DIFFERENTIAL ALGEBRAIC BIRKHOFF DECOMPOSITION AND THE RENORMALIZATION OF MULTIPLE ZETA VALUES

LI GUO AND BIN ZHANG

Abstract. In the Hopf algebra approach of Connes and Kreimer on renormalization of quantum field theory, the renormalization process is viewed as a special case of the Algebraic Birkhoff Decomposition. We give a differential algebra variation of this decomposition and apply this to the study of multiple zeta values.

1. Introduction

This paper applies the renormalization method in quantum field theory to the study of multiple zeta values when the defining sums of the multiple zeta values are divergent, with an emphasis on the differential structure underlying the renormalization process.

As we will see later, such divergence of the infinite sums cannot be cured by traditional mathematical methods, such as analytic continuation as in the case of one variable (Riemann) zeta function. On the other hand, theoretical physicists have dealt with similar divergencies in quantum field theory for several decades with such a great success that the related renormalization process is regarded as one of the greatest achievements in modern physics. Nevertheless mathematicians have been skeptical about the soundness of the mathematical foundation of the renormalization process. This situation is changed by the recent seminal work of Connes and Kreimer [12, 13] which puts the renormalization process in a more mathematical framework of Algebraic Birkhoff Decomposition. Such a framework not only reveals the mathematical structure underlying the physics process, it also makes it possible for this physics process to be adapted to treat other apparently unrelated problems in mathematics, in particular the divergence of multiple zeta values.

This is the goal of this and the companying paper [29] which complete each other. This paper consists of three sections. Section 2 motivates our renormalization approach through examples and special cases. In Section 3, we obtain the differential variation of the Algebraic Birkhoff Decomposition of Connes and Kreimer [12, 13]. We also extend the well-known quasi-shuffle Hopf algebra of Hoffman [32] in two directions. In one direction we consider such an algebra that is generated by a module instead of by a locally finite set. In the other direction, we consider quasi-shuffle Hopf algebras in the context of differential Hopf algebras instead of Hopf algebras. This context allows us to apply the Differential Algebraic Birkhoff Decomposition in Section 4 to study differential properties of renormalized multiple zeta values.

This paper highlights the differential aspect of the renormalization process. As is well-known, for the convergent multiple zeta values, the interaction between the quasi-shuffle product from their summation representations and the shuffle product from their integration representations plays a major role in their study. In the renormalized approach of divergent multiple zeta values, the integration representation is not available. On the other hand, differentiation is essential in the study of renormalized multiple zeta values. We interpret this in the Algebraic Birkhoff Decomposition and give an application on generating functions of renormalized multiple zeta values.

Parts of this paper, namely the examples in Section 2 and the general construction of quasi-shuffle Hopf algebras in Section 3, were included in an early version of [29], but was taken out of the final version of that paper to limit its size. Some readers have found the examples helpful in understanding the renormalization process in general. Further the general construction of quasi-shuffle algebra in Section 3 has been referred to or related to in a number of papers, such as that of D. Manchon and S. Payche [37] on renormalization of multiple zeta values, of J. Zhao [47] on renormalization of multiple q-zeta values and of F. Menous [40] on Birkhoff decomposition. Also considering its application in later part of this paper in the differential context, we have decided to provide details of the construction.

Acknowledgements. Both authors thank the Max-Planck Institute for Mathematics in Bonn for the stimulating environment. The first named author thanks the NSF grant DMS 0505445 for support. Thanks also go to Robert Sczech, especially for suggesting the term algebraic continuation, to Herbert Gangl, especially for suggesting the bar notation, and to the referee for careful reading and helpful comments.

2. Renormalization of MZVs: motivation and examples

Multiple zeta values (MZVs) are defined to be the convergent sums

$$\zeta(s_1, \cdots, s_k) = \sum_{n_1 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \tag{1}$$

where $s_1, \cdots, s_k$ are positive integers with $s_1 > 1$. Since the papers of Hoffman [30] and Zagier [45] in the early 1990s, their study have attracted interests from several areas of mathematics and physics [7, 8, 10, 22, 23, 32, 34], including number theory, combinatorics, algebraic geometry and mathematical physics.

In order to study of the multiple variable function $\zeta(s_1, \cdots, s_k)$ at integers $s_1, \cdots, s_k$ where the defining sum (1) is divergent, one first tries to use the analytic continuation, as the one variable case of the Riemann zeta function. Such an analytic continuation is achieved recently in [1, 3, 38, 39, 46]. Unfortunately, unlike in the one variable case, the multiple zeta function in Eq. (1) is still undefined at most non-positive integers even with the analytic continuation. Nevertheless, possible definitions of multiple zeta functions at certain non-positive integers were proposed in [1, 3] by making use of the analytic continuation. Let us briefly recall these previous progresses before introducing our approach by renormalization.

2.1. Earlier approach by analytic continuation.

Analytic continuation of $\zeta(s_1, s_2)$ was considered as early as 1949 by Atkinson [5] with applications to the study of the asymptotic behavior of the "mean values" of zeta-function.

Through the more recent work of Zhao [46] and Akiyama-Egami-Tanigawa [1], we know that $\zeta(s_1, \cdots, s_k)$ can be meromorphically continued to $\mathbb{C}^k$ with singularities on the subvarieties

$$\begin{aligned} s_1 &= 1; \\ s_1 + s_2 &= 2, 1, 0, -2, -4, \cdots; \text{ and} \\ \sum_{i=1}^{j} s_i &\in \mathbb{Z}_{\le j} \ (3 \le j \le k). \end{aligned}$$

We also have some control of the behavior near the singularities. For example, near $(0, 0)$,

$$\zeta(s_1, s_2) = \frac{5s_1 + 4s_2}{12(s_1 + s_2)} + R_2(s_1, s_2)$$

where $R_2(s_1, s_2)$ is an entire function near $(0, 0)$ with $R_2(0, 0) = 0$.

In [1, 3], several definitions were proposed for multiple zeta functions at $(s_1, \cdots, s_k)$ where s_i are all non-positive. Some of the definitions of $\zeta(s_1, \cdots, s_k)$ are

$$\lim_{r_1 \to s_1} \cdots \lim_{r_k \to s_k} \zeta(r_1, \cdots, r_k), \quad \lim_{r_k \to s_k} \cdots \lim_{r_1 \to s_1} \zeta(r_1, \cdots, r_k), \quad \lim_{r \to 0} \zeta(s_1 + r, \cdots, s_k + r). \tag{2}$$

Naturally they give different values. In the case of $\zeta(s_1, s_2)$ at $(s_1, s_2) = (0, 0)$, the proposed values are $5/12, 1/3, 3/8$ respectively according to the above three definitions. In fact, by letting (r_1, r_2) approach to $(0, 0)$ along different paths, one can get any value, as well as the infinity, to be the limit. Even though some good properties of the variously defined non-positive MZVs were obtained in these studies, they fell short of the analogous properties of the positive MZVs, especially the double shuffle relations.

2.2. **An illustration of the renormalization approach.** Our approach to define MZVs where the sum (1) is divergent is adapted from a renormalization procedure (dimensional regularization plus minimal subtraction) in quantum field theory (QFT). This definition coincides with the usual definition of MZVs when they are defined. Our extended MZVs also satisfy the quasi-shuffle relation. So our approach is an **algebraic continuation** in the sense that we extend the definition of MZVs that preserves the quasi-shuffle relation.

The QFT renormalization procedure was recently put into a more mathematical framework through the work of Connes and Kreimer [12, 13] and was thus possible to be applied to problems in mathematics. For our purpose, the dimensional regularization of Feynman integrals is replaced by a regularization (or deformation) of infinite series that has occurred in the study of Todd classes for toric varieties [9].

Here we illustrate this method by some special cases with the general cases given in § 4 and in [29]. First recall the following generating series of Bernoulli numbers that goes back to Euler.

$$\frac{\varepsilon}{e^{\varepsilon} - 1} = \sum_{k \ge 0} B_k \frac{\varepsilon^k}{k!} \tag{3}$$

It can be easily rewritten as

$$\frac{e^{\varepsilon}}{1 - e^{\varepsilon}} = -\frac{1}{\varepsilon} \frac{-\varepsilon}{e^{-\varepsilon} - 1} = -\frac{1}{\varepsilon} + \sum_{k \ge 0} \zeta(-k) \frac{\varepsilon^k}{k!} \tag{4}$$

since $B_0 = 1$ and $\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}$ for $k \ge 0$.

Now consider

$$Z(s;\varepsilon)=\sum_{n\geq 1}\frac{e^{n\varepsilon}}{n^s},$$

regarded as a deformation or "regularization" of the series defining the Riemann zeta function $\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}$. The regularized series converges for any integer s when $\mathrm{Re}(\varepsilon)<0$. In particular,

$$Z(0;\varepsilon)=\frac{e^{\varepsilon}}{1-e^{\varepsilon}}$$

and Eq. (4) gives the Laurent series expansion of the regularized sum $Z(0;\varepsilon)=\sum_{n\geq 1}e^{n\varepsilon}$ at $\varepsilon=0$. For a Laurent series $f(\varepsilon)=\sum_{n\geq N}a_n\varepsilon^n$ where $N\in\mathbb{Z}$, we denote the **power series part** (or the **finite part**) $\sum_{n\geq 0}a_n\varepsilon^n$ of the Laurent series by $\mathrm{fp}(f)$, a notation borrowed from [36] which, according to the authors, can be traced back to Hadamard. We tentatively call $\mathrm{fp}(f)|_{\varepsilon=0}$ the renormalized value of $f(\varepsilon)$ (see Section 4.1 for the general case). Then we have

$$\mathrm{fp}\big(\sum_{n\geq 1}e^{n\varepsilon}\big)\Big|_{\varepsilon=0}=\zeta(0).$$

So the renormalized value of $Z(0;\varepsilon)=\sum_{n\geq 1}e^{n\varepsilon}$ is $\zeta(0)$. Similarly, to evaluate $\zeta(-k)$ for an integer $k\geq 1$, consider the regularized sum

$$Z(-k;\varepsilon)=\sum_{n\geq 1}n^k e^{n\varepsilon}=\frac{d^k}{d\varepsilon}\big(\frac{e^{\varepsilon}}{1-e^{\varepsilon}}\big)$$

which converges uniformly on any compact subset in $\mathrm{Re}(\varepsilon)<0$. So its Laurent series expansion at $\varepsilon=0$ is obtained by termwise differentiation of Eq. (4), yielding

$$Z(-k;\varepsilon)=(-1)^{-k-1}(k)!\,\varepsilon^{-k-1}+\sum_{j=0}^{\infty}\zeta(-k-j)\frac{\varepsilon^j}{j!}. \tag{5}$$

We then have

$$\mathrm{fp}\big(\sum_{n\geq 1}n^k e^{n\varepsilon}\big)\Big|_{\varepsilon=0}=\zeta(-k).$$

Thus the renormalization method does give the correct Riemann zeta values at non-positive integers. We next extend this to multiple zeta functions and "evaluate" $\zeta(0,0)$, for example, by consider the regularized sum

$$Z(0,0;\varepsilon)=\sum_{n_1>n_2>0}e^{n_1\varepsilon}e^{n_2\varepsilon}=\frac{e^{\varepsilon}}{1-e^{\varepsilon}}\frac{e^{2\varepsilon}}{1-e^{2\varepsilon}}.$$

Naively taking the finite part as in the one variable case, we find

$$\mathrm{fp}\big(\sum_{n_1>n_2>0}e^{n_1\varepsilon}e^{n_2\varepsilon}\big)\big|_{\varepsilon=0}=\mathrm{fp}\big(\frac{1}{2\varepsilon^2}-\frac{3}{2}\zeta(0)\frac{1}{\varepsilon}+\big(-\frac{5}{2}\zeta(-1)+\zeta(0)^2\big)+o(\varepsilon)\big)\big|_{\varepsilon=0}=11/24.$$

This does not agree with any of the previously proposed values of $\zeta(0,0)$ in Eq. (2). Further, this value does not satisfy the well-known quasi-shuffle (stuffle) relation:

$$\zeta(0)\zeta(0)\neq 2\,\zeta(0,0)+\zeta(0)$$

since the left hand side is $1/4$ and the right hand side is $5/12$. Recall the well-known quasi-shuffle relation

$$\zeta(s)\zeta(s)=2\,\zeta(s,s)+\zeta(2s)$$

for any integer $s \geq 2$.

To improve this situation and obtain a more suitable definition of $\zeta(0,0)$, we recall that a key principle in the renormalization procedure of QFT is that if a divergent Feynman integral contains a component integral that is already divergent, then the divergency of the component integral should be removed before removing the divergency of the integral itself. For our example of $\zeta(0,0)$, the regularized sum

$$\sum_{n_1>n_2>0} e^{n_1\varepsilon}e^{n_2\varepsilon} = \sum_{n_2\geq 1} e^{n_2\varepsilon} \sum_{n_1\geq n_2+1} e^{n_1\varepsilon}$$

has a component sum $\sum_{n_1\geq n_2+1} e^{n_1\varepsilon}$ that is already divergent when ε goes to 0. By the renormalized process adopt to our case (see Eq. (32) and Remark 4.7) we found that the renormalized value should be defined by

$$\begin{aligned} & \mathrm{fp}\Big(\sum_{n_1>n_2>0} e^{n_1\varepsilon}e^{n_2\varepsilon} - \sum_{n_2>0} e^{n_2\varepsilon} \underbrace{\big(\sum_{n_1>0} e^{n_1\varepsilon} - \mathrm{fp}(\sum_{n_1>0} e^{n_1\varepsilon}) \big)}_{\text{subdivergence}} \Big)\Big|_{\varepsilon=0} \\ = \ & \mathrm{fp}\Big(\frac{1}{2}\frac{1}{\varepsilon^2} - \frac{3}{2}\zeta(0)\frac{1}{\varepsilon} + (-\frac{5}{2}\zeta(-1) + \zeta(0)^2 + o(\varepsilon)) - \big(\frac{1}{\varepsilon^2} - \frac{\zeta(0)}{\varepsilon} - \zeta(-1) + o(\varepsilon)\big)\Big)\Big|_{\varepsilon=0} \\ = \ & -\frac{3}{2}\zeta(-1) + \zeta(0)^2 = \frac{3}{8}. \end{aligned}$$

This value indeed satisfies the quasi-shuffle relation $\zeta(0)\zeta(0) = 2\zeta(0,0) + \zeta(0)$.

This renormalization process will be systematically carried out for all $\zeta(s_1, \cdots, s_k)$ in § 4 after the general setup in § 3. Remark 4.7 shows how the above example follows from the general case.

3. Differential Birkhoff decomposition and quasi-shuffle Hopf algebras

We formulate a general setup for our later applications to MZVs.

3.1. **Differential Birkhoff decomposition.** We review and extend the algebraic framework of Connes and Kreimer for renormalization of perturbative quantum field theory. For further details of physics applications, see [12, 13, 14, 18, 19, 21, 35].

Let $\mathbf{k}$ be a unitary commutative ring that we usually take to be $\mathbb{R}$ or a subring of $\mathbb{C}$. In the following an algebra means a unitary $\mathbf{k}$-algebra unless otherwise specified. A **connected filtered Hopf algebra** is a Hopf algebra (H, Δ) with $\mathbf{k}$-submodules $H_n,\ n \geq 0$, of H such that

(1) $H_n \subseteq H_{n+1}$,
(2) $\cup_{n\geq 0} H_n = H$,
(3) $H_pH_q \subseteq H_{p+q}$,
(4) $\Delta(H_n) \subseteq \bigoplus_{p+q=n} H_p \otimes H_q$,
(5) (connectedness) $H_0 = \mathbf{k}$.

A **Rota–Baxter algebra** [6, 42, 43] of weight λ is a pair (R, P) where R is an algebra and $P : R \to R$ is a linear operator such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \tag{6}$$

for any $x,\ y \in R$. Often $\theta = -\lambda$ is used, especially in the physics literature.

Now let H be a connected filtered Hopf algebra and let (R, P) be a commutative Rota-Baxter algebra. Consider the algebra $\mathcal{R} := \mathrm{Hom}(H, R)$ of linear maps from H to R where the product is given by the convolution

$$f \star g(x) := \sum_{(x)} f(x_{(1)}) g(x_{(2)}).$$

Here we have used the Sweedler's notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$. Then the operator

$$\mathcal{P} : \mathrm{Hom}(H, R) \to \mathrm{Hom}(H, R), f \mapsto P \circ f,$$

is a Rota-Baxter operator on $\mathcal{R}$.

Definition 3.1. (1) A **differential algebra** is a pair (R, d) where R is an algebra and d is a **differential operator**, that is, such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A differential algebra homomorphism $f : (R_1, d_1) \to (R_2, d_2)$ between two differential algebras (R_1, d_1) and (R_2, d_2) is an algebra homomorphism $f : R_1 \to R_2$ such that $f \circ d_1 = d_2 \circ f$.

(2) A **differential Hopf algebra** is a pair (H, d) where H is a Hopf algebra and $d : H \to H$ is a differential operator such that

$$\Delta(d(x)) = \sum_{(x)} \big(d(x_{(1)}) \otimes x_{(2)} + x_{(1)} \otimes d(x_{(2)}) \big). \tag{7}$$

(3) A **differential Rota-Baxter algebra** is a triple (R, P, d) where (R, P) is a Rota-Baxter algebra and $d : R \to R$ is a differential operator such that $P \circ d = d \circ P$.

Theorem 3.2. *Let H be a commutative connected filtered Hopf algebra. Let (R, P) be a Rota-Baxter algebra of weight -1. Let $\phi : H \to R$ be an algebra homomorphism.*

(1) **(Algebraic Birkhoff Decomposition)** *There are algebra homomorphisms $\phi_- : H \to \mathbf{k} + P(R)$ and $\phi_+ : H \to \mathbf{k} + (\mathrm{id}_R - P)(R)$ such that*

$$\phi = \phi_-^{\star(-1)} \star \phi_+.$$

Here $\phi_-^{\star(-1)}$ is the inverse of ϕ_- with respect to the convolution product. Further,

$$\phi_-(x) = -P\big(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')\big) \tag{8}$$

and

$$\phi_+(x) = (\mathrm{id} - P)\big(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')\big). \tag{9}$$

Here we have used the notation $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(x)} x' \otimes x''$.

(2) *If $P^2 = P$, then the decomposition in (1) is unique.*

(3) **(Differential Algebraic Birkhoff Decomposition)** *If in addition (H, d) is a differential Hopf algebra, (R, P, ∂) is a commutative differential Rota-Baxter algebra, and ϕ is a differential algebra homomorphism, then ϕ_- and ϕ_+ are also differential algebra homomorphisms.*

Proof. For the proof of (1), see [35, Theorem II.5.1]. For the proof of (2), see [20, Theorem 3.7]. So we just need to prove (3). We only verify that ϕ_- is a differential algebra homomorphism, the proof for ϕ_+ being the same. For this we use induction on n to prove

$$\partial \circ \phi_-(x) = \phi_- \circ d(x) \tag{10}$$

for $x \in H_n, n \geq 0$.

For $x \in H_0$, we have $x \in \mathbf{k}$. Since ϕ_- is an algebra homomorphism by (1), we have $\phi_-(x) = x$. So $\partial(\phi_-(x)) = 0 = \phi_-(d(x))$.

Assume Eq. (10) has been verified for $n \leq k$ and consider $x \in H_{k+1}$. Applying ∂ to Eq. (8), we have

$$\begin{aligned}
\partial(\phi_-(x)) &= \partial\big(-P(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x''))\big) \\
&= -P\big(\partial(\phi(x)) + \sum_{(x)} (\partial\phi_-(x')\phi(x'') + \phi_-(x')\partial(\phi((x''))))\big) \\
&= -P\big(\phi(d(x)) + \sum_{(x)} (\phi_-(d(x'))\phi(x'') + \phi_-(x')\phi(d(x'')))\big).
\end{aligned}$$

Here we have used the induction hypothesis in the last step. On the other hand,

$$\phi_-(d(x)) = -P(\phi(d(x)) + \sum_{(d(x))} \phi_-(d(x)')\phi(d(x)'').$$

Further

$$\Delta(d(x)) = d(x)\otimes 1 + 1\otimes d(x) + \sum_{(d(x))} d(x)'\otimes d(x)''$$

and

$$d(\Delta(x)) = d\big(x\otimes 1 + 1\otimes x + \sum_{(x)} x'\otimes x''\big) = d(x)\otimes 1 + 1\otimes d(x) + \sum_{(x)} \big(d(x')\otimes x'' + x'\otimes d(x'')\big).$$

Since H is a differential Hopf algebra with the operator d, by Eq. (7), we have

$$\sum_{(d(x))} d(x)'\otimes d(x)'' = \sum_{(x)} \big(d(x')\otimes x'' + x'\otimes d(x'')\big)$$

and then applying $\phi_- \star \phi$ to it, we get

$$\sum_{(d(x))} \phi_-(d(x)')\phi(d(x)'') = \sum_{(x)} \big(\phi_-(d(x'))\phi(x'') + \phi_-(x')\phi(d(x''))\big).$$

This completes the induction for Eq. (10). $\square$

3.2. **Mixable shuffle algebras.** Let A be a unitary or nonunitary $\mathbf{k}$-algebra. Consider the tensor product algebra

$$\mathcal{H}_A := T(A) = \oplus_{n\geq 0} A^{\blacksquare n} \tag{11}$$

with the convention that $A^{\blacksquare 0} = \mathbf{k}$. Here we have used $\blacksquare$ instead of $\otimes$ to denote the tensor product of A with itself since $\otimes$ will be used for the coproduct introduced below. Let $\bar{\blacksquare}$ be the product in $T(A)$, so for $\mathfrak{a} \in A^{\otimes m}$ and $\mathfrak{b} \in A^{\otimes n}$, we have

$$\mathfrak{a} \bar{\otimes} \mathfrak{b} = \begin{cases} \mathfrak{a} \otimes \mathfrak{b} \in A^{\otimes m+n}, & \text{if } m > 0, n > 0, \\ \mathfrak{a}\mathfrak{b} \in A^{\otimes n}, & \text{if } m = 0, n > 0, \\ \mathfrak{a}\mathfrak{b} \in A^{\otimes m}, & \text{if } m > 0, n = 0, \\ \mathfrak{a}\mathfrak{b} \in \mathbf{k}, & \text{if } m = n = 0. \end{cases} \tag{12}$$

Here the products in the second and third cases are scalar product and in the fourth case is the product in $\mathbf{k}$. Thus, $\bar{\otimes}$ identifies $\mathbf{k} \otimes A$ and $A \otimes \mathbf{k}$ with A by the structure maps $\mathbf{k} \otimes A \to A$ and $A \otimes \mathbf{k} \to A$ of the $\mathbf{k}$-module A.

We next equip $\mathcal{H}_A = T(A)$ with another product $*$ so that, together with the deconcatenation coproduct, $\mathcal{H}_A$ is a connected filtered Hopf algebra. In a special case, this recovers the quasi-shuffle Hopf algebra of Hoffman [32]. The product $*$ is defined by the **quasi-shuffle product** in its recursive form, and by the **mixable shuffle product** in its explicit form by one of the authors and Keigher [27]. Given two pure tensors $\mathfrak{a} \in A^{\otimes m}$ and $\mathfrak{b} \in A^{\otimes n}$, to define $\mathfrak{a} * \mathfrak{b}$ explicitly, recall that a shuffle of $\mathfrak{a}$ and $\mathfrak{b}$ is a tensor list of a_i and b_j without changing the order of the a_is and b_js. A mixable shuffle is a shuffle in which some (or none) of the pairs $a_i \otimes b_j$ are merged into $a_i b_j$. See [27] for the precise definition. Then $\mathfrak{a} * \mathfrak{b}$ is the sum of mixable shuffles of $\mathfrak{a}$ and $\mathfrak{b}$. It is shown in [15] that the quasi-shuffle product and the mixable shuffle agree with each other. We note that the mixable shuffle arises from the construction of free commutative Rota–Baxter algebras and has been denoted by $\diamond^+$ in the literature of Rota–Baxter algebras (previously known as Baxter algebras), such as [4, 15, 16, 17, 24, 25, 26, 27, 28]. We adapt the notation $*$ to be consistent with the current convention on multiple zeta values.

We give more details of the recursive definition of the product $*$. We only need to define $\mathfrak{a} * \mathfrak{b}$ for pure tensors $\mathfrak{a} \in A^{\otimes m}$ and $\mathfrak{b} \in A^{\otimes n}$, $m, n \geq 0$, and then to extend it to a product on $\mathcal{H}_A$ by biadditivity. If $m = 0$, then $\mathfrak{a} \in \mathbf{k}$. Then we define $\mathfrak{a} * \mathfrak{b}$ to be the scalar product. Similarly if $n = 0$. In particular, the identity $\mathbf{1}$ of $\mathbf{k}$ is also the identity for $*$. So we only need to define $\mathfrak{a} * \mathfrak{b}$ when $m > 0$ and $n > 0$. Then we have $\mathfrak{a} = a_1 \bar{\otimes} \mathfrak{a}'$ and $\mathfrak{b} = b_1 \bar{\otimes} \mathfrak{b}'$ for $a_1, b_1 \in A$ and $\mathfrak{a}' \in A^{\otimes (m-1)}$, $\mathfrak{b}' \in A^{\otimes (n-1)}$. With this notation, we define $\mathfrak{a} * \mathfrak{b}$ by induction on $m + n$. When $m + n = 0$, this has been done. Assuming that $\mathfrak{a} * \mathfrak{b}$ has been defined for $0 \leq m + n \leq k$ and consider $\mathfrak{a} \in A^{\otimes m}, \mathfrak{b} \in A^{\otimes n}$ with $m + n = k + 1$. If either m or n is zero, then again we are done. If $m > 0$ and $n > 0$, then define

$$\mathfrak{a} * \mathfrak{b} = a_1 \bar{\otimes} (\mathfrak{a}' * \mathfrak{b}) + b_1 \bar{\otimes} (\mathfrak{a} * \mathfrak{b}') + (a_1 b_1) \bar{\otimes} (\mathfrak{a}' * \mathfrak{b}'). \tag{13}$$

By the induction hypothesis, the three terms on the right hand side are defined.

Without resorting to the notation $\bar{\otimes}$, the product $*$ is defined by the following recursion. First define the multiplication by $A^{\otimes 0} = \mathbf{k}$ to be the scalar product. In particular, $\mathbf{1}$ is the identity. For any $m, n \geq 1$ and $\mathfrak{a} := a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$, $\mathfrak{b} := b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$, define $a * b$ by induction on the sum $m + n$. Then $m + n \geq 2$. When $m + n = 2$, we have $a = a_1$ and $b = b_1$. Define

$$a * b = a_1 \otimes b_1 + b_1 \otimes a_1 + a_1 b_1. \tag{14}$$

Assume that $\mathfrak{a}*\mathfrak{b}$ has been defined for $m+n \geq k \geq 2$ and consider $\mathfrak{a}$ and $\mathfrak{b}$ with $m+n = k+1$. Then $m+n \geq 3$ and so at least one of m and n is greater than 1. Then we define

$$\begin{aligned} \mathfrak{a}*\mathfrak{b} &= a_1 \blacksquare b_1 \blacksquare \cdots \blacksquare b_n + b_1 \blacksquare \big(a_1 * (b_2 \blacksquare \cdots \blacksquare b_n)\big) \\ &\quad +(a_1 b_1) \blacksquare b_2 \blacksquare \cdots \blacksquare b_n, \text{ when } m=1, n\geq 2, \end{aligned} \tag{15}$$

$$\begin{aligned} \mathfrak{a}*\mathfrak{b} &= a_1 \blacksquare \big((a_2 \blacksquare \cdots \blacksquare a_m) * b_1\big) + b_1 \blacksquare a_1 \blacksquare \cdots \blacksquare a_m \\ &\quad +(a_1 b_1) \blacksquare a_2 \blacksquare \cdots \blacksquare a_m, \text{ when } m\geq 2, n = 1, \end{aligned} \tag{16}$$

$$\begin{aligned} \mathfrak{a}*\mathfrak{b} &= a_1 \blacksquare \big((a_2 \blacksquare \cdots \blacksquare a_m) * (b_1 \blacksquare \cdots \blacksquare b_n)\big) + b_1 \blacksquare \big((a_1 \blacksquare \cdots \blacksquare a_m) * (b_2 \blacksquare \cdots \blacksquare b_n)\big) \\ &\qquad +(a_1, b_1)\big((a_2 \blacksquare \cdots \blacksquare a_m) * (b_2 \blacksquare \cdots \blacksquare b_n)\big), \text{ when } m, n\geq 2. \end{aligned} \tag{17}$$

Here the products by $*$ on the right hand side of each equation are well-defined by the induction hypothesis.

To relate to the quasi-shuffle algebra of Hoffman and to the algebra we will use later, we introduce the following concept.

Definition 3.3. A semigroup X is called a **filtered semigroup** if there is an increasing sequence $X_n \subseteq X$, $n \geq 1$, of subsets of X such that $X_m X_n \subseteq X_{m+n}$, $m, n \geq 1$.

Let $A_X = \mathbf{k}\, X$ be the semi-group ring of X with coefficients in $\mathbf{k}$. A special case is when X is **locally finite**, i.e., X has a grading given by the disjoint union $X = \coprod_{k\geq 1} X^{(k)}$ with each $X^{(k)}$ finite and such that $X^{(m)}X^{(n)} \subseteq X^{(m+n)}$, $m, n \geq 1$.

The following theorem is a simple generalization of [32, Theorem 3.1, 3.2] and [15, Theorem 2.5] where X is a locally finite set. The same proof works in our generality with the notation $\bar{\blacksquare}$.

Theorem 3.4. *Let A be a commutative nonunitary $\mathbf{k}$-algebra.*

(1) *The product $*$ equips $\mathcal{H}_A$ with the structure of a commutative unitary filtered $\mathbf{k}$-algebra.*
(2) *The product $*$ coincides with the mixable shuffle product in* [27].
(3) *Equip $\mathcal{H}_A$ with the deconcatenation coproduct*

$$\Delta : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_A,$$

$$\begin{aligned} \Delta(a_1 \blacksquare \cdots \blacksquare a_m) &= 1\otimes(a_1 \blacksquare \cdots \blacksquare a_m) + \sum_{i=1}^{m-1} (a_1 \blacksquare \cdots \blacksquare a_i)\otimes(a_{i+1} \blacksquare \cdots \blacksquare a_m) \\ &\quad +(a_1 \blacksquare \cdots \blacksquare a_m)\otimes 1 \end{aligned} \tag{18}$$

and the projection counit

$$\varepsilon : \mathcal{H}_A \to \mathbf{k}$$

onto the direct summand $\mathbf{k} = A^{\blacksquare 0} \subseteq \mathcal{H}_A$. Then $\mathcal{H}_A$ is a commutative cocommutative connected filtered Hopf algebra.

Proof. (1) The same proof for [32, Theorem 2.1] applies. Just replace the length of a word by the tensor degree of a tensor. Alternatively, one can use item (2) which is proved independently from item (1) and use the fact that the mixable shuffle product is commutative and associative [27, Theorem 3.5].

(2) The proof is the same as that for [15, Theorem 2.5] since the proof there only requires that A is a commutative $\mathbf{k}$-algebra.

(3) The proof that $\mathcal{H}_A$ is a bialgebra is the same as that for [32, Theorem 3.1] by considering tensor products in place of word concatenations. Since the result is essential for our later applications, we provide some details.

The coassociativity is clear. So to prove that $\mathcal{H}_A$ is a bialgebra, we only need to show that ε and Δ are algebra homomorphisms. For ε, this is clear. For Δ, we prove

$$\Delta(\mathfrak{a}) * \Delta(\mathfrak{b}) = \Delta(\mathfrak{a} * \mathfrak{b}) \tag{19}$$

for pure tensors $\mathfrak{a} \in A^{\otimes m}$ and $\mathfrak{b} \in A^{\otimes n}$ by induction on $m+n \geq 0$. If $m+n \leq 1$, then at least one of $\mathfrak{a}$ and $\mathfrak{b}$ is a scalar in $\mathbf{k}$ and Eq. (19) is clear. Suppose the equation has been proved for $0 \leq m+n \leq k$ and consider pure tensors $\mathfrak{a}$ and $\mathfrak{b}$ with $m+n = k+1$. Then Eq. (19) is again clear if either one of m or n is zero. So we can assume $m>0$ and $n>0$. Then we have $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$ with $a_1, b_1 \in A$ and $\mathfrak{a}' \in A^{\otimes (m-1)}$, $\mathfrak{b}' \in A^{\otimes (n-1)}$. Let

$$\Delta(\mathfrak{a}') = \sum_{(\mathfrak{a}')} \mathfrak{a}'_{(1)} \otimes \mathfrak{a}'_{(2)}, \qquad \Delta(\mathfrak{b}') = \sum_{(\mathfrak{b}')} \mathfrak{b}'_{(1)} \otimes \mathfrak{b}'_{(2)}$$

by Sweedler's notation. Then by Eq. (18), we have

$$\Delta(\mathfrak{a}) = 1 \otimes \mathfrak{a} + \sum_{(\mathfrak{a}')} (a_1 \otimes \mathfrak{a}'_{(1)}) \otimes \mathfrak{a}'_{(2)}, \qquad \Delta(\mathfrak{b}) = 1 \otimes \mathfrak{b} + \sum_{(\mathfrak{b}')} (b_1 \otimes \mathfrak{b}'_{(1)}) \otimes \mathfrak{b}'_{(2)}. \tag{20}$$

That is

$$\Delta(\mathfrak{a}) = 1 \otimes \mathfrak{a} + a_1 \otimes \Delta(\mathfrak{a}'), \qquad \Delta(\mathfrak{b}) = 1 \otimes \mathfrak{b} + b_1 \otimes \Delta(\mathfrak{b}'). \tag{21}$$

Here $a_1 \otimes \Delta(\mathfrak{a}')$ means a_1 is multiplied with the first tensor factors of $\Delta(\mathfrak{a}')$. Thus

$$\begin{aligned}
\Delta(\mathfrak{a}) * \Delta(\mathfrak{b}) \;=\;& \sum_{(\mathfrak{a}')} \big((a_1 \otimes \mathfrak{a}'_{(1)}) * (b_1 \otimes \mathfrak{b}'_{(1)})\big) \otimes \sum_{(\mathfrak{b}')} \big(\mathfrak{a}'_{(2)} * \mathfrak{b}'_{(2)}\big) \\
& + \sum_{(\mathfrak{a}')} (a_1 \otimes \mathfrak{a}'_{(1)}) \otimes (\mathfrak{a}'_{(2)} * \mathfrak{b}) + \sum_{(\mathfrak{b}')} (b_1 \otimes \mathfrak{b}'_{(1)}) \otimes (\mathfrak{a} * \mathfrak{b}'_{(2)}) + 1 \otimes (\mathfrak{a} * \mathfrak{b}).
\end{aligned}$$

Applying Eq. (13) to the first and fourth terms we get

$$\begin{aligned}
& \sum_{(\mathfrak{a}')} \big(a_1 \otimes (\mathfrak{a}'_{(1)} * (b_1 \otimes \mathfrak{b}'_{(1)})) + b_1 \otimes ((a_1 \otimes \mathfrak{a}'_{(1)}) * \mathfrak{b}'_{(1)}) + (a_1 b_1) \otimes (\mathfrak{a}'_{(1)} * \mathfrak{b}'_{(1)})\big) \otimes \sum_{(\mathfrak{b}')} \big(\mathfrak{a}'_{(2)} * \mathfrak{b}'_{(2)}\big) \\
& + \sum_{(\mathfrak{a}')} (a_1 \otimes \mathfrak{a}'_{(1)}) \otimes (\mathfrak{a}'_{(2)} * \mathfrak{b}) + \sum_{(\mathfrak{b}')} (b_1 \otimes \mathfrak{b}'_{(1)}) \otimes (\mathfrak{a} * \mathfrak{b}'_{(2)}) \\
& + \mathbf{1} \otimes (a_1 \otimes (\mathfrak{a}' * \mathfrak{b}) + b_1 \otimes (\mathfrak{a} * \mathfrak{b}') + (a_1 b_1) \otimes (\mathfrak{a}' * \mathfrak{b}')).
\end{aligned}$$

On the other hand, by Eq. (13), Eq. (21) and the induction hypothesis, we have

$$\begin{aligned}
\Delta(\mathfrak{a} * \mathfrak{b}) \;&=\; \Delta\big(a_1 \otimes (\mathfrak{a}' * \mathfrak{b}) + b_1 \otimes (\mathfrak{a} * \mathfrak{b}') + (a_1 b_1) \otimes (\mathfrak{a}' * \mathfrak{b})\big) \\
&= \mathbf{1} \otimes \big(a_1 \otimes (\mathfrak{a}' * \mathfrak{b})\big) + a_1 \otimes \Delta(\mathfrak{a}' * \mathfrak{b}) + \mathbf{1} \otimes \big(b_1 \otimes (\mathfrak{a} * \mathfrak{b}')\big) \\
&\quad + b_1 \otimes \Delta(\mathfrak{a} * \mathfrak{b}') + \mathbf{1} \otimes \big((a_1 b_1) \otimes (\mathfrak{a}' * \mathfrak{b})\big) + (a_1 b_1) \otimes \Delta(\mathfrak{a}' * \mathfrak{b}) \\
&= \mathbf{1} \otimes \big(a_1 \otimes (\mathfrak{a}' * \mathfrak{b})\big) + a_1 \otimes \big(\Delta(\mathfrak{a}') * \Delta(\mathfrak{b})\big) + \mathbf{1} \otimes \big(b_1 \otimes (\mathfrak{a} * \mathfrak{b}')\big) \\
&\quad + b_1 \otimes \big(\Delta(\mathfrak{a}) * \Delta(\mathfrak{b}')\big) + \mathbf{1} \otimes \big((a_1 b_1) \otimes (\mathfrak{a}' * \mathfrak{b})\big) + (a_1 b_1) \otimes \big(\Delta(\mathfrak{a}') * \Delta(\mathfrak{b}')\big)
\end{aligned}$$

Then by Eq. (20), the right hand side is

$$\begin{aligned}
& a_1 \overline{\blacksquare} \big((\sum_{(\mathfrak{a}')} \mathfrak{a}'_{(1)} \otimes \mathfrak{a}'_{(2)}) * (\mathbf{1} \otimes \mathfrak{b} + \sum_{(\mathfrak{b}')} (b_1 \blacksquare \mathfrak{b}'_{(1)}) \otimes \mathfrak{b}'_{(2)}) \big) \\
& + b_1 \overline{\blacksquare} \big((\mathbf{1} \otimes \mathfrak{a} + \sum_{(\mathfrak{a}')} (a_1 \blacksquare (\mathfrak{a}'_{(1)} \otimes \mathfrak{a}'_{(2)})) * \sum_{(\mathfrak{b}')} \mathfrak{b}'_{(1)} \otimes \mathfrak{b}'_{(2)} \big) \\
& + (a_1 b_1) \overline{\blacksquare} \big(\sum_{(\mathfrak{a}')} (\mathfrak{a}'_{(1)} \otimes \mathfrak{a}'_{(2)}) * \sum_{(\mathfrak{b}')} (\mathfrak{b}'_{(1)} \otimes \mathfrak{b}'_{(2)}) \big) \\
& + \mathbf{1} \otimes (a_1 \overline{\blacksquare} (\mathfrak{a}' * \mathfrak{b}) + b_1 \overline{\blacksquare} (\mathfrak{a} * \mathfrak{b}') + (a_1 b_1) \overline{\blacksquare} (\mathfrak{a}' * \mathfrak{b}')).
\end{aligned}$$

This agrees with $\Delta(\mathfrak{a}) * \Delta(\mathfrak{b})$.

Therefore $\mathcal{H}_A$ is a bialgebra. By the definition of $*$ and Δ, $\mathcal{H}_A$ is connected filtered. Then $\mathcal{H}_A$ is automatically a Hopf algebra by [21, Proposition 5.3], for example. □

We next give a differential version of quasi-shuffle Hopf algebras.

Theorem 3.5. *Let (A, d) be a commutative differential algebra. Extending d to $\mathcal{H}_A = \oplus_{k\geq 0} A^{\blacksquare k}$ by defining, for $\mathfrak{a} := a_1 \blacksquare \cdots \blacksquare a_k \in A^{\blacksquare k}$,*

$$d(\mathfrak{a}) = \sum_{i=1}^{k} a_{i,1} \blacksquare \cdots \blacksquare a_{i,k}, \tag{22}$$

where

$$a_{i,j} = \begin{cases} a_j, & j \neq i, \\ d(a_j), & j = i. \end{cases}$$

Then $(\mathcal{H}_A, d)$ is a differential Hopf algebra.

Proof. We have

$$\begin{aligned}
\Delta \circ d(\mathfrak{a}) &= \sum_{i=1}^{k} \Delta(a_{i,1} \blacksquare \cdots \blacksquare a_{i,k}) \\
&= \sum_{i=1}^{k} \sum_{j=0}^{k} (a_{i,1} \blacksquare \cdots \blacksquare a_{i,j}) \otimes (a_{i,j+1} \blacksquare \cdots \blacksquare a_{i,k}).
\end{aligned}$$

Here we have use the convention that if $j = 0$, then $a_{i,1} \blacksquare \cdots \blacksquare a_{i,j} = \mathbf{1} \in \mathbf{k}$ and if $j = k$, then $a_{i,j+1} \blacksquare \cdots \blacksquare a_{i,k} = \mathbf{1} \in \mathbf{k}$.

With the same convention, we also have

$$\begin{aligned}
d \circ \Delta(\mathfrak{a}) &= d(\sum_{j=0}^{k} (a_1 \blacksquare \cdots \blacksquare a_j) \otimes (a_{j+1} \blacksquare \cdots \blacksquare a_k)) \\
&= \sum_{j=0}^{k} \big(d(a_1 \blacksquare \cdots \blacksquare a_j) \otimes (a_{j+1} \blacksquare \cdots \blacksquare a_k) + (a_1 \blacksquare \cdots \blacksquare a_j) \otimes d(a_{j+1} \blacksquare \cdots \blacksquare a_k) \big) \\
&= \sum_{j=0}^{k} \big(\sum_{i=1}^{j} (a_{i,1} \blacksquare \cdots \blacksquare a_{i,j}) \otimes (a_{i,j+1} \blacksquare \cdots \blacksquare a_{i,k})
\end{aligned}$$

$$+\sum_{i=j+1}^{k}(a_{i,1}\,\rule{0.5em}{0.8em}\cdots\rule{0.5em}{0.8em}\,a_{i,j})\otimes(a_{i,j+1}\,\rule{0.5em}{0.8em}\cdots\rule{0.5em}{0.8em}\,a_{i,k})\Big)$$
$$= \Delta\circ d(\mathfrak{a}).$$

□

4. Renormalization of multiple zeta values and the differential structure

We now apply the general setup in § 3 to the study of multiple zeta values. In order to apply the Differential Algebraic Birkhoff Decomposition in Theorem 3.2, we need to construct a commutative connected differential Hopf algebra $(\mathcal{H}, d)$, a commutative differential Rota-Baxter algebra (R, P, ∂) of weight -1 and an algebra homomorphism $\phi : \mathcal{H} \to R$. We will provide these in § 4.1 which is modified with a differential twist from [29] for which we refer the reader for some of the details. We then use this decomposition to study the regularized and renormalized MZVs.

4.1. **Differential Algebraic Birkhoff Decomposition for multiple zeta values.**

4.1.1. *Directional regularized multiple zeta values.* As mentioned in § 2, the expression

$$\zeta(\vec{s}) = \sum_{n_1>\cdots>n_k>0} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}} \tag{23}$$

makes sense for integers $s_1, \cdots, s_k$ only when $s_i > 0$ and $s_1 > 1$. For other integers, we call the expression **formal multiple zeta values** since they are only formal expressions without a real meaning. In order to make sense of such formal expressions, we define the **directional regularized multiple zeta values**

$$Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right];\varepsilon) := \sum_{n_1>\cdots>n_k>0} \frac{e^{n_1\, r_1\varepsilon}\cdots e^{n_k\, r_k\varepsilon}}{n_1^{s_1}\cdots n_k^{s_k}} \tag{24}$$

where r_i are positive real numbers that are introduced so that the space spanned by the directional regularized multiple zeta values is closed under multiplication.

$Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right];\varepsilon)$ can be defined recursively as follows. Consider the following set of functions. For $(s, r) \in \mathbb{Z}\times\mathbb{R}_{>0}$ and $(\varepsilon, x)\in\mathbb{C}\times\mathbb{R}$ with $\mathrm{Re}(\varepsilon) < 0$, define

$$f_{\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right]}(\varepsilon, x) := f(\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right];\varepsilon, x) = \frac{e^{xr\varepsilon}}{x^s}.$$

For vectors $\vec{s} = (s_1, \cdots, s_k) \in \mathbb{Z}^k$ and $\vec{r} = (r_1, \cdots, r_k) \in (\mathbb{R}_{>0})^k$, define

$$Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right];\varepsilon, x) := \sum_{n_1>\cdots>n_k>0} \frac{e^{(n_1+x)\, r_1\varepsilon}\cdots e^{(n_k+x)\, r_k\varepsilon}}{(n_1+x)^{s_1}\cdots(n_k+x)^{s_k}}. \tag{25}$$

Then $Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right];\varepsilon, x)$ is also given by the recursive definition

$$Z(\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right];\varepsilon;x) = Q(f_{\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right]}(\varepsilon;x))$$

where Q is the summation operator [17, 48]

$$Q(f)(x) = \sum_{n\geq 1} f(x+n), \tag{26}$$

and, for $\vec{s} = (s_1, \cdots, s_k) \in \mathbb{Z}^k$ and $\vec{r} = (r_1, \cdots, r_k) \in (\mathbb{R}_{>0})^k$,

$$Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right]; \varepsilon, x) = Q\big(f_{\left[\begin{smallmatrix} s_k\\ r_k\end{smallmatrix}\right]}(\varepsilon, x) Z(\left[\begin{smallmatrix} s_1, \cdots, s_{k-1}\\ r_1, \cdots, r_{k-1}\end{smallmatrix}\right]; \varepsilon, x)\big), k \geq 2.$$

These are related to the multiple Lerch functions [10, 17] which are generalizations of the multiple polylogarithms

$$\mathrm{Li}_{s_1, \cdots, s_k}(z_1, \cdots, z_k) = \sum_{n_1 > \cdots n_k > 0} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

We then have

$$Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right]; \varepsilon, 0) = Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right]; \varepsilon).$$

4.1.2. *The (differential) Hopf algebra for regularized multiple zeta values.* We next construct a Hopf algebra from the regularized MZVs to capture the algebra properties of these values.

We consider the commutative semigroup

$$\mathfrak{M} = \{f_{\left[\begin{smallmatrix} s\\ r\end{smallmatrix}\right]} \mid (s, r) \in \mathbb{Z} \times \mathbb{R}_{>0}\} \tag{27}$$

with the multiplication

$$f_{\left[\begin{smallmatrix} s\\ r\end{smallmatrix}\right]} f_{\left[\begin{smallmatrix} s'\\ r'\end{smallmatrix}\right]} = f_{\left[\begin{smallmatrix} s+s'\\ r+r'\end{smallmatrix}\right]}.$$

We similarly define the commutative semigroups

$$\mathfrak{M}_+ = \{f_{\left[\begin{smallmatrix} s\\ r\end{smallmatrix}\right]} \mid (s, r) \in \mathbb{Z}_{>0} \times \mathbb{R}_{>0}\}, \quad \mathfrak{M}_- = \{f_{\left[\begin{smallmatrix} s\\ r\end{smallmatrix}\right]} \mid (s, r) \in \mathbb{Z}_{<0} \times \mathbb{R}_{>0}\}. \tag{28}$$

For each of these semigroups $\mathfrak{N}$, let $A_{\mathfrak{N}} = \mathbf{k}\,\mathfrak{N}$ be the semigroup algebra. By Theorem 3.4,

$$\mathcal{H}_{\mathfrak{N}} := \sum_{n\geq 0} (A_{\mathfrak{N}})^{\blacksquare n} = \sum_{n \geq 0} \mathbf{k}\,\mathfrak{N}^n,$$

with the quasi-shuffle product and deconcatenation coproduct, is a connected filtered Hopf algebra. Further, for $\mathfrak{N} = \mathfrak{M}$ or $\mathfrak{M}_-$, the operator

$$d: A_{\mathfrak{N}} \to A_{\mathfrak{N}}, \ \left[\begin{smallmatrix} s\\ r\end{smallmatrix}\right] = r\left[\begin{smallmatrix} s-1\\ r\end{smallmatrix}\right]$$

is a differential operator. Thus by Theorem 3.5, $\mathcal{H}_{\mathfrak{N}}$ is a differential Hopf algebra.

4.1.3. *The Laurent series for directional regularized multiple zeta values.* Let $\mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}]$ be the algebra of convergent Laurent series, regarded as a subalgebra of the algebra of (germs of) complex valued functions meromorphic in a neighborhood of $\varepsilon = 0$. Since $\ln(-\varepsilon)$ is transcendental over $\mathbb{C}\{\{\varepsilon\}\}[\varepsilon]$ by [29, Lemma 3.1], we have

$$\mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}][\ln(-\varepsilon)] \cong \mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}][T],$$

the polynomial algebra with the variable $T = -\ln(-\varepsilon)$ and with coefficients in $\mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}]$ which embeds into the ring of Laurent series $\mathbb{C}[T][\varepsilon^{-1},\varepsilon]]$ with coefficients in $\mathbb{C}[T]$ by the remark after Lemma 3.2 in [29]. Thus we obtain an injective algebra homomorphism

$$u : \mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}][T] \to \mathbb{C}[T][\varepsilon^{-1},\varepsilon]]. \tag{29}$$

Then with the decomposition

$$\mathbb{C}[\varepsilon^{-1},\varepsilon]] = \varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}] \oplus \mathbb{C}[[\varepsilon]]$$

of $\mathbb{C}[\varepsilon^{-1},\varepsilon]]$ into a direct sum of subalgebras, we have the vector space direct sum decomposition

$$\mathbb{C}[T][\varepsilon^{-1},\varepsilon]] = \mathbb{C}[T][\varepsilon^{-1},\varepsilon]] = \varepsilon^{-1}\mathbb{C}[T][\varepsilon^{-1}] \oplus \mathbb{C}[T][[\varepsilon]]$$

of the subalgebras $\varepsilon^{-1}\mathbb{C}[T][\varepsilon^{-1}]$ and $\mathbb{C}[T][[\varepsilon]]$. Thus $\mathbb{C}[T][\varepsilon^{-1},\varepsilon]]$ is a Rota-Baxter algebra with the Rota-Baxter operator P to be the projection to $\varepsilon^{-1}\mathbb{C}[T][\varepsilon^{-1}]$:

$$P\left(\sum_{n\geq N}\alpha_k(T)\varepsilon^k\right) = \sum_{k\leq -1}\alpha_k(T)\varepsilon^k.$$

This can also be directly verified as with the case of $\mathbb{C}[\varepsilon^{-1},\varepsilon]]$ (see [12, 18]).

The following facts are easy to verify.

Proposition 4.1. (1) *Define the operator*

$$\partial : \mathbb{C}[T][\varepsilon^{-1},\varepsilon]] \to \mathbb{C}[T][\varepsilon^{-1},\varepsilon]]$$

by $\partial(\varepsilon^k) = k\varepsilon^{k-1}$ *and* $\partial(T^n) = nT^{n-1}/\varepsilon$. *Then* ∂ *is a differential operator on* $\mathbb{C}[T][\varepsilon^{-1},\varepsilon]]$.

(2) *We have*

$$\partial(Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right];\varepsilon,x)) = \sum_{i=1}^{k} r_i Z(\left[\begin{smallmatrix}\vec{s}-\vec{e}_i\\ \vec{r}\end{smallmatrix}\right];\varepsilon,x), \tag{30}$$

where $\vec{e}_i \in \mathbb{Z}^k$ *is the* i*-th unit vector.*

(3) $(\mathbb{C}[\varepsilon^{-1},\varepsilon]], P, \partial)$ *is a commutative differential Rota-Baxter algebra in the sense of Definition 3.1.*

Remark 4.2. *We note that the differential operator* ∂ *and the Rota-Baxter operator* P *do not commute in general. For example,* $\partial(P(T)) = \partial(0) = 0$*, but* $P(\partial(T)) = P(1/\varepsilon) = 1/\varepsilon$*.*

By [29, Theorem 3.3],

$$Z(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right];\varepsilon) = \sum_{n_1>\cdots>n_k>0}\frac{e^{n_1r_1\varepsilon}e^{n_2r_2\varepsilon}\cdots e^{n_kr_k\varepsilon}}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}$$

has a Laurent series expansion in $\mathbb{C}[\ln(-\varepsilon)]\{\{\varepsilon\}\}[\varepsilon^{-1}]$ for $\vec{s}\in\mathbb{Z}^k$, $\vec{r}\in\mathbb{Z}^k_{>0}$ and a Laurent series expansion in $\mathbb{C}\{\{\varepsilon\}\}[\varepsilon^{-1}]$ for $\vec{s}\in\mathbb{Z}^k_{\le 0}$, $\vec{r}\in\mathbb{Z}^k_{>0}$. Combining this with Eq. (29), we obtained an algebra homomorphism

$$\tilde{Z}:\mathcal{H}_{\mathfrak{M}}\to\mathbb{C}[T][\varepsilon^{-1},\varepsilon]],\quad \tilde{Z}(f_{\left[\substack{\vec{s}\\ \vec{r}}\right]})=u(Z(\left[\substack{\vec{s}\\ \vec{r}}\right];\varepsilon))$$

which restricts to a differential algebra homomorphism

$$\tilde{Z}:\mathcal{H}_{\mathfrak{M}^-}\to\mathbb{C}[\varepsilon^{-1},\varepsilon]].$$

Then by the Differential Algebraic Birkhoff Decomposition in Theorem 3.2, we have

Corollary 4.3. *There is the unique decomposition*

$$\tilde{Z}=\tilde{Z}_-^{-1}\star\tilde{Z}_+$$

where the map $\tilde{Z}_+:\mathcal{H}_{\mathfrak{M}}\to\mathbb{C}[T][[\varepsilon]]$ is an algebra homomorphism which restricts to a differential algebra homomorphism $\tilde{Z}_+:\mathcal{H}_{\mathfrak{M}^-}\to\mathbb{C}[[\varepsilon]]$.

By Theorem 3.8 in [29],

$$\tilde{Z}_+\big(\left[\substack{s_1,\,s_2\\ r_1,\,r_2}\right];\varepsilon\big)=(\mathrm{id}-P)\Big(\tilde{Z}\big(\left[\substack{s_1,\,s_2\\ r_1,\,r_2}\right];\varepsilon\big)-P\big(\tilde{Z}\big(\left[\substack{s_1\\ r_1}\right];\varepsilon\big)\big)\tilde{Z}\big(\left[\substack{s_2\\ r_2}\right];\varepsilon\big)\Big).\tag{31}$$

Definition 4.4. *For $\vec{s}=(s_1,\cdots,s_k)\in\mathbb{Z}^k$ and $\vec{r}=(r_1,\cdots,r_k)\in\mathbb{R}^k_{>0}$, define the* **renormalized directional multiple zeta value (MZV)**

$$\zeta\big(\left[\substack{\vec{s}\\ \vec{r}}\right]\big)=\lim_{\varepsilon\to 0}\tilde{Z}_+\big(\left[\substack{\vec{s}\\ \vec{r}}\right];\varepsilon\big).\tag{32}$$

The definition makes sense because of Corollary 4.3. Furthermore, since $\tilde{Z}_+$ is an algebra homomorphism, we have the quasi-shuffle relation

$$\zeta\big(\left[\substack{\vec{s}\\ \vec{r}}\right]\big)\zeta\big(\left[\substack{\vec{s}'\\ \vec{r}'}\right]\big)=\zeta\big(\left[\substack{\vec{s}\\ \vec{r}}\right]*\left[\substack{\vec{s}'\\ \vec{r}'}\right]\big)$$

meaning if

$$\left[\substack{\vec{s}\\ \vec{r}}\right]*\left[\substack{\vec{s}'\\ \vec{t}'}\right]=\sum_{\left[\substack{\vec{s}''\\ \vec{t}''}\right]}\left[\substack{\vec{s}''\\ \vec{t}''}\right]$$

is the quasi-shuffle product in $\mathcal{H}_{\mathfrak{M}}$ defined in Eq. (13) – (17), then

$$\zeta\big(\left[\substack{\vec{s}\\ \vec{r}}\right]\big)\zeta\big(\left[\substack{\vec{s}'\\ \vec{r}'}\right]\big)=\sum_{\left[\substack{\vec{s}''\\ \vec{t}''}\right]}\zeta\big(\left[\substack{\vec{s}''\\ \vec{t}''}\right]\big).$$

Definition 4.5. *For $\vec{s}\in\mathbb{Z}^k_{>0}\cup\mathbb{Z}^k_{\le 0}$, define*

$$\bar{\zeta}(\vec{s})=\lim_{\delta\to 0^+}\zeta\big(\left[\substack{\vec{s}\\ |\vec{s}|+\delta}\right]\big),\tag{33}$$

where, for $\vec{s}=(s_1,\cdots,s_k)$ and $\delta\in\mathbb{R}_{>0}$, we denote $|\vec{s}|=(|s_1|,\cdots,|s_k|)$ and $|\vec{s}|+\delta=(|s_1|+\delta,\cdots,|s_k|+\delta)$. These $\bar{\zeta}(\vec{s})$ are called the **renormalized MZVs** *of the multiple zeta function $\zeta(u_1,\cdots,u_k)$ at $\vec{s}$.*

We show in [29, Theorem4.2] that our renormalized MZVs are well-defined and are compatible with the known MZVs defined by either convergence, analytic continuation [46] or the Ihara-Kaneko-Zagier regularization [33]. It is also proved that the renormalized MZVs satisfy the quasi-shuffle relation.

We next show that our examples in Section 2 follows from the above general set up. Let Σ_k denote the symmetric group on k letters. For $\sigma \in \Sigma_k$ and $\vec{r} = (r_1, \cdots, r_k)$, denote $\sigma(\vec{r}) = (r_{\sigma(1)}, \cdots, r_{\sigma(k)})$ and $f(\vec{r})^{(\Sigma_k)} = \sum_{\sigma\in\Sigma_k} f(\sigma(\vec{r}))$.

Proposition 4.6. ([29, Proposition 4.9]) *For any $k \geq 1$, $\zeta\big(\big[{\vec{0}_k \atop \vec{r}}\big]\big)^{(\Sigma_k)}$ is independent of the choice of $\vec{r} \in \mathbb{R}^k_{>0}$ and*

$$\bar{\zeta}(\vec{0}_k) = \frac{1}{k!}\zeta\big(\big[{\vec{0}_k \atop \vec{r}}\big]\big)^{(\Sigma_k)}.$$

Remark 4.7. Taking $k = 2$ and $\vec{r} = (1, 1)$ in the above proposition, we have $\bar{\zeta}(0, 0) = \zeta\big(\big[{0,0 \atop 1,1}\big]\big)$. Then by Eq. (31) and Eq. (32), we obtain the renormalized value for $\zeta(0, 0)$ discussed in Section 2.

4.2. **Differential structures on renormalized multiple zeta values.** We now consider some differential properties of the renormalized MZVs. By Eq. (5),

$$\tilde{Z}(\big[{0 \atop r}\big]; \varepsilon) = -\frac{1}{r\varepsilon} + \sum_{i=0}^{\infty} \zeta(-i)\frac{(r\varepsilon)^i}{i!}.$$

So

$$\tilde{Z}_+(\big[{0 \atop r}\big]; \varepsilon) = \sum_{i=0}^{\infty} \zeta(-i)\frac{(r\varepsilon)^i}{i!} \tag{34}$$

is the generating function for $\zeta(s)$, $s \in \mathbb{Z}_{\leq 0}$. We next generalize this to multiple variables.

Theorem 4.8.

$$\tilde{Z}_+\big(\big[{\vec{0}_k \atop \vec{r}}\big]; \varepsilon\big) = \sum_{n\geq 0}\sum_{i_1+\cdots+i_k=n} \zeta\big(\big[{-(i_1,\cdots,i_k) \atop \vec{r}}\big]\big)\frac{(r_1\varepsilon)^{i_1}}{i_1!}\cdots\frac{(r_k\varepsilon)^{i_k}}{i_k!}, \tag{35}$$

where the sum runs over ordered partitions of n.

Proof. Let

$$\tilde{Z}_+(\big[{\vec{s} \atop \vec{r}}\big]; \varepsilon) = \sum_{n\geq 0} a_n \frac{\varepsilon^n}{n!}.$$

We find a formula for a_n. Of course a_n is the constant term of

$$\tilde{Z}_+^{(n)}(\big[{\vec{s} \atop \vec{r}}\big]; \varepsilon) := \frac{d^n}{d\varepsilon^n}\tilde{Z}_+(\big[{\vec{s} \atop \vec{r}}\big]; \varepsilon).$$

But since $\tilde{Z}_+$ is a differential algebra homomorphism by Corollary 4.3, we have

$$\frac{d}{d\varepsilon}\tilde{Z}_+(\big[{\vec{s} \atop \vec{r}}\big]; \varepsilon) = \sum_{i=1}^{k} r_i\tilde{Z}_+(\big[{\vec{s}-\vec{1}_i \atop \vec{r}}\big]; \varepsilon)$$

where $\vec{e}_i$ is the i-th unit vector of length k. By an inductive argument, we have in general

$$\tilde{Z}_+^{(n)}(\big[{\vec{s} \atop \vec{r}}\big];\varepsilon) = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\cdots,i_k} r_1^{i_1}\cdots r_k^{i_k}\tilde{Z}_+(\big[{\vec{s}-(i_1,\cdots,i_k) \atop \vec{r}}\big];\varepsilon)$$

where the sum runs over ordered partitions of n. Thus we have

$$\begin{aligned} a_n &= \tilde{Z}_+^{(n)}\big(\big[{\vec{s} \atop \vec{r}}\big];0\big) \\ &= \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\cdots,i_k} r_1^{i_1}\cdots r_k^{i_k}\tilde{Z}_+(\big[{\vec{s}-(i_1,\cdots,i_k) \atop \vec{r}}\big];0) \\ &= \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\cdots,i_k} r_1^{i_1}\cdots r_k^{i_k}\zeta\big(\big[{\vec{s}-(i_1,\cdots,i_k) \atop \vec{r}}\big]\big) \end{aligned}$$

Therefore we have the generating function

$$\begin{aligned} \tilde{Z}_+\big(\big[{\vec{s} \atop \vec{r}}\big]\big) &= \sum_{n\geq 0}\sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\cdots,i_k} r_1^{i_1}\cdots r_k^{i_k}\zeta\big(\big[{\vec{s}-(i_1,\cdots,i_k) \atop \vec{r}}\big]\big)\frac{\varepsilon^n}{n!} \\ &= \sum_{n\geq 0}\sum_{i_1+\cdots+i_k=n} \zeta\big(\big[{\vec{s}-(i_1,\cdots,i_k) \atop \vec{r}}\big]\big)\frac{r_1^{i_1}}{i_1!}\cdots\frac{r_k^{i_k}}{i_k!}\varepsilon^n \\ &= \sum_{n\geq 0}\sum_{i_1+\cdots+i_k=n} \zeta\big(\big[{\vec{s}-(i_1,\cdots,i_k) \atop \vec{r}}\big]\big)\frac{(r_1\varepsilon)^{i_1}}{i_1!}\cdots\frac{(r_k\varepsilon)^{i_k}}{i_k!} \end{aligned}$$

This proves Eq. (35). □

We also consider the regularized series before the renormalization. In the one variable case, we have

$$\tilde{Z}(\big[{0 \atop r}\big];\varepsilon) = -\frac{1}{r\varepsilon} + \sum_{i=0}^{\infty}\zeta(-i)\frac{(r\varepsilon)^i}{i!}$$

For the two variable case, we have

$$\tilde{Z}(\big[{0,0 \atop r_1,r_2}\big];\varepsilon) = P_0 + \sum_{n\geq 0} a_n\frac{\varepsilon^n}{n!}$$

where P_0 is the negative power part. Let us find a formula for a_n. First by n-th derivative, we have a_n =constant term of $\tilde{Z}^{(n)}(\big[{s_1,s_2 \atop r_1,r_2}\big];\varepsilon)$. But

$$\tilde{Z}^{(n)}(\big[{0,0 \atop r_1,r_2}\big];\varepsilon) = \sum_{i=0}^{n}\binom{n}{i} r_1^i r_2^{n-i}\tilde{Z}(\big[{-i,i-n \atop r_1,r_2}\big];\varepsilon),$$

and, by [29, Proposition 4.8], the constant term of $\tilde{Z}(\big[{-i,i-n \atop r_1,r_2}\big];\varepsilon)$ is

$$\zeta\big(\big[{-i,i-n \atop r_1,r_2}\big]\big) + (-1)^{i-1}\frac{1}{i+1}(\frac{r_2}{r_1})^{i+1}\zeta(-n-1).$$

So

$$\begin{aligned} a_n &= \sum_{i=0}^{n} \binom{n}{i} r_1^i r_2^{n-i} \zeta\left(\left[\begin{smallmatrix} -i, i-n \\ r_1, r_2 \end{smallmatrix}\right]\right) + \left(\sum_{i=0}^{n} \binom{n}{i} (-1)^{i-1} \frac{1}{i+1}\right) \frac{r_2^{n+1}}{r_1} \zeta(-n-1) \\ &= \sum_{i=0}^{n} \binom{n}{i} r_1^i r_2^{n-i} \zeta\left(\left[\begin{smallmatrix} -i, i-n \\ r_1, r_2 \end{smallmatrix}\right]\right) - \frac{r_2^{n+1}}{r_1} \frac{\zeta(-n-1)}{n+1} \end{aligned}$$

References

[1] S. Akiyama, S. Egami and Y. Tanigawa, Analytic continuation of multiple zeta-functions and their values at non-positive integers, *Acta Arith.* **98** (2001), 107–116.

[2] S Akiyama and Y. Tanigawa, Multiple zeta values at non-positive integers, *Ramanujan J.* **5** (2001), 327–351.

[3] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, *Nagoya Math. J.* **153** (1999), 189–209.

[4] G. E. Andrews, L. Guo, W. Keigher and K. Ono, Baxter algebras and Hopf algebras, *Trans. AMS*, **355** (2003), no. 11, 4639–4656, math.RA/0407181

[5] F. V. Atkinson, The mean value of the Riemann zeta fuction, *Acta Math.*, **81** (1949), 353-376.

[6] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.*, **10**, (1960), 731–742.

[7] J. M. Borwein, D. J. Broadhurst, D. M. Bradley, and P. Lisoněk, Special values of multiple polylogarithms, *Trans. Amer. Math. Soc.*, , (2001), no. 3, 907–941. math.CA/9910045

[8] D. J. Broadhurst and D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, *Phys. Lett. B*, **393**, (1997), no. 3-4, 403–412.

[9] J.-L. Bryliski, B. Zhang, Equivariant Todd classes for toric varieties, arXiv:math.AT/0311318.

[10] P. Cartier, Fonctions polylogarithmes, nombres polyztas et groupes pro-unipotents, *Astrisque*, **282**, (2002), 137–173, (Sm. Bourbaki no. 885).

[11] P. Cartier, On the structure of free Baxter algebras, *Adv. in Math.*, **9**, (1972), 253–265.

[12] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, *Comm. Math. Phys.*, **210**, (2000), no. 1, 249–273. hep-th/9912092

[13] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The β-function, diffeomorphisms and the renormalization group, *Comm. Math. Phys.*, **216**, (2001), no. 1, 215–241. hep-th/0003188

[14] A. Connes and M. Marcolli, *From Physics to Number Theory via Noncommutative Geometry, Part II: Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory,* in "Frontiers in Number Theory, Physics, and Geometry, II" pp.617–713, Springer Verlag, 2006, arXiv:hep-th/0411114.

[15] K. Ebrahimi-Fard and L. Guo Mixable Shuffles, Quasi-shuffles and Hopf Algebras, to appear in *J. Alg. Combinatorics*, math.RA/0506418.

[16] K. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras and dendriform algebras, to appear in *J. Pure Appl. Algebra*, math.RA/0503647.

[17] K. Ebrahimi-Fard and L. Guo, Multiple zeta values and Rota-Baxter algebras, preprint, arXiv:math.NT/0601558.

[18] K. Ebrahimi-Fard, L. Guo and D. Kreimer, Integrable Renormalization I: the Ladder case, *J. Math. Phys.*, **45**, (2004), 3758–3769, hep-th/0402095

[19] K. Ebrahimi-Fard, L. Guo and D. Kreimer, Integrable Renormalization II: the General case, Annales Henri Poincaré, **6**, (2005), 369–395. hep-th/0403118

[20] K. Ebrahimi-Fard, L. Guo and D. Kreimer, Spitzer's Identity and the Algebraic Birkhoff Decomposition in pQFT, *J. Phys. A: Math. Gen.*, **37**, (2004) 11037–11052. hep-th/0407082

[21] H. Figueroa, J. M. Gracia-Bondía, Combinatorial Hopf algebras in quantum field theory I, *Reviews of Mathematical Physics*, **17**, 881–976, (2005), arXiv:hep-th/0408145

[22] A. G. Goncharov, Periods and mixed motives, preprint: Feb. 2002, math.AG/0202154.

[23] A. Goncharov and Y. Manin, Multiple ζ-motives and moduli spaces $\overline{\mathcal{M}}_{0,n}$, *Comp. Math.* **140** (2004), 1–14.
[24] L. Guo, Baxter algebra and differential algebra, in: “Differential Algebra and Related Topics”, World Scientific Publishing Company, (2002), 281–305.
[25] L. Guo, Baxter algebras and the umbral calculus, *Adv. in Appl. Math.*, **27**, (2001), no. 2-3, 405–426.
[26] L. Guo, Baxter algebras, Stirling numbers and partitions, *J. Algebra Appl.*, **4**, (2005), no. 2, 153–164, math.AC/0402348
[27] L. Guo and W. Keigher, Free Baxter algebras and shuffle products, *Adv. in Math.,* **150**, (2000), 117–149.
[28] L. Guo and W. Keigher, On Baxter algebras: completions and the internal construction, *Adv. in Math.*, **151**, (2000), 101–127.
[29] L. Guo and B. Zhang, Renormalization of multiple zeta values, arXiv:math.NT/0606076.
[30] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.*, **152** (1992), no. 2, 275–290.
[31] M. E. Hoffman, The algebra of multiple harmonic series, *J. Algebra*, **194**, no. 2, (1997), 477–495.
[32] M. E. Hoffman, Quasi-shuffle products, *J. Algebraic Combin.*, **11** (2000), 49–68.
[33] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, *Compos. Math.* **142** (2006), 307–338.
[34] D. Kreimer, Knots and Feynman diagrams, Cambridge Lecture Notes in Physics, **13**. Cambridge University Press, Cambridge, (2000). xii+259 pp.
[35] D. Manchon, Hopf algebras, from basics to applications to renormalization, Comptes-rendus des Rencontres mathmématiques de Glanon 2001, arXiv:math.QA/0408405
[36] D. Manchon and S. Paycha, Shuffle relations for regularized integrals of symbols, *Comm. Math. Phys.* **270** (2007), 13–51, arXiv:math-ph/0510067.
[37] D. Manchon and S. Paycha, Renormalized Chen integrals for symbols on $\mathbb{R}^n$ and renormlized polyzeta functions, arXiv:math.NT/0604562.
[38] K. Matsumoto, Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, *Nagoya Math. J.*, **172** (2003), 59–102.
[39] K. Matsumoto, The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I, *J. Number Theory*, **101** (2003), 223–243.
[40] F. Menous, Formulas for Birkhoff-(Rota-Baxter) decompositions related to connected bialgebras, preprint.
[41] G. Racinet, Doubles mélanges des polylogarithmes multiples aux racines de l’unité, *Pub. Math. IHES*, **95** (2002), 185-231.
[42] G.-C. Rota, Baxter algebras and combinatorial identities I & II, *Bull. Amer. Math. Soc.*, **75**, (1969), 325–329, 330–334.
[43] G.-C. Rota, Baxter operators, an introduction, In: “Gian-Carlo Rota on Combinatorics, Introductory Papers and Commentaries”, Joseph P.S. Kung, Editor, Birkhäuser, Boston, 1995.
[44] T. Terasoma, Mixed Tate motives and multiple zeta values, *Invent. Math.*, **149**, (2002), no. 2, 339–369. arXiv: math.AG/0104231
[45] D. Zagier, Values of zeta functions and their applications, First European Congress of Mathematics, Vol. II (Paris, 1992), 497–512, *Progr. Math.*, **120**, Birkhuser, Basel, 1994
[46] J. Zhao, Analytic continuation of multiple zeta functions. *Proc. Amer. Math. Soc.* **128** (2000), 1275-1283.
[47] J. Zhao, Renormalization of multiple q-zeta values, to appear in *Acta Math. Sinica*, arXiv:math/0612093.
[48] W. Zudelin, Algebraic relations for multiple zeta values, *Uspekhi Mat. Nauk*, **58**, no.1 , 3–32, 2003, translation in *Russian Math. Survey*, **58**, vol. 1, 1–29,(2003).

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102
E-mail address: liguo@newark.rutgers.edu

Yangtze Center of Mathematics, Sichuan University, Chengdu, 610064, P. R. China
E-mail address: binzhang@mpim-bonn.mpg.de