\[
\nabla_X Y = XY,
\]

then the identity (1) for a pre-Lie algebra exactly amounts to Eq. (6).
What is a pre-Lie algebra?

- The origins and roles of pre-Lie algebras:
  - Cohomology and deformations of associative algebras: Gerstenhaber (1964), ···
  - Complex structures on Lie groups: Andrada and Salamon (2003), ···
  - Rooted trees: Cayley (1896), ···
  - Combinatorics: K. Ebrahimi-Fard (2002), ···
  - Operads: Chapoton and Livernet (2001), ···
  - Vertex algebras: Bakalov and Kac (2002), ···
What is a pre-Lie algebra?

- Phase spaces: Kuperschmidt (1994), ...
- Integrable systems: Bordemann (1990), Winterhalder (1997), ...
- Classical and quantum Yang-Baxter equations: Svinolupov and Sokolov (1994), Etingof and Soloviev (1999), Golubschik and Sokolov (2000), ...
- Poisson brackets and infinite-dimensional Lie algebras: Gel’fand and Dorfman (1979), Dubrovin and Novikov (1984), Balinskii and Novikov (1985), ...
- Quantum field theory and noncommutative geometry: Connes and Kreimer (1998), ...

Chengming Bai  Pre-Lie algebras
What is a pre-Lie algebra?

○ Interesting structures:

Let $A$ be a pre-Lie algebra and $\mathfrak{g}(A)$ be the sub-adjacent Lie algebra.

- On $A \oplus A$ as a direct sum of vector spaces:
  1. $\mathfrak{g}(A) \ltimes_{\text{ad}} \mathfrak{g}(A) \implies$ Complex structures
  2. $\mathfrak{g}(A) \ltimes_{\text{L}} \mathfrak{g}(A) \implies$ Manin Triple (Lie bialgebra)

- On $A \oplus A^*$ as a direct sum of vector spaces ($A^*$ is the dual space of $A$):
  1. $\mathfrak{g}(A) \ltimes_{\text{ad}^*} \mathfrak{g}(A)^* \implies$ Manin Triple (Lie bialgebra)
  2. $\mathfrak{g}(A) \ltimes_{\text{L}^*} \mathfrak{g}(A)^* \implies$ Symplectic structures
What is a pre-Lie algebra?

- Main Problems: Non-associativity!
  1. There is not a suitable (and computable) representation theory.
  2. There is not a complete (and good) structure theory.

Example: There exists a transitive simple pre-Lie algebra which combines “simplicity” and certain “nilpotence”:

\[ e_1 e_2 = e_2, e_1 e_3 = -e_3, e_2 e_3 = e_3 e_2 = e_1. \]

- Ideas: Try to find more examples!
  1. The relations with other topics (including application).
  2. Realize by some known structures.
Examples of pre-Lie algebras

Construction from commutative associative algebras

Proposition

(S. Gel’fand) Let \((A, \cdot)\) be a commutative associative algebra, and \(D\) be its derivation. Then the new product

\[
a \ast b = a \cdot Db, \ \forall a, b \in A
\]

makes \((A, \ast)\) become a pre-Lie algebra.

In fact, \((A, \ast)\) is a Novikov algebra which is a pre-Lie algebra satisfying an additional condition \(R(a)R(b) = R(b)R(a)\), where \(R(a)\) is the right multiplication operator for any \(a \in A\).

\[
R(a)R(b) = R(b)R(a)
\]
defines a NAP algebra
Examples of pre-Lie algebras

- **Background:** Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus.

\[
\{u(x_1), v(x_2)\} = \frac{\partial}{\partial x_1} ((uv)(x_1)) x_1^{-1} \delta\left(\frac{x_1}{x_2}\right)
\]

\[+(uv + vu)(x_1) \frac{\partial}{\partial x_1} x_1^{-1} \delta\left(\frac{x_1}{x_2}\right).\]  

- **Generalization:** (Bai-Meng’s conjecture) Every Novikov algebra can be realized as the algebras (9) and their (compatible) linear deformations.

- **Conclusion:**

  1. **(Bai-Meng)** The conjecture holds in dimension \( \leq 3 \).

  2. **(Dzhumadil’daev-Lofwall)** Any Novikov algebra is a quotient of a subalgebra of an algebra given by Eq. (9).
Examples of pre-Lie algebras

○ Construction from linear functions

**Proposition**

Let $V$ be a vector space over the complex field $\mathbb{C}$ with the ordinary scalar product $(\cdot, \cdot)$ and $a$ be a fixed vector in $V$, then

$$u \ast v = (u, v)a + (u, a)v, \forall u, v \in V,$$

(11)

defines a pre-Lie algebra on $V$.

- Background: The integrable (generalized) Burgers equation

$$U_t = U_{xx} + 2U \ast U_x + (U \ast (U \ast U)) - ((U \ast U) \ast U).$$

(12)

- Generalization: Pre-Lie algebras from linear functions.

- Conclusion: The pre-Lie algebra given by Eq. (11) is simple.
Examples of pre-Lie algebras

○ Construction from associative algebras

Proposition

Let \( (A, \cdot) \) be an associative algebra and \( R : A \to A \) be a linear map satisfying

\[
R(x) \cdot R(y) + R(x \cdot y) = R(R(x) \cdot y + x \cdot R(y)), \forall x, y \in A. \quad (13)
\]

Then

\[
x \ast y = R(x) \cdot y - y \cdot R(x) - x \cdot y, \quad \forall x, y \in A \quad (14)
\]
defines a pre-Lie algebra.

- The above \( R \) is called \textit{Rota-Baxter map of weight 1}.
- Generalization: Approach from associative algebras.
- Related to the so-called “modified classical Yang-Baxter equation”.
Examples of pre-Lie algebras

- **Construction from Lie algebras (I)**

  **Question 1:** Whether there is a compatible pre-Lie algebra on every Lie algebra?
  
  **Answer:** No!

  Necessary condition: The sub-adjacent Lie algebra of a finite-dimensional pre-Lie algebra $A$ over an algebraically closed field with characteristic 0 satisfies

  $$[\mathfrak{g}(A), \mathfrak{g}(A)] \neq \mathfrak{g}(A).$$  

  $\Rightarrow \mathfrak{g}(A)$ can not be semisimple.

  **Question 2:** How to construct a compatible pre-Lie algebra on a Lie algebra?
  
  **Answer:** etale affine representation $\iff$ bijective 1-cocycle.
Examples of pre-Lie algebras

Let \( \mathfrak{g} \) be a Lie algebra and \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \) be a representation of \( \mathfrak{g} \). A 1-cocycle \( q \) associated to \( \rho \) (denoted by \((\rho, q)\)) is defined as a linear map from \( \mathfrak{g} \) to \( V \) satisfying

\[
q[x, y] = \rho(x)q(y) - \rho(y)q(x), \quad \forall x, y \in \mathfrak{g}. \tag{16}
\]

Let \((\rho, q)\) be a bijective 1-cocycle of \( \mathfrak{g} \), then

\[
x \ast y = q^{-1}\rho(x)q(y), \quad \forall x, y \in A, \tag{17}
\]

defines a pre-Lie algebra structure on \( \mathfrak{g} \). Conversely, for a pre-Lie algebra \( A \), \((L, id)\) is a bijective 1-cocycle of \( \mathfrak{g}(A) \).

**Proposition**

*There is a compatible pre-Lie algebra on a Lie algebra \( \mathfrak{g} \) if and only if there exists a bijective 1-cocycle of \( \mathfrak{g} \).*
Examples of pre-Lie algebras

- Application: Providing a linearization procedure of classification.
- Conclusion: Classification of complex pre-Lie algebras up to dimension 3.
- Generalization: classical r-matrices! $\Rightarrow$ a realization by Lie algebras
Let $\mathfrak{g}$ be a Lie algebra and $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. $r$ is called a solution of \textit{classical Yang-Baxter equation (CYBE)} in $\mathfrak{g}$ if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \text{ in } U(\mathfrak{g}),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1; \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i; \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$  

$r$ is said to be \textit{skew-symmetric} if

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i).$$

We also denote $r^{21} = \sum_i b_i \otimes a_i$. 
Pre-Lie algebras and classical Yang-Baxter equation

Definition

Let $\mathfrak{g}$ be a Lie algebra and $\rho: \mathfrak{g} \rightarrow gl(V)$ be a representation of $\mathfrak{g}$. A linear map $T: V \rightarrow \mathfrak{g}$ is called an $O$-operator if $T$ satisfies

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \forall u, v \in V. \quad (21)$$
○ Relationship between $\mathcal{O}$-operators and CYBE

From CYBE to $\mathcal{O}$-operators

**Proposition**

Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Suppose that $r$ is skew-symmetric. Then $r$ is a solution of CYBE in $\mathfrak{g}$ if and only if $r$ regarded as a linear map from $\mathfrak{g}^* \to \mathfrak{g}$ is an $\mathcal{O}$-operator associated to $\text{ad}^*$. 
From $\mathcal{O}$-operators to CYBE

Notation: let $\rho : \mathfrak{g} \rightarrow gl(V)$ be a representation of the Lie algebra $\mathfrak{g}$. On the vector space $\mathfrak{g} \oplus V$, there is a natural Lie algebra structure (denoted by $\mathfrak{g} \ltimes_\rho V$) given as follows:

$$[x_1 + v_1, x_2 + v_2] = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1, \quad (22)$$

for any $x_1, x_2 \in \mathfrak{g}, v_1, v_2 \in V$.

**Proposition**

Let $\mathfrak{g}$ be a Lie algebra. Let $\rho : \mathfrak{g} \rightarrow gl(V)$ be a representation of $\mathfrak{g}$ and $\rho^* : \mathfrak{g} \rightarrow gl(V^*)$ be the dual representation. Let $T : V \rightarrow \mathfrak{g}$ be a linear map which is identified as an element in $\mathfrak{g} \otimes V^* \subset (\mathfrak{g} \ltimes_\rho V^*) \otimes (\mathfrak{g} \ltimes_\rho V^*)$. Then $r = T - T^{21}$ is a skew-symmetric solution of CYBE in $\mathfrak{g} \ltimes_\rho V^*$ if and only if $T$ is an $\mathcal{O}$-operator.
○ Relationship between $O$-operators and pre-Lie algebras

Proposition

Let $\mathfrak{g}$ be a Lie algebra and $\rho: \mathfrak{g} \to gl(V)$ be a representation. Let $T: V \to \mathfrak{g}$ be an $O$-operator associated to $\rho$, then

$$u \ast v = \rho(T(u))v, \quad \forall u, v \in V$$

(23)

defines a pre-Lie algebra on $V$. 
Lemma

Let \( \mathfrak{g} \) be a Lie algebra and \((\rho, V)\) is a representation. Suppose \( f : \mathfrak{g} \rightarrow V \) is invertible. Then \( f \) is a 1-cocycle of \( \mathfrak{g} \) associated to \( \rho \) if and only if \( f^{-1} \) is an \( O \)-operator.

Corollary

Let \( \mathfrak{g} \) be a Lie algebra. There is a compatible pre-Lie algebra structure on \( \mathfrak{g} \) if and only if there exists an invertible \( O \)-operator of \( \mathfrak{g} \).
Proposition

Let $A$ be a pre-Lie algebra. Then

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i)$$ (24)

is a solution of the classical Yang-Baxter equation in the Lie algebra $\mathfrak{g}(A) \ltimes_{L^*} \mathfrak{g}(A)^*$, where $\{e_1, ..., e_n\}$ is a basis of $A$ and $\{e_1^*, ..., e_n^*\}$ is the dual basis.
From CYBE to pre-Lie algebras (Construction from Lie algebras (II))

Lemma

Let $\mathfrak{g}$ be a Lie algebra and $f$ be a linear transformation on $\mathfrak{g}$. Then on $\mathfrak{g}$ the new product

$$x * y = [f(x), y], \forall x, y \in \mathfrak{g}$$

defines a pre-Lie algebra if and only if

$$[f(x), f(y)] - f([f(x), y] + [x, f(y)]) \in C(\mathfrak{g}), \forall x, y \in \mathfrak{g},$$

where $C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0, \forall y \in \mathfrak{g}\}$ is the center of $\mathfrak{g}$. 
Corollary

Let \( \mathfrak{g} \) be a Lie algebra. Let \( r : \mathfrak{g} \to \mathfrak{g} \) be an \( O \)-operator associated to \( \text{ad} \), that is,

\[
[r(x), r(y)] - r([r(x), y] + [x, r(y)]) = 0, \quad \forall x, y \in \mathfrak{g}.
\] (27)

Then Eq. (25) defines a pre-Lie algebra.

Remark

1. Eq. (27) is called operator form of CYBE;
2. Eq. (27) is the Rota-Baxter operator of weight zero in the context of Lie algebras.
3. If there is a nondegenerate invariant bilinear form on \( \mathfrak{g} \) (that is, as representations, \( \text{ad} \) is isomorphic to \( \text{ad}^* \)) and \( r \) is skew-symmetric, then \( r \) satisfies CYBE if and only if \( r \) regarded as a linear transformation on \( \mathfrak{g} \) satisfies Eq. (27).
An algebraic interpretation of “left-symmetry”:

Let \( \{e_i\} \) be a basis of a Lie algebra \( \mathfrak{g} \). Let \( r : \mathfrak{g} \to \mathfrak{g} \) be an \( O \)-operator associated to \( \text{ad} \). Set \( r(e_i) = \sum_{j \in I} r_{ij} e_j \). Then the basis-interpretation of Eq. (25) is given as

\[
e_i \ast e_j = \sum_{l \in I} r_{il} [e_l, e_j]. \tag{28}
\]

Such a construction of left-symmetric algebras can be regarded as a Lie algebra “left-twisted” by a classical \( r \)-matrix.

On the other hand, let us consider the right-symmetry. We set

\[
e_i \cdot e_j = [e_i, r(e_j)] = \sum_{l \in I} r_{jl} [e_i, e_l]. \tag{29}
\]

Then the above product defines a right-symmetric algebra on \( \mathfrak{g} \), which can be regarded as a Lie algebra “right-twisted” by a classical \( r \)-matrix.

Application: Realization of pre-Lie algebras by Lie algebras.
A **vertex algebra** is a vector space $V$ equipped with a linear map

$$Y : V \rightarrow \text{Hom}(V, V((x))), v \rightarrow Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

and equipped with a distinguished vector $1 \in V$ such that

$$Y(1, x) = 1;$$

$$Y(v, x)1 \in V[[x]] \text{ and } Y(v, x)1|_{x=0}(= v_{-1}1) = v, \ \forall v \in V,$$

and for $u, v \in V$, there is Jacobi identity:

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1)$$

$$= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2).$$
Pre-Lie algebras and vertex (operator) algebras

**Proposition**

Let \((V, Y, 1)\) be a vertex algebra. Then

\[ a \ast b = a_{-1}b \]  

(32)

defines a pre-Lie algebra.

That is,

\[ (a_{-1}b)_{-1}c - a_{-1}(b_{-1}c) = (b_{-1}a)_{-1}c - b_{-1}(a_{-1}c). \]

(33)

In fact, by Borcherd’s identities:

\[
(a_m(b))_n(c) = \sum_{i \geq 0} (-1)^i C_m^i ((a_{m-i}(b_{n+i}(c)) - (-1)^m b_{m+n-i}(a_i(c))),
\]

(34)

let \(m = n = -1\), we have

\[
(a_{-1}b)_{-1}c - a_{-1}(b_{-1}c) = \sum_{i \geq 0} (a_{-2-i}(b_ic) + b_{-2-i}(a_ic)).
\]

(35)
Proposition

A vertex algebra is equivalent to a pre-Lie algebra and an algebra names Lie conformal algebra with some compatible conditions.
Definition

A **Lie conformal algebra** is a \( \mathbb{C}[D] \)-module \( V \) endowed a \( \mathbb{C} \)-linear map \( V \otimes V \rightarrow \mathbb{C}[\lambda] \otimes V \) denoted by \( a \otimes b \rightarrow [a_{\lambda}b] \), satisfying

\[
[(D a)_{\lambda}b] = -\lambda [a_{\lambda}b];
\]

\[
[a_{\lambda}(D b)] = (\lambda + D)[a_{\lambda}b];
\]

\[
[b_{\lambda}a] = [a - \lambda - T b];
\]

\[
[a_{\lambda}b]_{\lambda + \mu}c = [a_{\lambda}[b_{\mu}c] - [b_{\mu}[a_{\lambda}c]].
\]

Remark

*Let \( (V, Y, 1) \) be a vertex algebra. Then*

\[
a_{\lambda}b = \text{Res}_x e^{\lambda x}Y(a, x)b = \sum_{n \geq 0, \text{finite}} \lambda^n a_n b/n!.
\]
Pre-Lie algebras and vertex (operator) algebras

**Proposition**

A vertex algebra is a pair \((V, 1)\), where \(V\) is a \(\mathbb{C}[D]\)-module and \(1 \in V\), endowed two operations:

1. \(V \otimes V \to \mathbb{C}[[\lambda]] \otimes V, a \otimes b \to [a_\lambda b]\), Lie conformal algebra;
2. \(V \otimes V \to V, a \otimes b \to ab\), a differential algebra with unit \(1\) and derivation \(D\).

They satisfy the following conditions:

1. \((ab)c - a(bc) = (\int_0^D d\lambda a)[b_\lambda c] + (\int_0^D d\lambda b)[a_\lambda c]\);
2. \(ab - ba = \int_0^D d\lambda a[a_\lambda b]\);
3. \([a_\lambda(bc)] = [a_\lambda b]c + b[a_\lambda c] + \int_0^\lambda d\mu[a_\lambda b] - \mu c\).
Roughly speaking,

\[ Y(a, x)b = Y(a, x)_+ b + Y(a, x)_- b, \]

where

\[ Y(a, x)_+ b = (e^{xD}a)_- 1 b, \]
\[ Y(a, x)_- b = (a - \partial_x b)(x^{-1}). \]
Novikov algebras and infinite-dimensional Lie algebras and vertex (operator) algebras

Proposition

Let $A$ be a finite-dimensional algebra with a bilinear product $(a, b) \to ab$. Set

$$A = A \otimes \mathbb{C}[t, t^{-1}].$$

(37)

Then the bracket

$$[a \otimes t^m, b \otimes t^n] = (-mab + nba) \otimes t^{m+n-1}, \quad \forall a, b \in A, \; m, n \in \mathbb{Z}$$

(38)

defines a Lie algebra structure on $A$ if and only if $A$ is a Novikov algebra with the product $ab$. 
Let $A$ be a Novikov algebra. There is a $\mathbb{Z}$–grading on the Lie algebra $A$ defined by Eqs. (37) and (38):

$$A = \bigoplus_{n \in \mathbb{Z}} A(n), \quad A(n) = A \otimes t^{-n+1}.$$ 

Moreover,

$$A(n \leq 1) = \bigoplus_{n \leq 1} A(n) = \bigoplus_{n \leq 0} A(n) \oplus A(1)$$

is a Lie subalgebra of $A$. For any $a \in A$, we define the generating function:

$$a(x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1} = \sum_{n \in \mathbb{Z}} (a \otimes t^n) x^{-n-1} \in A[[x, x^{-1}]].$$

Then we have

$$[a(x_1), b(x_2)] = (ab+ba)(x_1) \frac{\partial}{\partial x_1} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) + \left[\frac{\partial}{\partial x_1} ab(x_1)\right] x_2^{-1} \delta\left(\frac{x_1}{x_2}\right).$$
Let $C$ be the trivial module of $\mathcal{A}_{(n \leq 1)}$ and we can get the following (Verma) module of $\mathcal{A}$:

$$\hat{\mathcal{A}} = U(\mathcal{A}) \otimes U(\mathcal{A}_{(n \leq 1)}) \ C,$$

where $U(\mathcal{A})$ ($U(\mathcal{A}_{(n \leq 1)})$) is the universal enveloping algebra of $\mathcal{A}$ ($\mathcal{A}_{(n \leq 1)}$). $\hat{\mathcal{A}}$ is a natural $\mathbb{Z}$-graded $\mathcal{A}$-module:

$$\hat{\mathcal{A}} = \bigoplus_{n \geq 0} \hat{\mathcal{A}}(n),$$

where

$$\hat{\mathcal{A}}(n) = \{ a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r} 1 | m_1 + \cdots + m_r = n-r, \ m_1 \geq \cdots \geq m_r \geq 1, \ r \geq 0 \}.$$
Theorem

Let $A$ be a Novikov algebra. Then there exists a unique vertex algebra structure $(\hat{A}, Y, 1)$ on $\hat{A}$ such that $1 = 1 \in \mathbb{C}$ and $Y(a, x) = a(x), \forall a \in A$, and

$$Y(a_{(n_1)}^{(1)} \cdots a_{(n_r)}^{(r)} 1, x) = a_{(x)^{n_1}}^{(1)} \cdots a_{(x)^{n_r}}^{(r)} 1_{\hat{A}}$$ (39)

for any $r \geq 0, a^{(i)} \in A, n_i \in \mathbb{Z}$, where

$$a(x)_n b(x) = \text{Res}_{x_1} \left( (x_1 - x)^n a(x_1) b(x) - (-x + x_1)^n b(x) a(x_1) \right).$$ (40)
Pre-Lie algebras and vertex (operator) algebras

**Theorem**

Let $(V, Y, 1)$ be a vertex algebra with the following properties:

1. $V = \bigoplus_{n \geq 0} V(n)$, $V(0) = \mathbb{C}1$, $V(1) = 0$;
2. $V$ is generated by $V(2)$ with the following property

$$V = \text{span}\{a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r} 1 | m_i \geq 1, r \geq 0, a^{(i)} \in V(2)\}$$

3. $\text{Ker} D = \mathbb{C}$, where $D$ is a linear transformation of $V$ given by $D(v) = v_{-2}1$, $\forall v \in V$.

4. $V$ is a graded (by the integers) vertex algebra, that is, $1 \in V(0)$ and $V(i)_n V(j) \subset V(i+j-n-1)$.
(Continued) Then $V$ is generated by $V_{(2)}$ with the following property

\[ V = \text{span}\{a_{-m_1}^{(1)} \cdots a_{-m_r}^{(r)} 1 | m_1 \geq \cdots \geq m_r \geq 1, r \geq 0, a^{(i)} \in V_{(2)} \} \]

and $V_{(2)}$ is a Novikov algebra with a product $(a, b) \rightarrow a \ast b$ given by

\[ a \ast b = -D^{-1}(b_0 a). \]
Furthermore, if the Novikov algebra $A$ has an identity $e$, then $-e$ corresponds to the Virasoro element which gives a vertex operator algebra structure with zero central charge. With a suitable central extension, the non-zero central charge will lead to a commutative associative algebra with a nondegenerate symmetric invariant bilinear form (the so-called Frobenius algebra).

**Corollary**

*For the above vertex algebra $(V, Y, 1)$, the algebra given by*

$$a \ast b = a_{-1}b$$

*is a graded pre-Lie algebra, that is $V_m \ast V_n \subset V_{m+n}$.**