BAXTER ALGEBRAS AND DIFFERENTIAL ALGEBRAS

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In memory of Professor Chuan-Yan Hsiong

A Baxter algebra is a commutative algebra A that carries a generalized integral operator. In the first part of this paper we review past work of Baxter, Miller, Rota and Cartier in this area and explain more recent work on explicit constructions of free Baxter algebras that extended the constructions of Rota and Cartier. In the second part of the paper we will use these explicit constructions to relate Baxter algebras to Hopf algebras and give applications of Baxter algebras to the umbral calculus in combinatorics.

0 Introduction

This is a survey article on Baxter algebras, with emphasis on free Baxter algebras and their applications in probability theory, Hopf algebra and umbral calculus. This article can be read in conjunction with the excellent introductory article of Rota [28]. See also [31,29].

A Baxter algebra is a commutative algebra R with a linear operator P that satisfies the Baxter identity

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy), \forall x, y \in R, \tag{0.1}$$

where λ is a pre-assigned constant, called the weight, from the base ring of R.

0.1 Relation with differential algebra

The theory of Baxter algebras is related to differential algebra just as the integral analysis is related to the differential analysis.

Differential algebra originated from differential equations, while the study of Baxter algebras originated from the algebraic study by Baxter[5] on integral equations arising from fluctuation theory in probability theory. Differential algebra has provided the motivation for some of the recent studies on Baxter algebras. A Baxter algebra of weight zero is an integration algebra, i.e., an algebra with an operator that satisfies the integration by parts formula (see Example 1 below). The motivation of the recent work in [13,14,15] is to extend the beautiful theory of differential algebra [19] to integration algebras. On the other hand, the Baxter operators, regarded as a twisted family of integration operators, motivated the study of a twisted family of differential operators, generalizing the differential operator (when the weight is 0) and the difference operator (when the weight is 1).

0.2 Some history

In the 1950's and early 1960's, several spectacular results were obtained in the fluctuation theory of sums of independent random variables by Anderson [1], Baxter [4], Foata [9] and Spitzer [32]. The most important result is Spitzer's identity (see Proposition 2.16) which was applied to show that certain functionals of sums of independent random variables, such as the maximum and the number of positive partial sum, were independent of the particular distribution. In an important paper [5], Baxter deduced Spitzer's identity and several other identities from identity (0.1). This identity was further studied by Wendel [34], Kingman [18] and Atkinson [3].

Rota [27] realized the algebraic and combinatorial significance of this identity and started a systemic study of the algebraic structure of Baxter algebras. Free Baxter algebras were constructed by him [27,31] and Cartier [6]. Baxter algebras were also applied to the study of Schur functions [33,22,35], hypergeometric functions and symmetric functions, and are closely related to several areas in algebra and geometry, such as quantum groups and iterated integrals, as well as differential algebra. The two articles by Rota [28,29] include surveys in this area and further references. Rota's articles helped to revive the study of Baxter algebras in recent years: Baxter sequences in [35], free Baxter algebras in [14,15,10,11] and applications in [2,12]

Despite the close analogy between differential algebras and Baxter algebras, in particular integration algebras, relatively little is known about Baxter algebras in comparison with differential algebras. It is our hope that this article will further promote the study of Baxter algebras and related algebraic structures.

0.3 Outline

After introducing notations and examples in Section 1, we will focus on the construction of free Baxter algebras in Section 2. We will give three constructions of free Baxter algebras. We first explain Cartier's construction using brackets, followed by a similar construction using a generalization of shuffle products. These two constructions are "external" in the sense that each is a free Baxter algebra obtained without reference to any other Baxter algebra.

We then explain Rota's standard Baxter algebra which chronologically came first. Rota's construction is an "internal" construction, obtained as a Baxter subalgebra inside a naturally defined Baxter algebra whose construction traces back to Baxter [5].

In Section 3, we give two applications of Baxter algebras. We use free Baxter algebras to construct a new class of Hopf algebras, generalizing the classical divided power Hopf algebra. We then use Baxter algebras to give an interpretation and generalization of the umbral calculus. Other applications of free Baxter algebras can be found in Section 1, relating Baxter operators to integration and summation, and in Section 2, proving the famous formula of Spitzer.

We are not able to include some other work on Baxter algebras, for example on Baxter sequences and the Young tableau [33,35], and on zero divisors and chain conditions in free Baxter algebras [10,11]. We refer the interested readers to the original literature.

1 Definitions, examples and basic properties

1.1 Definitions and examples

We will only consider rings and algebras with identity in this paper. If R is the ring or algebra, the identity will be denoted by $\mathbf{1}_R$, or by 1 if there is no danger of confusion.

Definiton 1.1 Let C be a commutative ring. Fix a λ in C. A *Baxter C-algebra* (*of weight* λ) is a commutative C-algebra R together with a *Baxter operator* (*of weight* λ) on R, that is, a C-linear operator $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy),$$

for any $x, y \in R$.

Let $\mathbf{Bax}_C = \mathbf{Bax}_{C,\lambda}$ denote the category of Baxter C-algebras of weight λ in which the morphisms are algebra homomorphisms that commute with the Baxter operators.

There are many examples of Baxter algebras.

Example 1.2 (**Integration**) Let R be $\mathrm{Cont}(\mathbb{R})$, the ring of continuous functions on $\mathbb{R}$. For f in $\mathrm{Cont}(\mathbb{R})$, define $P(f) \in \mathrm{Cont}(\mathbb{R})$ by

$$P(f)(x) = \int_0^x f(t)dt,\ x \in \mathbb{R}.$$

Then $(\mathrm{Cont}(\mathbb{R}), P)$ is a Baxter algebra of weight zero.

Example 1.3 (**Divided power algebra**) This is the algebra

$$R = \bigoplus_{n\geq 0} Ce_n$$

on which the multiplication is defined by

$$e_m e_n = \binom{m+n}{m} e_{m+n},\ m,\ n \geq 0.$$

The operator $P : R \to R$, where $P(e_n) = e_{n+1},\ n \geq 0$, is a Baxter operator of weight zero.

Example 1.4 (**Hurwitz series)** Let R be

$$HC := \{(a_n) | a_n \in C, n \in \mathbb{N}\},$$

the ring of Hurwitz series [17]. The addition is defined componentwise, and the multiplication is given by $(a_n)(b_n) = (c_n)$, where $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. Define

$$P : HC \to HC,\ P((a_n)) = (a_{n-1}),\ \text{where } a_{-1} = 0.$$

Then HC is a Baxter algebra of weight 0 which is the completion of the divided power algebra.

We will return to Example 1.3 and 1.4 in Section 2.2.

Example 1.5 (**Scalar multiplication**) Let R be any C-algebra. For a given $\lambda \in C$, define

$$P_\lambda : R \to R, x \mapsto -\lambda x, \forall\, x \in R.$$

Then P_λ is a Baxter operator of weight λ on R.

Example 1.6 (**Partial sums**) This is one of the first examples of a Baxter algebra, introduced by Baxter [5]. Let A be any C-algebra. Let

$$R = \prod_{n\in\mathbb{N}_+} A = \{(a_1, a_2, \ldots) | a_n \in A,\ n \in \mathbb{N}_+\}.$$

with addition, multiplication and scalar product defined entry by entry. Define $P : R \to R$ to be the "partial sum" operator:

$$P(a_1, a_2, \ldots) = \lambda(0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots).$$

Then P is a Baxter operator of weight λ on R.

We will return to this example in Section 2.3

Example 1.7 (**Distributions [5]**) Let R be the Banach algebra of functions

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$$

where F is a function such that $||\varphi|| := \int_{-\infty}^{\infty} |dF(x)| < \infty$ and such that $F(-\infty) := \lim_{x \to -\infty} F(x)$ exists. The addition and multiplication are defined pointwise. Let $P(\varphi)(t) = \int_0^{\infty} e^{itx} dF(x) + F(0) - F(-\infty)$. Then (R, P) is a Baxter algebra of weight -1.

We will come back to this example in Proposition 2.16.

Example 1.8 This is an important example in combinatorics [31]. Let R be the ring of functions $f : \mathbb{R} \to \mathbb{R}$ with finite support in which the product is the convolution:

$$(fg)(x) := \sum_{y \in \mathbb{R}} f(y) g(x - y), \ \ x \in \mathbb{R}.$$

Define $P : R \to R$ by

$$P(f)(x) = \sum_{y \in \mathbb{R}, \max(0, y) = x} f(y), \ \ x \in \mathbb{R}.$$

Then (R, P) is a Baxter algebra of weight -1.

1.2 Integrations and summations

Define a system of polynomials $\Phi_n(x) \in \mathbb{Q}[x]$ by the generating function

$$\frac{t(e^{xt} - 1)}{e^t - 1} = \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!}.$$

Then we have

$$\Phi_n(x) = B_n(x) - B_n,$$

where $B_n(x)$ is the n-th Bernoulli polynomial, defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ ,$$

and $B_n = B_n(0)$ is the n-th Bernoulli number. It is well-known (see [16, Section 15.1]) that

$$\Phi_{n+1}(k + 1) = (n + 1) \sum_{r=1}^{k} r^n \qquad (1.9)$$

for any integer $k \geq 1$. The following property of Baxter algebras is due to Miller [21].

Proposition 1.10 *Let C be a $\mathbb{Q}$-algebra and let $R = C[t]$. Let P be a C-linear operator on R such that $P(1) = t$.*

(a) *The operator P is a Baxter operator of weight 0 if and only if $P(t^n) = \frac{1}{n+1} t^{n+1}$.*

(b) *The operator P is a Baxter operator of weight -1 if and only if $P(t^n) = \frac{1}{n+1} \Phi_{n+1}(t+1)$.*

(c) *The operator P is a Baxter operator of weight -1 if and only if $P(t^n)(k) = \sum_{r=1}^{k} r^n$ for every integer $k \geq 1$.*

Proof. (a): The only if part can be easily proved by induction. For details see the proof of Theorem 3 in [21]. The if part is obvious.

(b) is Theorem 4 in [21].

(c): Because of (b), we only need to show

$$P(t^n) = \frac{1}{n+1} \Phi_{n+1}(t+1) \Leftrightarrow P(t^n)(k) = \sum_{r=1}^{k} r^n, \forall k \geq 1.$$

($\Rightarrow$) follows from (1.9). ($\Leftarrow$) can be seen easily, say from [21, Lemma 5]. $\square$

Using this proposition and free Baxter algebras that we will construct in the next section, we will prove the following property of Baxter algebras in Section 2.2.

Proposition 1.11 *Let C be a $\mathbb{Q}$-algebra and let (R, P) be a Baxter C-algebra of weight λ. Let t be $P(1)$.*

(a) *If $\lambda = 0$, then $P(t^n) = \frac{1}{n+1} t^{n+1}$ for all $n \geq 1$.*

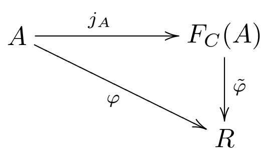
(b) *If $\lambda = -1$, then $P(t^n) = \frac{1}{n+1} \Phi_{n+1}(t+1)$.*

Because of Proposition 1.10 and 1.11, a Baxter operator of weight zero is also called an anti-derivation or integration, and a Baxter operator of weight -1 is also called a summation operator [21].

2 Free Baxter algebras

Free objects are usually defined to be generated by sets. We give the following more general definition.

Definiton 2.1 Let A be a C-algebra. A Baxter C-algebra $(F_C(A), P_A)$, together with a C-algebra homomorphism $j_A : A \to F_C(A)$, is called a *free Baxter C-algebra on A (of weight λ)*, if, for any Baxter C-algebra (R, P) of weight λ and any C-algebra homomorphism $\varphi : A \to R$, there exists a unique Baxter C-algebra homomorphism $\tilde{\varphi} : (F_C(A), P_A) \to (R, P)$ such that the diagram



commutes.

Let X be a set. One can define a free Baxter C-algebra $(F_C(X), P_X)$ on X in a similar way. This Baxter algebra is naturally isomorphic to $(F_C(C[X]), P_{C[X]})$, where $C[X]$ is the polynomial algebra over C generated by the set X.

Free Baxter algebras play a central role in the study of Baxter algebras. Even though the existence of free Baxter algebras follows from the general theory of universal algebras [20], in order to get a good understanding of free Baxter algebras, it is desirable to find concrete constructions. This is in analogy to the important role played by the ring of polynomials in the study of commutative algebra.

Free Baxter algebras on sets were first constructed by Rota [27] and Cartier [6] in the category of Baxter algebras with no identities (with some restrictions on the weight and the base ring). Two more general constructions have been obtained recently [14,15]. The first construction is in terms of *mixable shuffle products* which generalize the well-known *shuffle products* of path integrals developed by Chen [7] and Ree [23]. The second construction is modified after the construction of Rota.

The shuffle product construction of a free Baxter algebra has the advantage that its module structure and Baxter operator can be easily described. The construction of a free Baxter algebra as a standard Baxter algebra has the advantage that its multiplication is very simple. There is a canonical isomorphism between the shuffle Baxter algebra and the standard Baxter algebra. This isomorphism enables us to make use of properties of both constructions.

2.1 Free Baxter algebras of Cartier

We first recall the construction of free Baxter algebras by Cartier[6,14]. The original construction of Cartier is for free objects on a set, in the category of algebras with no identity and with weight -1. We will extend his construction to free objects in the category of algebras with identity and with arbitrary weight. We will still consider free objects on sets. In order to consider free objects on a C-algebra A, it will be more convenient to use the tensor product notation to be introduced in Section 2.2.

Let X be a set. Let λ be a fixed element in C. Let M be the free commutative semigroup with identity on X. Let $\widetilde{X}$ denote the set of symbols of the form

$$u_0 \cdot [\ ],\ u_0 \in M,$$

and

$$u_0 \cdot [u_1, \ldots, u_m], m \geq 1,\ u_0, u_1, \ldots, u_m \in M.$$

Let $\mathfrak{B}(X)$ be the free C-module on $\widetilde{X}$. Cartier gave a C-bilinear multiplication $\diamond_c$ on $\mathfrak{B}(X)$ by defining

$$\begin{aligned}
(u_0 \cdot [\ ]) \diamond_c (v_0 \cdot [\ ]) &= u_0 v_0 \cdot [\ ],\\
(u_0 \cdot [\ ]) \diamond_c (v_0 \cdot [v_1, \ldots, v_n]) &= (v_0 \cdot [v_1, \ldots, v_n]) \diamond_c (u_0 \cdot [\ ])\\
&= u_0 v_0 \cdot [v_1, \ldots, v_n],
\end{aligned}$$

and

$$\begin{aligned}
&(u_0 \cdot [u_1, \ldots, u_m]) \diamond_c (v_0 \cdot [v_1, \ldots, v_n])\\
&= \sum_{(k,P,Q)\in \overline{S}_c(m,n)} \lambda^{m+n-k} u_0 v_0 \cdot \Phi_{k,P,Q}([u_1, \ldots, u_m], [v_1 \ldots, v_n]).
\end{aligned}$$

Here $\overline{S}_c(m,n)$ is the set of triples (k, P, Q) in which k is an integer between 1 and $m+n$, P and Q are ordered subsets of $\{1, \ldots, k\}$ with the natural ordering such that $P \cup Q = \{1, \ldots, k\}$, $|P| = m$ and $|Q| = n$. For each $(k, P, Q) \in \overline{S}_c(m,n)$, $\Phi_{k,P,Q}([u_1, \ldots, u_m], [v_1, \ldots, v_n])$ is the element $[w_1, \ldots, w_k]$ in $\widetilde{X}$ defined by

$$w_j = \begin{cases} u_\alpha, & \text{if } j \text{ is the } \alpha\text{-th element in } P \text{ and } j \notin Q;\\ v_\beta, & \text{if } j \text{ is the } \beta\text{-th element in } Q \text{ and } j \notin P;\\ u_\alpha v_\beta, & \text{if } j \text{ is the } \alpha\text{-th element in } P \text{ and the } \beta\text{-th element in } Q. \end{cases}$$

Define a C-linear operator P_X^c on $\mathfrak{B}(X)$ by

$$P_X^c(u_0 \cdot [\,]) = 1 \cdot [u_0],$$
$$P_X^c(u_0 \cdot [u_1, \ldots, u_m]) = 1 \cdot [u_0, u_1, \ldots, u_m].$$

The following theorem is a modification of Theorem 1 in [6] and can be proved in the same way.

Theorem 2.2 *The pair $(\mathfrak{B}(X), P_X^c)$ is a free Baxter algebra on X of weight λ in the category* $\mathbf{Bax}_C$.

2.2 Mixable shuffle Baxter algebras

We now describe the mixable shuffle Baxter algebras. It gives another construction of free Baxter algebras. An advantage of this construction is that it is related to the well-known shuffle products. It also enables us to consider the free object on a C-algebra A.

Intuitively, to form the shuffle product, one starts with two decks of cards and puts together all possible shuffles of the two decks. Similarly, to form the mixable shuffle product, one starts with two decks of charged cards, one deck positively charged and the other negatively charged. When a shuffle of the two decks is taken, some of the adjacent pairs of cards with opposite charges are allowed to be merged into one card. When one puts all such "mixable shuffles" together, with a proper measuring of the number of pairs that have been merged, one gets the mixable shuffle product. We now give a precise description of the construction.

Mixable shuffles. For $m, n \in \mathbb{N}_+$, define the set of (m,n)*-shuffles* by

$$S(m,n) = \left\{ \sigma \in S_{m+n} \;\middle|\; \begin{array}{l} \sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(m), \\ \sigma^{-1}(m+1) < \sigma^{-1}(m+2) < \ldots < \sigma^{-1}(m+n) \end{array} \right\}.$$

Here S_{m+n} is the symmetric group on $m+n$ letters. Given an (m,n)-shuffle $\sigma \in S(m,n)$, a pair of indices $(k, k+1)$, $1 \leq k < m+n$, is called an *admissible pair* for σ if $\sigma(k) \leq m < \sigma(k+1)$. Denote $\mathcal{T}^\sigma$ for the set of admissible pairs for σ. For a subset T of $\mathcal{T}^\sigma$, call the pair (σ, T) a *mixable* (m,n)*-shuffle.* Let $|\, T \,|$ be the cardinality of T. Identify (σ, T) with σ if T is the empty set. Denote

$$\overline{S}(m,n) = \{(\sigma, T) \mid \sigma \in S(m,n),\ T \subset \mathcal{T}^\sigma\}$$

for the set of *mixable* (m,n)*-shuffles.*

Example 2.3 There are three $(2,1)$ shuffles:

$$\sigma_1 = \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix}.$$

The pair $(2,3)$ is an admissible pair for σ_1. The pair $(1,2)$ is an admissible pair for σ_2. There are no admissible pairs for σ_3.

For $A \in \mathbf{Alg}_C$ and $n \geq 0$, let $A^{\otimes n}$ be the n-th tensor power of A over C with the convention $A^{\otimes 0} = C$. For $x = x_1 \otimes \ldots \otimes x_m \in A^{\otimes m}$, $y = y_1 \otimes \ldots \otimes y_n \in A^{\otimes n}$ and $(\sigma, T) \in \overline{S}(m,n)$, the element

$$\sigma(x \otimes y) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(m+n)} \in A^{\otimes(m+n)},$$

where

$$u_k = \begin{cases} x_k, & 1 \leq k \leq m, \\ y_{k-m}, & m+1 \leq k \leq m+n, \end{cases}$$

is called a *shuffle* of x and y; the element

$$\sigma(x \otimes y; T) = u_{\sigma(1)} \widehat{\otimes} u_{\sigma(2)} \widehat{\otimes} \ldots \widehat{\otimes} u_{\sigma(m+n)} \in A^{\otimes(m+n-|T|)},$$

where for each pair $(k, k+1)$, $1 \leq k < m+n$,

$$u_{\sigma(k)} \widehat{\otimes} u_{\sigma(k+1)} = \begin{cases} u_{\sigma(k)} u_{\sigma(k+1)}, & (k,k+1) \in T \\ u_{\sigma(k)} \otimes u_{\sigma(k+1)}, & (k,k+1) \notin T, \end{cases}$$

is called a *mixable shuffle* of x and y.

Example 2.4 For $x = x_1 \otimes x_2 \in A^{\otimes 2}$ and $y = y_1 \in A$, there are three shuffles of x and y:

$$x_1 \otimes x_2 \otimes y_1,\ x_1 \otimes y_1 \otimes x_2,\ y_1 \otimes x_1 \otimes x_2.$$

For the mixable shuffles of x and y, we have in addition,

$$x_1 \otimes x_2 y_1,\ x_1 y_1 \otimes x_2.$$

Now fix $\lambda \in C$. Define, for x and y above,

$$x \diamond^+ y \ = \sum_{(\sigma,T) \in \overline{S}(m,n)} \lambda^{|T|} \sigma(x \otimes y; T) \in \bigoplus_{k \leq m+n} A^{\otimes k}. \tag{2.5}$$

Example 2.6 For x, y in our previous example,

$$x \diamond^+ y = x_1 \otimes x_2 \otimes y_1 + x_1 \otimes y_1 \otimes x_2 + y_1 \otimes x_1 \otimes x_2 + \lambda(x_1 \otimes x_2 y_1 + x_1 y_1 \otimes x_2).$$

Shuffle Baxter algebras. The operation $\diamond^+$ extends to a map

$$\diamond^+ : A^{\otimes m} \times A^{\otimes n} \to \bigoplus_{k \leq m+n} A^{\otimes k},\ m,\ n \in \mathbb{N}$$

by C-linearity. Let

$$Ш_C^+(A) = Ш_C^+(A, \lambda) = \bigoplus_{k \in \mathbb{N}} A^{\otimes k} = C \oplus A \oplus A^{\otimes 2} \oplus \ldots.$$

Extending by additivity, the binary operation $\diamond^+$ gives a C-bilinear map

$$\diamond^+ : Ш_C^+(A) \times Ш_C^+(A) \to Ш_C^+(A)$$

with the convention that

$$C \times A^{\otimes m} \to A^{\otimes m}$$

is the scalar multiplication.

Theorem 2.7 [14] *The mixable shuffle product $\diamond^+$ defines an associative, commutative binary operation on $Ш_C^+(A) = \bigoplus_{k \in \mathbb{N}} A^{\otimes k}$, making it into a C-algebra with the identity $\mathbf{1}_C \in C = A^{\otimes 0}$.*

Define $Ш_C(A) = Ш_C(A, \lambda) = A \otimes_C Ш_C^+(A)$ to be the tensor product algebra. Define a C-linear endomorphism P_A on $Ш_C(A)$ by assigning

$$P_A(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = \mathbf{1}_A \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_n,$$

for all $x_0 \otimes x_1 \otimes \ldots \otimes x_n \in A^{\otimes (n+1)}$ and extending by additivity. Let $j_A : A \to Ш_C(A)$ be the canonical inclusion map. Call $(Ш_C(A), P_A)$ the (*mixable*) *shuffle Baxter C-algebra on A of weight λ.*

For a given set X, we also let $(Ш_C(X), P_X)$ denote the shuffle Baxter C-algebra $(Ш_C(C[X]), P_{C[X]})$, called the (*mixable*) *shuffle Baxter C-algebra on X* (*of weight* λ). Let $j_X : X \to Ш_C(X)$ be the canonical inclusion map.

Theorem 2.8 [14] *The shuffle Baxter algebra $(Ш_C(A), P_A)$, together with the natural embedding j_A, is a free Baxter C-algebra on A of weight λ. Similarly, $(Ш_C(X), P_X)$, together with the natural embedding j_X, is a free Baxter C-algebra on X of weight λ.*

Relation with Cartier's construction. The mixable shuffle product construction of free Baxter algebras is canonically isomorphic to Cartier's construction. Using the notations from Section 2.1, we define a map $f : \widetilde{X} \to Ш_C(X)$ by

$$f(u_0 \cdot [\ ]) = u_0;$$
$$f(u_0 \cdot [u_1, \ldots, u_m]) = u_0 \otimes u_1 \otimes \ldots \otimes u_m,$$

and extend it by C-linearity to a C-linear map

$$f : \mathfrak{B}(X) \to Ш_C(X).$$

The same argument as for Proposition 5.1 in [14, Prop. 5.1] can be used to prove the following

Proposition 2.9 *f is an isomorphism in* $\mathbf{Bax}_C$.

Let A be a C-algebra. Using the tensor product notation in the construction of $Ш_C(A)$, one can extend Cartier's construction (Theorem 2.2) and Proposition 2.9 for free Baxter algebras on A.

Special cases. We next consider some special cases of the shuffle Baxter algebras.

Case 1: $\lambda = 0$. In this case, $Ш_C(A)$ is the usual shuffle algebra generated by the C-module A. It played a central role in the work of K.T. Chen [7] on path integrals and is related to many areas of pure and applied mathematics.

Case 2: $X = \phi$. Taking $A = C$, we get

$$Ш_C(C) = \bigoplus_{n=0}^{\infty} C^{\otimes(n+1)} = \bigoplus_{n=0}^{\infty} C\mathbf{1}^{\otimes(n+1)},$$

where $\mathbf{1}^{\otimes(n+1)} = \underbrace{\mathbf{1}_C \otimes \ldots \otimes \mathbf{1}_C}_{(n+1)-\text{factors}}$. In this case the mixable shuffle product formula (2.5) gives

Proposition 2.10 *For any $m, n \in \mathbb{N}$,*

$$\mathbf{1}^{\otimes(m+1)}\mathbf{1}^{\otimes(n+1)} = \sum_{k=0}^{m} \binom{m+n-k}{n} \binom{n}{k} \lambda^k \mathbf{1}^{\otimes(m+n+1-k)}.$$

We are now ready to prove Proposition 1.11 : By Proposition 4.2 in [12], the map

$$\begin{aligned} Ш_C(C) &\to C[x], \\ \mathbf{1}^{\otimes(n+1)} &\mapsto \frac{x(x-\lambda)\cdots(x-\lambda(n-1))}{n!},\ n \geq 0, \end{aligned}$$

is an isomorphism of C-algebras. Then P_C enables us to define a Baxter operator Q on $C[x]$ through this isomorphism and we have $Q(1) = x$. Then by Proposition 1.10, we have

$$Q(x^n) = \begin{cases} \frac{1}{n+1} x^{n+1}, & \text{if } \lambda = 0, \\ \frac{1}{n+1} \Phi_{n+1}(x+1), & \text{if } \lambda = -1. \end{cases}$$

Now let (R, P) be any Baxter C-algebra. By the universal property of $(C[x], Q)$ $(\cong (Ш_C(C), P_C))$ stated in Theorem 2.8, there is a unique homomorphism $\widetilde{\varphi} : (C[x], Q) \to (R, P)$ of Baxter algebras such that $\widetilde{\varphi}(x) = P(1)$. Let $t = P(1) = \widetilde{\varphi}(Q(1)) = \widetilde{\varphi}(x)$ and since $\Phi_{n+1}(x+1)$ has coefficients in $\mathbb{Q} \subset C$, we have

$$P(t^n) = \widetilde{\varphi}(Q(x^n)) = \widetilde{\varphi}\left(\frac{1}{n+1} x^{n+1}\right) = \frac{1}{n+1} t^{n+1}$$

when $\lambda = 0$ and

$$P(t^n) = \widetilde{\varphi}(Q(x^n)) = \widetilde{\varphi}\left(\frac{1}{n+1} \Phi_{n+1}(x+1)\right) = \frac{1}{n+1} \Phi_{n+1}(t+1)$$

when $\lambda = -1$. This proves Proposition 1.11.

Case 3: $\lambda = 0$ and $X = \phi$. Taking the "pull-back" of Cases 1 and 2, we get

$$\begin{array}{ccc} \{Ш_C(\phi, 0)\} & \xrightarrow{\subset} & \{Ш_C(X, 0) | X \in \textbf{Sets}\} \\ \cap \downarrow & & \downarrow \cap \\ \{Ш_C(\phi, \lambda) | \lambda \in C\} & \xrightarrow{\subset} & \{Ш_C(X, \lambda) | X \in \textbf{Sets}, \lambda \in C\} \end{array} \tag{2.11}$$

Thus $Ш_C(\phi, 0)$ is the *divided power algebra*

$$Ш_C(\phi, 0) = \bigoplus_{k \in \mathbb{N}} C e_k, \ e_n e_m = \binom{m+n}{m} e_{m+n}$$

in Example 1.3.

Variation: Complete shuffle Baxter algebras. We now consider the completion of $Ш_C(A)$.

Given $k \in \mathbb{N}$, $\mathrm{Fil}^k Ш_C(A) := \bigoplus_{n \geq k} A^{\otimes(n+1)}$ is a Baxter ideal of $Ш_C(A)$. Consider the infinite product of C-modules $\widehat{Ш}_C(A) = \prod_{k \in \mathbb{N}} A^{\otimes(k+1)}$. It contains $Ш_C(A)$ as a dense subset with respect to the topology defined by the filtration $\{\mathrm{Fil}^k Ш_C(A)\}$. All operations of the Baxter C-algebra $Ш_C(A)$ are continuous with respect to this topology. Hence they extend uniquely to operations on $\widehat{Ш}_C(A)$, making $\widehat{Ш}_C(A)$ a Baxter algebra of weight λ, with the Baxter operator denoted by $\widehat{P}$. It is called the *complete shuffle Baxter* *algebra on A*. It naturally contains $Ш_C(A)$ as a Baxter subalgebra and is a free object in the category of Baxter algebras that are complete with respect to a canonical filtration defined by the Baxter operator [15].

When $A = C$ and $\lambda = 0$, we have

$$\widehat{Ш}_C(C, \lambda) = \prod_{k \in \mathbb{N}} C e_k \cong HC,$$

the ring of Hurwitz series in Example 1.4.

2.3 Standard Baxter algebras

The standard Baxter algebra constructed by Rota in [27] is a free object in the category $\mathbf{Bax}_C^0$ of Baxter algebras not necessarily having an identity. It is described as a Baxter subalgebra of another Baxter algebra whose construction goes back to Baxter [5]. In Rota's construction, there are further restrictions that C be a field of characteristic zero, the free Baxter algebra obtained be on a finite set X, and the weight λ be 1. By making use of shuffle Baxter algebras, we will show that Rota's description can be modified to yield a free Baxter algebra on an algebra in the category $\mathbf{Bax}_C$ of Baxter algebras with an identity, with a mild restriction on the weight λ. We can also provide a similar construction for algebras not necessarily having an identity, and for complete Baxter algebras, but we will not explain it here. See [15].

We will first present Rota's construction, modified to give free objects in the category of Baxter algebras with identity. We then give the general construction.

The standard Baxter algebra of Rota. For details, see [27,31].

As before, let C be a commutative ring with an identity, and fix a λ in C. Let X be a given set. For each $x \in X$, let $t^{(x)}$ be a sequence $(t_1^{(x)}, \ldots, t_n^{(x)}, \ldots)$ of distinct symbols $t_n^{(x)}$. We also require that the sets $\{t_n^{(x_1)}\}_n$ and $\{t_n^{(x_2)}\}_n$ be disjoint for $x_1 \neq x_2$ in X. Denote

$$\overline{X} = \bigcup_{x \in X} \{t_n^{(x)} \mid n \in \mathbb{N}_+\}$$

and denote by $\mathfrak{A}(X)$ the ring of sequences with entries in $C[\overline{X}]$, the C-algebra of polynomials with variables in $\overline{X}$. Thus the addition, multiplication and scalar multiplication by $C[\overline{X}]$ in $\mathfrak{A}(X)$ are defined componentwise. Alternatively, for $k \in \mathbb{N}_+$, denote γ_k for the sequence $(\delta_{n,k})_n$, where $\delta_{n,k}$ is the

Kronecker delta. Then we can identify a sequence $(a_n)_n$ in $\mathfrak{A}(X)$ with a series

$$\sum_{n=1}^{\infty} a_n \gamma_n = a_1\gamma_1 + a_2\gamma_2 + \dots .$$

Then the addition, multiplication and scalar multiplication by $C[\overline{X}]$ are given termwise.

Define

$$P_X^r = P_{X,\lambda}^r : \mathfrak{A}(X) \to \mathfrak{A}(X)$$

by

$$P_X^r(a_1, a_2, a_3, \dots) = \lambda(0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots).$$

In other words, each entry of $P_X^r(a),\ a = (a_1, a_2, \dots)$, is λ times the sum of the previous entries of a. If elements in $\mathfrak{A}(X)$ are described by series $\sum_{n=1}^{\infty} a_n\gamma_n$ given above, then we simply have

$$P_X^r\left(\sum_{n=1}^{\infty} a_n\gamma_n\right) = \lambda \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n-1} a_i\right)\gamma_n.$$

It is well-known [5,27] that, for $\lambda = 1$, P_X^r defines a Baxter operator of weight 1 on $\mathfrak{A}(X)$. It follows that, for any $\lambda \in C$, P_X^r defines a Baxter operator of weight λ on $\mathfrak{A}(X)$. Hence $(\mathfrak{A}(X), P_X^r)$ is in $\mathbf{Bax}_C$.

Definiton 2.12 The *standard Baxter algebra* on X is the Baxter subalgebra $\mathfrak{S}(X)$ of $\mathfrak{A}(X)$ generated by the sequences $t^{(x)} = (t_1^{(x)}, \dots, x_n^{(x)}, \dots),\ x \in X$.

An important result of Rota [27,31] is

Theorem 2.13 $(\mathfrak{S}(X), P_X^r)$ *is a free Baxter algebra on* X *in the category* $\mathbf{Bax}_C$.

The standard Baxter algebra in general. Given $A \in \mathbf{Alg}_C$, we now give an alternative construction of a free Baxter algebra on A in the category $\mathbf{Bax}_C$.

For each $n \in \mathbb{N}_+$, denote by $A^{\otimes n}$ the n-th tensor power algebra where the tensor product is taken over C. Note that the multiplication on $A^{\otimes n}$ here is different from the multiplication on $A^{\otimes n}$ when it is regarded as a C-submodule of $\text{Ш}_C(A)$.

Consider the direct limit algebra

$$\overline{A} = \varinjlim A^{\otimes n}$$

where the transition map is given by

$$A^{\otimes n} \longrightarrow A^{\otimes (n+1)},\ x \mapsto x \otimes \mathbf{1}_A.$$

Let $\mathfrak{A}(A)$ be the set of sequences with entries in $\overline{A}$. Thus we have

$$\mathfrak{A}(A) = \prod_{n=1}^{\infty} \overline{A}\gamma_n = \left\{ \sum_{n=1}^{\infty} a_n\gamma_n,\ a_n \in \overline{A} \right\}.$$

Define addition, multiplication and scalar multiplication on $\mathfrak{A}(A)$ componentwise, making $\mathfrak{A}(A)$ into a $\overline{A}$-algebra, with the sequence $(1, 1, \ldots)$ as the identity. Define

$$P_A^r = P_{A,\lambda}^r : \mathfrak{A}(A) \to \mathfrak{A}(A)$$

by

$$P_A^r(a_1, a_2, a_3, \ldots) = \lambda(0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots).$$

Then $(\mathfrak{A}(A), P_A^r)$ is in $\mathbf{Bax}_C$. For each $a \in A$, define $t^{(a)} = (t_k^{(a)})_k$ in $\mathfrak{A}(A)$ by

$$t_k^{(a)} = \otimes_{i=1}^k a_i (= \otimes_{i=1}^{\infty} a_i),\ a_i = \begin{cases} a,\ i = k\,, \\ 1,\ i \neq k\,. \end{cases}$$

Definiton 2.14 The *standard Baxter algebra* on A is the Baxter subalgebra $\mathfrak{S}(A)$ of $\mathfrak{A}(A)$ generated by the sequences $t^{(a)} = (t_1^{(a)}, \ldots, t_n^{(a)}, \ldots),\ a \in A$.

Since $Ш_C(A)$ is a free Baxter algebra on A, the C-algebra morphism

$$A \to \mathfrak{A}(A), a \mapsto t^{(a)}$$

extends uniquely to a morphism in $\mathbf{Bax}_C$

$$\Phi : Ш_C(A) \to \mathfrak{A}(A).$$

Theorem 2.15 [15] *Assume that $\lambda \in C$ is not a zero divisor in $\overline{A}$. The morphism in* $\mathbf{Bax}_C$

$$\Phi : Ш_C(A) \to \mathfrak{S}(A)$$

induced by sending $a \in A$ to $t^{(a)} = (t_1^{(a)}, \ldots, t_n^{(a)}, \ldots)$ is an isomorphism.

Consequently, when λ is not a zero divisor in $\overline{A}$, $(\mathfrak{S}(A), P_A^r)$ is a free Baxter algebra on A in the category $\mathbf{Bax}_C$.

Spitzer's identity. As an application of the standard Baxter algebra, we recall the proof of Spitzer's identity by Rota [27,31]. Spitzer's identity is regarded as a remarkable stepping stone in the theory of sums of independent random variables and motivates of Baxter's identity. For other proofs of Spitzer's identity, see [5,18,34,3,6]. We first present an algebraic formulation.

Proposition 2.16 [31] *Let C be a $\mathbb{Q}$-algebra. Let (R, P) be a Baxter C-algebra of weight 1. Then for $b \in R$, we have*

$$\exp\left(-P(\log(1+tb)^{-1})\right) = \sum_{n=0}^{\infty} t^n (Pb)^{[n]} \tag{2.17}$$

in the ring of power series $R[[t]]$. Here

$$(Pb)^{[n]} = \underbrace{P(b(P(b\ldots(Pb)\ldots)))}_{n\text{-iteration}}$$

with the convention that $(Pb)^{[1]} = P(b)$ and $(Pb)^{[0]} = 1$.

Proof. First let $x = (x_1, x_2, \ldots)$ where $x_i,\ i \geq 1$, are symbols and let $X = \{x\}$. Consider the standard Baxter algebra $\mathfrak{S}(X)$. It is easy to verify that

$$(Px)^{[n]} = (0, e_n(x_1), e_n(x_1, x_2), e_n(x_1, x_2, x_3), \ldots)$$

where $e_n(x_1, \ldots, x_m)$ is the elementary symmetric function of degree n in the variables $x_1, \ldots, x_m$. By definition,

$$P(x^k) = (0, x_1^k, x_1^k + x_2^k, x_1^k + x_2^k + x_3^k, \ldots, p_k(x_1, \ldots, x_m), \ldots),$$

where $p_k(x_1, \ldots, x_m) = x_1^k + x_2^k + \ldots, x_m^k$ is the power sum symmetric function of degree k in the variables $x_1, \ldots, x_m$. These two classes of symmetric functions are related by the well-known Waring's formula [31]

$$\exp\left(-\sum_{k=1}^{\infty}(-1)^k t^k p_k(x_1, \ldots, x_m)/k\right) = \sum_{n=0}^{\infty} e_n(x_1, \ldots, x_m) t^n,\ \forall\ m \geq 1.$$

This proves

$$\exp\left(-P_X^r(\log(1+tx)^{-1})\right) = \sum_{n=0}^{\infty} t^n (P_X^r x)^{[n]}. \tag{2.18}$$

Next let (R, P) be any Baxter C-algebra and let b be any element in R. By the universal property of the free Baxter algebra $(\mathfrak{S}, P_X^r)$, there is a unique Baxter algebra homomorphism $\tilde{\varphi} : \mathfrak{S} \to R$ such that $\tilde{\varphi}(x) = b$. Since all the coefficients in the expansion of $\log(1+u)$ and $\exp(u)$ are rational, and $\tilde{\varphi} \circ P_X^r = P \circ \tilde{\varphi}$, applying $\tilde{\varphi}$ to (2.18) gives the desired equation. □

We can now specialize to the original identity of Spitzer, following Baxter [5] and Rota [31]. Consider the Baxter algebra (R, P) in Example 1.7. Let $\{X_k\}$ be a sequence of independent random variables with identical distribution function $F(x)$ and characteristic function

$$\psi(s) = \int_{-\infty}^{\infty} e^{isx} dF(x).$$

Let $S_n = X_1 + \ldots + X_n$ and let $M_n = \max(0, S_1, S_2, \ldots, S_n)$. Let $F_n(x) = \text{Prob}(M_n < x)$ (Prob for probability) be the distribution function of M_n. We note that, if $f(s)$ is the characteristic function of the random variable of X, then $P(f)(s)$ is the characteristic function of the random variable $\max(0, X)$. Applying Proposition 2.16 to $b = \psi(s)$, we obtain the identity first obtained by Spitzer:

$$\sum_{n=0}^{\infty} \int_0^{\infty} e^{isx} dF_n(x) = \exp\left(\sum_{k=1}^{\infty} \left(\int_0^{\infty} e^{isx} dF(x) + F(0) \right) \right).$$

We refer the reader to [31] for the application of the standard Baxter algebra to the proof of some other identities, such as the Bohnenblust-Spitzer formula.

3 Further applications of free Baxter algebras

3.1 Overview

Recall that the free Baxter algebra $\text{Ш}_C(A, \lambda)$ in the special case when $A = C$ and $\lambda = 0$ is the divided power algebra. The divided power algebra and its completion are known to be related to

- crystalline cohomology and rings of p-adic periods in number theory,
- shuffle products in differential geometry and topology,
- Hopf algebra in commutative algebra,
- Hurwitz series in differential algebra,
- umbral calculus in combinatorics, and
- incidence algebra in graph theory.

By the "pull-back" diagram (2.11), free (complete) Baxter algebras give a vast generalization of the (complete) divided power algebra, and so suggest a framework in which these connections and applications of the divided power algebra can be extended. We give two such connections and applications in the next two sections, one to Hopf algebras (Section 3.2) and one to the umbral calculus in combinatorics (Section 3.3).

Definition of Hopf algebra. We recall some basic definitions and facts. Recall that a *cocommutative C-coalgebra* is a triple (A, Δ, ε) where A is a C-module, and $\Delta : A \to A \otimes A$ and $\varepsilon : A \to C$ are C-linear maps that make the following diagrams commute.

$$\begin{array}{ccc} A & \stackrel{\Delta}{\longrightarrow} & A \otimes A \\ \downarrow \Delta & & \downarrow \mathrm{id}\otimes\Delta \\ A \otimes A & \stackrel{\Delta\otimes\mathrm{id}}{\longrightarrow} & A \otimes A \otimes A \end{array} \tag{3.1}$$

$$\begin{array}{ccccc} C \otimes A & \stackrel{\varepsilon\otimes\mathrm{id}}{\longleftarrow} & A \otimes A & \stackrel{\mathrm{id}\otimes\varepsilon}{\longrightarrow} & A \otimes C \\ & \cong\nwarrow & \uparrow \Delta & \nearrow\cong & \\ & & A & & \end{array} \tag{3.2}$$

$$\begin{array}{ccc} & A & \\ {}^{\Delta}\swarrow & & \searrow^{\Delta} \\ A \otimes A & \stackrel{\tau_{A,A}}{\longrightarrow} & A \otimes A \end{array} \tag{3.3}$$

where $\tau_{A,A} : A \otimes A \to A \otimes A$ is defined by $\tau_{A,A}(x \otimes y) = y \otimes x$.

Recall that a *C-bialgebra* is a quintuple $(A, \mu, \eta, \Delta, \varepsilon)$ where (A, μ, η) is a C-algebra and (A, Δ, ε) is a C-coalgebra such that μ and η are morphisms of coalgebras.

Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a C-bialgebra. For C-linear maps f, $g : A \to A$, the convolution $f \star g$ of f and g is the composition of the maps

$$A \stackrel{\Delta}{\longrightarrow} A \otimes A \stackrel{f\otimes g}{\longrightarrow} A \otimes A \stackrel{\mu}{\longrightarrow} A.$$

A C-linear endomorphism S of A is called an *antipode* for A if

$$S \star \mathrm{id}_A = \mathrm{id}_A \star S = \eta \circ \varepsilon. \tag{3.4}$$

A *Hopf algebra* is a bialgebra A with an antipode S.

The main theorem. On the Baxter algebra $Ш_C(C, \lambda)$, let μ be the canonical multiplication and let $\eta : C \hookrightarrow Ш_C(C, \lambda)$ be the unit map. Define a comultiplication Δ, a counit ε and an antipode S by

$$\Delta = \Delta_\lambda : \text{Ш}_C(C,\lambda) \to \text{Ш}_C(C,\lambda) \otimes \text{Ш}_C(C,\lambda),$$

$$a_n \mapsto \sum_{k=0}^{n}\sum_{i=0}^{n-k} (-\lambda)^k a_i \otimes a_{n-k-i},$$

$$\varepsilon = \varepsilon_\lambda : \text{Ш}_C(C,\lambda) \to C,\ a_n \mapsto \begin{cases} \mathbf{1}, & n=0, \\ \lambda\mathbf{1}, & n=1, \\ 0, & n \geq 2, \end{cases}$$

$$S = S_\lambda : \text{Ш}_C(C,\lambda) \to \text{Ш}_C(C,\lambda), \quad a_n \mapsto (-1)^n \sum_{v=0}^{n} \binom{n-3}{v-3} \lambda^{n-v} a_v.$$

The following result is proved in [2].

Theorem 3.5 *The sextuple* $(\text{Ш}_C(C,\lambda), \mu, \eta, \Delta, \varepsilon, S)$ *is a Hopf* C*-algebra.*

3.3 The umbral calculus

Definition and examples. For simplicity, we assume that C is a $\mathbb{Q}$-algebra for the rest of the paper.

The *umbral calculus* is the study and application of *polynomial sequences of binomial type*, i.e., polynomial sequences $\{p_n(x) \mid n \in \mathbb{N}\}$ in $C[x]$ such that

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$$

in $C[x,y]$ for all n. Such a sequence behaves as if its terms are powers of x and has found applications in several areas of pure and applied mathematics, including number theory and combinatorics, since the 19th century. There are many well-known sequences of binomial types.

Examples 3.6 (a) Monomials. x^n.

(b) Lower factorial polynomials. $(x)_n = x(x-1)\cdots(x-n+1)$.

(c) Exponential polynomials. $\phi_n(x) = \sum_{k=0}^{n} S(n,k)x^k$, where $S(n,k)$ with $n,k \geq 0$ are the Stirling numbers of the second kind.

(d) Abel polynomials. Fix $a \neq 0$. $A_n(x) = x(x-an)^{n-1}$.

(e) Mittag-Leffler polynomials. $M_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n-1}{n-k} 2^k (x)_k$.

(f) Bessel polynomials. $y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!}(\frac{x}{2})^k$ (a solution to the Bessel equation $x^2y'' + (2x+2)y' + n(n+1)y = 0$).

There are also Bell polynomials, Hermite polynomials, Bernoulli polynomials, Euler polynomials, ...

As useful as umbral calculus is in many areas of mathematics, the foundations of umbral calculus were not firmly established for over a hundred years. Vaguely speaking, the difficulty in the study is that such sequences do not observe the *algebra* rules of $C[x]$. Rota embarked on laying down the foundation of umbral calculus during the same period of time as when he started the algebraic study of Baxter algebras. Rota's discovery is that these sequences do observe the algebra rules of the dual algebra (the umbral algebra), or in a fancier language, the coalgebra rules of $C[x]$. Rota's pioneer work [26] was completed over the next decade by Rota and his collaborators [30,25,24]. Since then, there have been a number of generalizations of the umbral calculus.

We will give a characterization of umbral calculus in terms of free Baxter algebras by showing that the umbral algebra is the free Baxter algebra of weight zero on the empty set. We also characterize the polynomial sequences studied in umbral calculus in terms of operations in free Baxter algebras.

We will then use the free Baxter algebra formulation of the umbral calculus to give a generalization of the umbral calculus, called the λ-*umbral calculus* for each constant λ. The umbral calculus of Rota is the special case when $\lambda = 0$.

Rota's umbral algebra. In order to describe the binomial sequences, Rota and his collaborators identify $C[x]$ as the dual of the algebra $C[[t]]$, called the *umbral algebra.* (algebra plus the duality). To identify $C[[t]]$ with the dual of $C[x]$, let $t_n = \frac{t^n}{n!},\ n \in \mathbb{N}$. Then

$$t_m t_n = \binom{m+n}{m} t_{m+n},\ m,\ n \in \mathbb{N}. \tag{3.7}$$

The C-algebra $C[[t]]$, together with the basis $\{t_n\}$ is called the *umbral algebra.*

We can identify $C[[t]]$ with the dual C-module of $C[x]$ by taking $\{t_n\}$ to be the dual basis of $\{x^n\}$. In other words, t_k is defined by

$$t_k : C[x] \to C,\ x^n \mapsto \delta_{k,n},\ k,\ n \in \mathbb{N}.$$

Rota and his collaborators removed the mystery of sequences of binomial type and Sheffer sequences by showing that such sequences have a simple characterization in terms of the umbral algebra.

Let $f_n,\ n \geq 0$, be a pseudo-basis of $C[[t]]$. That is, $f_n,\ n \geq 0$ are linearly independent and generate $C[[t]]$ as a topological C-module where the topology on $C[[t]]$ is defined by the filtration

$$F^n = \left\{ \sum_{k=n}^{\infty} c_k t_k \right\}.$$

A pseudo-basis $f_n,\ n \geq 0$, of $C[[t]]$ is called a *divided power* pseudo-basis if

$$f_m f_n = \binom{m+n}{m} f_{m+n},\ m,\ n \geq 0.$$

Theorem 3.8 [26,25]

(a) *A polynomial sequence $\{p_n(x)\}$ is of binomial type if and only if it is the dual basis of a divided power pseudo-basis of $C[[t]]$.*

(b) *Any divided power pseudo-basis of $C[[t]]$ is of the form $f_n(t) = \frac{f^n(t)}{n!}$ for some $f \in C[[t]]$ with $\operatorname{ord} f = 1$ (that is, $f(t) = \sum_{k=1}^{\infty} c_k t^k,\ c_1 \neq 0$).*

This theorem completely determines all polynomial sequences of binomial type. Algorithms to determine such sequences effectively have also been developed. See the book by Roman [24] for details.

λ-umbral calculus. Our first observation is that, with the operator

$$P : C[[t]] \to C[[t]],\ t_n \mapsto t_{n+1},$$

$C[[t]]$ becomes a Baxter algebra of weight zero, isomorphic to $\widehat{\text{Ш}}_C(C, 0)$. More generally, we have

Theorem 3.9 [12] *A sequence $\{f_n(t)\}$ in $C[[t]]$ is a divided power pseudo-basis if and only if the map $f_n(t) \mapsto t_n, n \geq 0$, defines an automorphism of the Baxter algebra $C[[t]]$.*

This theorem provides a link between umbral calculus and Baxter algebra. This characterization of the umbral calculus in terms of Baxter algebra also motivates us to study a generalization of binomial type sequences.

Definiton 3.10 A sequence $\{p_n(x) \mid n \in \mathbb{N}\}$ of polynomials in $C[x]$ is a sequence of λ-binomial type if

$$p_n(x+y) = \sum_{k=0}^{n} \lambda^k \sum_{i=0}^{n} \binom{n}{i} \binom{i}{k} p_i(x) p_{n+k-i}(y),\ \forall\, y \in C,\ n \in \mathbb{N}.$$

When $\lambda = 0$, we recover the sequences of binomial type. Denote

$$e_\lambda(x) = \frac{e^{\lambda x} - 1}{\lambda}$$

for the series

$$\sum_{k=1}^{\infty} \frac{\lambda^{k-1} x^k}{k!} .$$

When $\lambda = 0$, we get $e_\lambda(x) = x$. We verify that

$$\mathfrak{q} := \{(e_\lambda(x))^n\}_n$$

is a sequence of λ-binomial type of $C[[x]]$. Let $C<\mathfrak{q}>$ be the C-submodule of $C[[x]]$ generated by elements in $\mathfrak{q}$.

Let $f(t)$ be a power series of order 1. Define

$$d_n(f)(t) = \frac{f(t)(f(t)-\lambda)\cdots(f(t)-(n-1)\lambda)}{n!}, \ n \geq 0 .$$

Then $P : C[[t]] \to C[[t]],\ d_n(f) \mapsto d_{n+1}(f)$ defines a weight λ Baxter operator on $C[[t]]$. Such a sequence is called a *Baxter pseudo-basis* of $C[[t]]$.

Definiton 3.11 Fix a $\lambda \in C$. The algebra $C[[t]]$, together with the weight λ Baxter pseudo-basis $\{d_n(t)\}_n$, is called the λ-*umbral algebra*.

As in the classical case, we identify $C[[t]]$ with the dual algebra of $C<\mathfrak{q}>$ by taking $\{d_n(t)\}$ to be the dual basis of $\{(e_\lambda(x))^n\}$. We then extend the classical theory of the umbral calculus to the λ-umbral calculus. In particular, Theorem 3.8 is generalized to

Theorem 3.12 [12]

(a) *A pseudo-basis $\{s_n(x)\}$ of $C[[x]]$ is of λ-binomial type if and only if $\{s_n(x)\}$ is the dual basis of a Baxter pseudo-basis of $C[[t]]$.*

(b) *Any Baxter pseudo-basis of $C[[t]]$ is of the form $\{d_n(f)\}$ for some $f(t)$ in $C[[t]]$ of order 1.*

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