

QUASI-SHUFFLES, MIXABLE SHUFFLES AND HOPF ALGEBRAS

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ABSTRACT. The quasi-shuffle product and mixable shuffle product are both generalizations of the shuffle product and have both been studied quite extensively recently. We relate these two generalizations and realize quasi-shuffle product algebras as subalgebras of mixable shuffle product algebras. This allows us to extend a previous result of Hopf algebra structure on Baxter algebras.

1. INTRODUCTION

Let C be a commutative ring and let $\lambda \in C$ be fixed. A Baxter C -algebra of weight λ (also called Rota-Baxter algebra) is a pair (R, P) in which R is a C -algebra and $P : R \rightarrow R$ is a C -linear map, such that

$$(1) \quad P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \quad \forall x, y \in R.$$

Mixable shuffles arise from the study of Baxter algebras. Baxter algebra was introduced by the mathematician Glen Baxter [**Ba**] in 1960 to study the theory of fluctuations in probability. Rota greatly contributed to the study of the Baxter algebra by his pioneer work in the late 1960s and early 1970s [**Ro1**, **Ro2**, **Ro3**] and by his survey articles in late 1990s [**Ro4**, **Ro5**]. Unaware of these works, in the early 1980s the school around Faddeev, especially Semenov-Tian-Shansky [**STS**], developed a whole theory around the Lie algebraic version of equation (1), which is nowadays well-known in the realm of the theory of integrable systems under the name of (modified) classical Yang-Baxter equation.¹ In recent years, Baxter algebras have found applications in quantum field theory [**C-K1**, **C-K2**, **E-G-K1**, **E-G-K2**], dendriform algebras [**A-L**, **EF1**, **E-G1**, **Le**], number theory [**Gu3**], Hopf algebras [**A-G-K-O**] and combinatorics [**Gu2**].

Key to much of these applications is the realization of the free objects in which the product is defined by mixable shuffles [**G-K1**, **G-K2**] as a generalization of the shuffle product. The shuffle product is a natural generalization of the integration by parts formula and its construction can be traced back to Chen's path integrals [**Ch**] in 1950s. It has been defined and studied in many areas of mathematics, such as Lie and Hopf algebras, algebraic K -theory, algebraic topology and combinatorics. Its applications can also be found in chemistry and biology. It naturally carries the notion of a Baxter operator of weight zero.

Date: May 26, 2004.

¹The latter Baxter is the Australian physicist Rodney Baxter.

Another paper on a generalization of the shuffle product was published [**Ho2**] in the same year as the work on mixable shuffle products. It was on the quasi-shuffle product by Hoffman². Hoffman's quasi-shuffle product plays a prominent rôle in the recent studies of harmonic functions, quasi-symmetric functions, multiple zeta values [**Ho1**, **Ho3**, **H-O**, **B-B**] (where in special cases it is also called stuffle product or harmonic product) and q -multiple zeta values [**Br**].

Despite the extensive works on the two generalizations of shuffle products, they appeared to be carried out without being aware of each other. In particular, the relation of quasi-shuffles to Baxter algebras seems unnoticed. For example, in the numerous applications of quasi-shuffles in multiple zeta values in the current literature, no connections with Baxter algebras and mixable shuffles have been mentioned. The situation is similar in the theory of dendriform algebras. Even though both quasi-shuffles and Baxter algebras have been used to give examples of dendriform algebras [**A-L**, **L-R**], no connection of the two have been made. Also, in the work of Kreimer, and Connes and Kreimer [**Kr1**, **Kr2**, **C-K1**, **C-K2**] on renormalization theory in perturbative quantum field theory, both the shuffle and its generalization in form of the quasi-shuffle, and Baxter algebras appeared, under different contexts.

It was noted in [**EF2**] that the two constructions should be related. Our first goal of this paper is to make this connection precise. We show that the recursive formula for the quasi-shuffle product has its explicit form in the mixable shuffle product. Both can be derived from the Baxter relation (1) that defines a Baxter algebra of weight 1. We further show that the quasi-shuffle algebra on a locally finite set is a subalgebra of a mixable shuffle algebra on the corresponding locally finite algebra. With this connection, the concept of quasi-shuffle algebras can be defined for a larger class of algebras.

This connection allows us to use the Hopf algebra structure on quasi-shuffle algebras to obtain Hopf algebra structures on free Baxter algebras, generalizing a previous work [**A-G-K-O**] on this topic. In the other direction, considering the critical rôle played by the quasi-shuffle (stuffle) product in the recent studies of multiple zeta values and quasi-symmetric functions, this connection allows us to use the theory of Baxter algebras in the studies of these exciting areas [**E-G2**].

The paper is organized as follows. In the next section, we recall the concepts of shuffles, quasi-shuffles and mixable shuffles, and describe their relations (Theorem 2.5). In Section 3, we use these connections to obtain Hopf algebra structures on free Baxter algebras (Theorem 3.3).

2. SHUFFLES, QUASI-SHUFFLES, AND MIXABLE SHUFFLES

For the convenience of the reader and for the ease of later references, we recall the definition of each product before giving the relation among them.

2.1. Shuffle product. The shuffle product can be defined in two ways, one recursively, one explicitly. We will see that Hoffman's quasi-shuffle product is a generalization of the

²Hoffman mentioned in [**Ho2**], that there was also a generalization in the thesis of F. Fares [**Fa**].

recursive definition and the mixable shuffle product is a generalization of the explicit definition.

Let C be a commutative ring with identity 1_C . Let V be a C -module. Consider the C -module

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}.$$

Here the tensor products are taken over C and we take $V^{\otimes 0} = C$.

Usually the shuffle product on $T(V)$ starts with the shuffles of permutations [Re, Sw]. For $m, n \in \mathbb{N}_+$, define the set of (m, n) -**shuffles** by

$$S(m, n) = \left\{ \sigma \in S_{m+n} \mid \begin{array}{l} \sigma^{-1}(1) < \sigma^{-1}(2) < \dots < \sigma^{-1}(m), \\ \sigma^{-1}(m+1) < \sigma^{-1}(m+2) < \dots < \sigma^{-1}(m+n) \end{array} \right\}.$$

Here S_{m+n} is the symmetric group on $m+n$ letters.

For $a = a_1 \otimes \dots \otimes a_m \in V^{\otimes m}$, $b = b_1 \otimes \dots \otimes b_n \in V^{\otimes n}$ and $\sigma \in S(m, n)$, the element

$$\sigma(a \otimes b) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \dots \otimes u_{\sigma(m+n)} \in V^{\otimes(m+n)},$$

where

$$u_k = \begin{cases} a_k, & 1 \leq k \leq m, \\ b_{k-m}, & m+1 \leq k \leq m+n, \end{cases}$$

is called a **shuffle** of a and b . The sum

$$(2) \quad a \text{ III } b \stackrel{\text{def}}{=} \sum_{\sigma \in S(m, n)} \sigma(a \otimes b)$$

is called the **shuffle product** of a and b . Also, by convention, $a \text{ III } b$ is the scalar product if either $m = 0$ or $n = 0$. The operation III extends to a commutative and associative binary operation on $T(V)$, making $T(V)$ into a commutative algebra with identity, called the **shuffle product algebra** generated by V .

The shuffle product on $T(V)$ can also be recursively defined as follows. As above we choose two elements $a_1 \otimes \dots \otimes a_m \in V^{\otimes m}$ and $b_1 \otimes \dots \otimes b_n \in V^{\otimes n}$, and define

$$\begin{aligned} a_0 \text{ III } (b_1 \otimes b_2 \otimes \dots \otimes b_n) &= a_0 b_1 \otimes b_2 \otimes \dots \otimes b_n, \\ (a_1 \otimes a_2 \otimes \dots \otimes a_m) \text{ III } b_0 &= b_0 a_1 \otimes a_2 \otimes \dots \otimes a_m, \quad a_0, b_0 \in V^{\otimes 0} = C, \end{aligned}$$

and

$$(3) \quad \begin{aligned} &(a_1 \otimes \dots \otimes a_m) \text{ III } (b_1 \otimes \dots \otimes b_n) \\ &= a_1 \otimes ((a_2 \otimes \dots \otimes a_m) \text{ III } (b_1 \otimes \dots \otimes b_n)) \\ &\quad + b_1 \otimes ((a_1 \otimes \dots \otimes a_m) \text{ III } (b_2 \otimes \dots \otimes b_n)), \quad a_i, b_j \in V. \end{aligned}$$

Lemma 2.1. *For every element $v \in V$, the C -linear map $P_{(v)} : (T(V), \text{III}) \rightarrow (T(V), \text{III})$, $P_{(v)}(a) := v \otimes a$ is a Baxter operator of weight zero.*

Proof. This is evident from the recursive definition of the shuffle product. \square

2.2. Quasi-shuffle product. We recall the construction of quasi-shuffle algebras [Ho2]. Let X be a locally finite set, that is, X is the disjoint union of finite sets X_n , $n \geq 1$. The elements of X_n are defined to have degree n . Elements in X are called letters and monomials in the letters are called words. For a subfield \mathbf{k} of \mathbb{C} , consider the \mathbf{k} -module underlying the noncommutative polynomial algebra $\mathfrak{A} = \mathbf{k}\langle X \rangle$, that is, the free \mathbf{k} -algebra generated by X . The identity 1 of \mathfrak{A} is called the empty word. Define $\bar{X} = X \cup \{0\}$. Suppose that there is a pairing

$$(4) \quad [\cdot, \cdot] : \bar{X} \times \bar{X} \rightarrow \bar{X}$$

with the properties

- S0. $[a, 0] = 0$ for all $a \in \bar{X}$;
- S1. $[a, b] = [b, a]$ for all $a, b \in \bar{X}$;
- S2. $[[a, b], c] = [a, [b, c]]$ for all $a, b, c \in \bar{X}$;
- S3. either $[a, b] = 0$ for all $a, b \in \bar{X}$, or $\deg([a, b]) = \deg(a) + \deg(b)$ for all $a, b \in \bar{X}$.

We define a **Hoffman set** to be a locally finite set X with a pairing (4) that satisfies conditions S0-S3.

Definition 2.2. Let $\mathbf{k} \subset \mathbb{C}$ be a field and let X be a Hoffman set. The **quasi-shuffle product** $*$ on \mathfrak{A} is defined recursively by

- $1 * w = w * 1 = w$ for any word w ;
- $(aw_1) * (bw_2) = a(w_1 * (bw_2)) + b((aw_1) * w_2) + [a, b](w_1 * w_2)$, for any words w_1, w_2 and letters a, b .

When $[\cdot, \cdot]$ is identically zero, $*$ is the usual shuffle product \mathbb{III} defined in Eq. (3).

Theorem 2.3. (Hoffman)[Ho2]

- (1) $(\mathfrak{A}, *)$ is a commutative graded \mathbf{k} -algebra.
- (2) When $[\cdot, \cdot] \equiv 0$, $(\mathfrak{A}, *)$ is the shuffle product algebra $(T(V), \mathbb{III})$, where V is the vector space generated by X .
- (3) Together with the coconcatenation comultiplication

$$\Delta : \mathfrak{A} \rightarrow \mathfrak{A} \otimes \mathfrak{A}, w \mapsto \sum_{uv=w} u \otimes v$$

where wv is the concatenation of words, and counit

$$\epsilon : \mathfrak{A} \rightarrow \mathbf{k}, w \mapsto \delta_{w,1},$$

$(\mathfrak{A}, *)$ becomes a graded, connected bialgebra, in fact a Hopf algebra.

2.3. Mixable shuffle product. We next turn to the construction of mixable shuffle algebras and their properties [G-K1]. The adjective mixable suggests that certain elements in the shuffles can be mixed or merged. We first give an explicit formula of the product before giving a recursive definition which, under proper restrictions, will be seen to be equivalent to Hoffman's quasi-shuffle product.

Intuitively, to form the shuffle product, one starts with two decks of cards and puts together all possible shuffles of the two decks. Suppose a shuffle of the two decks is taken and suppose a card from the first deck is followed immediately by a card from the second deck, we allow the option to merge the two cards and call the result a mixable shuffle. To get the mixable shuffle product of the two decks of cards, one puts together all possible mixable shuffles.

Given an (m, n) -shuffle $\sigma \in S(m, n)$, a pair of indices $(k, k + 1)$, $1 \leq k < m + n$, is called an **admissible pair** for σ if $\sigma(k) \leq m < \sigma(k + 1)$. Denote \mathcal{T}^σ for the set of admissible pairs for σ . For a subset T of \mathcal{T}^σ , call the pair (σ, T) a **mixable (m, n) -shuffle**. Let $|T|$ be the cardinality of T . By convention, $(\sigma, T) = \sigma$ if $T = \emptyset$. Denote

$$(5) \quad \bar{S}(m, n) = \{(\sigma, T) \mid \sigma \in S(m, n), T \subset \mathcal{T}^\sigma\}$$

for the set of **mixable (m, n) -shuffles**.

Let A be a commutative C -algebra not necessarily having an identity. For $a = a_1 \otimes \dots \otimes a_m \in A^{\otimes m}$, $b = b_1 \otimes \dots \otimes b_n \in A^{\otimes n}$ and $(\sigma, T) \in \bar{S}(m, n)$, the element

$$\sigma(a \otimes b; T) = u_{\sigma(1)} \hat{\otimes} u_{\sigma(2)} \hat{\otimes} \dots \hat{\otimes} u_{\sigma(m+n)} \in A^{\otimes(m+n-|T|)},$$

where for each pair $(k, k + 1)$, $1 \leq k < m + n$,

$$u_{\sigma(k)} \hat{\otimes} u_{\sigma(k+1)} = \begin{cases} u_{\sigma(k)} u_{\sigma(k+1)}, & (k, k + 1) \in T \\ u_{\sigma(k)} \otimes u_{\sigma(k+1)}, & (k, k + 1) \notin T, \end{cases}$$

is called a **mixable shuffle** of the words a and b .

Now fix $\lambda \in C$. Define, for a and b as above, the **mixable shuffle product**

$$(6) \quad a \diamond^\lambda b := a \diamond_\lambda^+ b = \sum_{(\sigma, T) \in \bar{S}(m, n)} \lambda^{|T|} \sigma(a \otimes b; T) \in \bigoplus_{k \leq m+n} A^{\otimes k}.$$

As in the case of the shuffle product, the operation \diamond^+ extends to a commutative and associative binary operation on

$$\bigoplus_{k \geq 1} A^{\otimes k} = A \oplus A^{\otimes 2} \oplus \dots$$

Making it a commutative algebra without identity. Note that this is so even when A has an identity $\mathbf{1}_A$. In any case, we extend the product to

$$(7) \quad \mathbb{H}^+(A) := \mathbb{H}_{C, \lambda}^+(A) := \bigoplus_{k \in \mathbb{N}} A^{\otimes k} = C \oplus A \oplus A^{\otimes 2} \oplus \dots,$$

making $\mathbb{H}^+(A)$ a commutative algebra with identity $\mathbf{1} \in C$ [**G-K1**].

Suppose A has an identity $\mathbf{1}_A$. Define

$$(8) \quad \mathbb{H}(A) := \mathbb{H}_{C, \lambda}(A) := A \otimes \mathbb{H}_C^+(A)$$

to be the tensor product algebra, i.e., the **augmented mixable shuffle product** \diamond on $\mathbb{H}(A)$ is defined by:

$$(9) \quad (a_0 \otimes a) \diamond (b_0 \otimes b) := (a_0 b_0) \otimes (a \diamond^+ b), \quad a_0, b_0 \in A, \quad a, b \in \mathbb{H}^+(A).$$

Thus we have the algebra isomorphism (embedding of the second tensor factor)

$$(10) \quad \alpha : (\mathbb{H}^+(A), \diamond^+) \rightarrow (\mathbf{1}_A \otimes \mathbb{H}^+(A), \diamond).$$

Define the C -linear endomorphism P_A on $\mathbb{H}_C(A)$ by assigning

$$\begin{aligned} P_A(a_0 \otimes a) &= \mathbf{1}_A \otimes a_0 \otimes a, \quad a \in A^{\otimes n}, \quad n \geq 1, \\ P_A(a_0 \otimes c) &= \mathbf{1}_A \otimes ca_0, \quad c \in A^{\otimes 0} = C \end{aligned}$$

and extending by additivity. Let $j_A : A \rightarrow \mathbb{H}_C(A)$ be the canonical inclusion map. Call $(\mathbb{H}_C(A), P_A)$ the **(mixable) shuffle Baxter C -algebra on A of weight λ** .

Theorem 2.4. [G-K1] *The shuffle Baxter algebra $(\mathbb{H}_C(A), P_A)$, together with the natural embedding j_A , is a free Baxter C -algebra on A of weight λ . More precisely, for any Baxter C -algebra (R, P) of weight λ and algebra homomorphism $f : A \rightarrow R$, there is a Baxter C -algebra homomorphism $\tilde{f} : (\mathbb{H}_C(A), P_A) \rightarrow (R, P)$ such that $f = \tilde{f} \circ j_A$.*

We will suppress C and λ from $\mathbb{H}_{C,\lambda}(A)$ when there is no danger of confusion.

2.4. The connection. We now establish the relation between quasi-shuffle algebra and mixable shuffle algebra.

Let $C = \mathbf{k}$ be a subfield of \mathbb{C} . Let $X = \cup_{n \geq 1} X_n$ be a Hoffman set. Then the pairing $[\cdot, \cdot]$ in (4) extends by C -linearity to a binary operation on the free C -module $A = C\{X\}$ on X , making A into a commutative C -algebra without identity, with grading $A_n = C\{X_n\}$. Let $\tilde{A} = C \oplus A$ be the unitary C -algebra spanned by A . Then $\tilde{A} = C\{\tilde{X}\}$ where $\tilde{X} = \{\mathbf{1}_A\} \cup X$. Here and in the rest of the paper, we will use $\mathbf{1}_A$ (instead of $\mathbf{1}_{\tilde{A}}$) to denote the unit of \tilde{A} . We will call A (resp. \tilde{A}) the **algebra** (resp. **unitary algebra**) **generated by X** . Theorem 2.3 and Theorem 3.1 give us Hopf algebras.

With the notations in Eq. (7) and (8), we have embeddings

$$(11) \quad \begin{array}{ccccc} \beta : \mathbb{H}^+(A) & \rightarrow & \mathbb{H}^+(\tilde{A}) & \rightarrow & \mathbb{H}(\tilde{A}), \\ a & \mapsto & a & \mapsto & \mathbf{1}_A \otimes a. \end{array}$$

of C -algebras. Here the first embedding is induced by the embedding $A \hookrightarrow \tilde{A}$ and the second embedding is the natural one, $\mathbb{H}^+(\tilde{A}) \rightarrow \mathbb{H}(\tilde{A}) := \tilde{A} \otimes \mathbb{H}^+(\tilde{A})$.

Theorem 2.5. *For a subfield $\mathbf{k} \subset \mathbb{C}$ and Hoffman set X , the quasi-shuffle algebra $\mathfrak{A} = \mathbf{k}\langle X \rangle$ is isomorphic to the algebra $\mathbb{H}^+(A)$ and thus to the subalgebra $\mathbf{1}_A \otimes \mathbb{H}^+(A)$ of $\mathbb{H}(\tilde{A})$ where the weight λ is 1.*

Proof. We define

$$f : X \rightarrow X \subseteq A = A^{\otimes 1} \subset \mathbb{H}^+(A)$$

to be the canonical embedding. We note that both \mathfrak{A} , with the concatenation product, and $\mathbb{H}^+(A)$, with the tensor concatenation, are the free unitary non-commutative \mathbf{k} -algebra on X . Thus f extends uniquely to a bijective map $\bar{f} : \mathfrak{A} \rightarrow \mathbb{H}^+(A)$ such that for any letters $a_1, \dots, a_n \in X$, we have

$$\bar{f}(a_1 \cdots a_n) = a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}.$$

To distinguish elements, we will use $a = a_1 \cdots a_n$ for an element in \mathfrak{A} and use $a^\otimes := a_1 \otimes \cdots \otimes a_n$ for an element in $\mathfrak{III}^+(A)$ in the rest of the proof.

To prove that \bar{f} is also an isomorphism between \mathfrak{A} , with the quasi-shuffle product $*$, and $\mathfrak{III}^+(A)$, with the mixable shuffle product \diamond^+ , we just need to show that both products satisfy the same recursive relations. We first note that the recursive relation of $*$ in Definition 2.2 can be rewritten as follows. For any $m, n \geq 1$ and $a := a_1 \cdots a_m$, $b := b_1 \cdots b_n$ with $a_i, b_j \in X$, $1 \leq i \leq m$, $1 \leq j \leq n$, then

$$(12) \quad \begin{aligned} (1). & a * b = a_1 b_1 + b_1 a_1 + [a_1, b_1], \quad \text{when } m, n = 1, \\ (2). & a * b = a_1 b_1 \cdots b_n + b_1 (a_1 * (b_2 \cdots b_n)) \\ & \quad + [a_1, b_1] b_2 \cdots b_n, \quad \text{when } m = 1, n \geq 2, \\ (3). & a * b = a_1 ((a_2 \cdots a_m) * b_1) + b_1 a_1 \cdots a_m \\ & \quad + [a_1, b_1] a_2 \cdots a_m, \quad \text{when } m \geq 2, n = 1, \\ (4). & a * b = a_1 ((a_2 \cdots a_m) * (b_1 \cdots b_n)) + b_1 ((a_1 \cdots a_m) * (b_2 \cdots b_n)) \\ & \quad + [a_1, b_1] ((a_2 \cdots a_m) * (b_2 \cdots b_n)), \quad \text{when } m, n \geq 2. \end{aligned}$$

On the other hand, consider the set $\bar{S}(m, n)$ of mixable (m, n) -shuffles defined in Eq. (5). By [G-K1, Eq. (4)], we have

$$\bar{S}(m, n) = \bar{S}_{1,0}(m, n) \dot{\cup} \bar{S}_{0,1}(m, n) \dot{\cup} \bar{S}_{1,1}(m, n)$$

where

$$\begin{aligned} \bar{S}_{1,0}(m, n) &= \{(\sigma, T) \in \bar{S}(m, n) \mid (1, 2) \notin T, \sigma^{-1}(1) = 1\}, \\ \bar{S}_{0,1}(m, n) &= \{(\sigma, T) \in \bar{S}(m, n) \mid (1, 2) \notin T, \sigma^{-1}(m+1) = 1\}, \\ \bar{S}_{1,1}(m, n) &= \{(\sigma, T) \in \bar{S}(m, n) \mid (1, 2) \in T\}. \end{aligned}$$

Further, by [G-K1, p. 137], for any $\lambda \in C$,

$$\begin{aligned} \sum_{(\sigma, T) \in \bar{S}_{1,0}(m, n)} \lambda^{|T|} \sigma(a^\otimes \otimes b^\otimes) &= a_1 \otimes \sum_{(\sigma, T) \in \bar{S}(m-1, n)} \lambda^{|T|} \sigma((a_2 \otimes \cdots \otimes a_m) \otimes b^\otimes; T) \\ &= \begin{cases} a_1 \otimes ((a_2 \otimes \cdots \otimes a_m) \diamond^+ b^\otimes), & m \geq 2, \\ a_1 \otimes b^\otimes, & m = 1. \end{cases} \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{(\sigma, T) \in \bar{S}_{0,1}(m, n)} \lambda^{|T|} \sigma(a^\otimes \otimes b^\otimes) &= b_1 \otimes \sum_{(\sigma, T) \in \bar{S}(m, n-1)} \lambda^{|T|} \sigma(a^\otimes \otimes (b_2 \otimes \cdots \otimes b_n); T) \\ &= \begin{cases} b_1 \otimes (a^\otimes \diamond^+ (b_2 \otimes \cdots \otimes b_n)), & n \geq 2, \\ b_1 \otimes a^\otimes, & n = 1; \end{cases} \end{aligned}$$

and

$$\begin{aligned}
& \sum_{(\sigma, T) \in \bar{S}_{1,1}(m,n)} \lambda^{|T|} \sigma(a^\otimes \otimes b^\otimes; T) \\
&= \sum_{(\sigma, T) \in \bar{S}(m-1, n-1)} \lambda[a_1, b_1] \otimes \lambda^{|T|} \sigma((a_2 \otimes \dots \otimes a_m) \otimes (b_2 \otimes \dots \otimes b_n); T) \\
&= \begin{cases} \lambda[a_1, b_1] \otimes ((a_2 \otimes \dots \otimes a_m) \diamond^+ (b_2 \otimes \dots \otimes b_n)), & m \geq 2, n \geq 2, \\ \lambda[a_1, b_1] \otimes (b_2 \otimes \dots \otimes b_n), & m = 1, n \geq 2, \\ \lambda[a_1, b_1] \otimes (a_2 \otimes \dots \otimes a_m), & m \geq 2, n = 1, \\ \lambda[a_1, b_1], & m = n = 1. \end{cases}
\end{aligned}$$

Since

$$\begin{aligned}
a^\otimes \diamond^+ b^\otimes &:= \sum_{(\sigma, T) \in \bar{S}(m,n)} \lambda^{|T|} \sigma(a^\otimes \otimes b^\otimes) \\
&= \sum_{(\sigma, T) \in \bar{S}_{1,0}(m,n)} \lambda^{|T|} \sigma(a^\otimes \otimes b^\otimes) + \sum_{(\sigma, T) \in \bar{S}_{0,1}(m,n)} \lambda^{|T|} \sigma(a^\otimes \otimes b^\otimes) + \sum_{(\sigma, T) \in \bar{S}_{1,1}(m,n)} \lambda^{|T|} \sigma(a^\otimes \otimes b^\otimes),
\end{aligned}$$

we get

$$(13) \quad \begin{aligned}
(1). & \quad a^\otimes \diamond^+ b^\otimes = a_1 \otimes b_1 + b_1 \otimes a_1 + \lambda[a_1, b_1], \quad \text{when } m, n = 1, \\
(2). & \quad a^\otimes \diamond^+ b^\otimes = a_1 \otimes b_1 \otimes \dots \otimes b_n + b_1 \otimes (a_1 \diamond^+ (b_2 \otimes \dots \otimes b_n)) \\
& \quad + \lambda[a_1, b_1] \otimes b_2 \otimes \dots \otimes b_n, \quad \text{when } m = 1, n \geq 2, \\
(3). & \quad a^\otimes \diamond^+ b^\otimes = a_1 \otimes ((a_2 \otimes \dots \otimes a_m) \diamond^+ b_1) + a_1 \otimes \dots \otimes a_m \otimes b_1 \\
& \quad + \lambda[a_1, b_1] \otimes a_2 \otimes \dots \otimes a_m, \quad \text{when } m \geq 2, n = 1, \\
(4). & \quad a^\otimes \diamond^+ b^\otimes = a_1 \otimes ((a_2 \otimes \dots \otimes a_m) \diamond^+ (b_1 \otimes \dots \otimes b_n)) \\
& \quad + b_1 \otimes ((a_1 \otimes \dots \otimes a_m) \diamond^+ (b_2 \otimes \dots \otimes b_n)) \\
& \quad + \lambda[a_1, b_1] \otimes ((a_2 \otimes \dots \otimes a_m) \diamond^+ (b_2 \otimes \dots \otimes b_n)), \quad \text{when } m, n \geq 2.
\end{aligned}$$

Thus when $\lambda = 1$, we have $\bar{f}(a * b) = a^\otimes \diamond^+ b^\otimes$ for all words a and b with $m, n \geq 1$, and hence for all a and b with $m, n \geq 0$ since when $m = 0$ or $n = 0$, we have $a = 1$ or $b = 1$ and the multiplications through $*$ and \diamond^+ are both given by the identity. This proves the first isomorphism. The second one then follows from Eq. (11). \square

Corollary 2.6. *Under the same assumptions of Theorem 2.5, for any $\lambda \in C$, the subalgebra $\mathbb{H}^+(A)$ of $\mathbb{H}^+(\tilde{A})$ and the subalgebra $\mathbf{1}_A \otimes \mathbb{H}^+(A)$ of $\mathbb{H}(\tilde{A})$ are Hopf algebras.*

In the next section, we will address the question on whether this Hopf algebra can be extended to a larger Hopf algebra in $\mathbb{H}(\tilde{A})$.

Proof. Because of the isomorphism (10), we only need to prove the first part of the statement. When $\lambda = 1$, this follows from Theorem 2.5 and Theorem 2.3. When $\lambda = 0$, this is well-known (see [Ho2], for example).

Now assume $\lambda \neq 1, 0$. We define a map

$$g : \mathbb{H}_\lambda^+(A) \rightarrow \mathbb{H}_1^+(A)$$

by $g(a_1 \otimes \cdots \otimes a_n) = \lambda^n a_1 \otimes \cdots \otimes a_n, n \geq 1$ and $g(\mathbf{1}) = \mathbf{1}$. Then by the first equation in (13), we have

$$g(a_1 \diamond_\lambda^+ b_1) = g(a_1 \otimes b_1 + b_1 \otimes a_1 + \lambda[a_1, b_1]) = \lambda^2(a_1 \otimes b_1 + b_1 \otimes a_1 + [a_1, b_1]) = \lambda^2(a_1 \diamond_1^+ b_1)$$

which is just $g(a_1) \diamond_1^+ g(b_1)$. Using the other three equations in (13) and the induction on $m + n$, we verify that

$$g((a_1 \otimes \cdots \otimes a_m) \diamond_\lambda^+ (b_1 \otimes \cdots \otimes b_n)) = g(a_1 \otimes \cdots \otimes a_m) \diamond_1^+ g(b_1 \otimes \cdots \otimes b_n)$$

for all $m, n \geq 1$. Thus we have $\mathbb{H}_\lambda^+(A)$ is isomorphic to $\mathbb{H}_1^+(A)$ and thus carries a Hopf algebra structure. \square

3. HOPF ALGEBRAS IN BAXTER ALGEBRAS

We first recall the following theorem from [A-G-K-O].

Theorem 3.1. (Andrews-Guo-Keigher-Ono) *For any commutative ring C with identity and for any $\lambda \in C$, the free Baxter algebra $\mathbb{H}_\lambda(C)$ is a Hopf C -algebra.*

As shown in [A-G-K-O], when $\lambda = 0$, we have the divided power Hopf algebra.

We now extend this result to $\mathbb{H}(\tilde{A})$ for a C -algebra \tilde{A} coming from a Hoffman set X . To avoid confusion, we will use 1 for the identity of C and $\mathbf{1}_A$ for the identity of \tilde{A} even though they are often identified under the structure map $C \rightarrow \tilde{A}$ of the unitary C -algebra \tilde{A} .

Fix a $\lambda \in C$. First note that, as a C -module,

$$\mathbb{H}^+(C) = \bigoplus_{n \geq 0} C^{\otimes n} = C \oplus C \oplus C^{\otimes 2} + \cdots .$$

There are two copies of C in the sum since $C^{\otimes 0} = C = C^{\otimes 1}$. The identity of $\mathbb{H}^+(C)$ is the 1 in the first copy, which we will denote by $1^{\otimes 0} = \mathbf{1}$ for distinction. Thus we have

$$\mathbb{H}^+(C) = C\mathbf{1} \oplus C1 \oplus C1^{\otimes 2} \oplus \cdots = \bigoplus_{n \geq 0} C1^{\otimes n}.$$

Then

$$\mathbb{H}(C) = C \otimes \mathbb{H}^+(C) = \bigoplus_{n \geq 0} C(1 \otimes 1^{\otimes n})$$

with the identity $1 \otimes \mathbf{1}$. Since C is the base ring, the algebra homomorphism (11) gives

$$\beta : (\mathbb{H}^+(C), \diamond^+) \cong (\mathbb{H}(C), \diamond).$$

Thus, by Theorem 3.1 we get

Lemma 3.2. *For any $\lambda \in C$, $(\mathbb{H}^+(C), \diamond^+)$ is a Hopf algebra.*

For now let \tilde{A} be any unitary C -algebra with unit $\mathbf{1}_A$. Then

$$\mathbb{H}^+(\tilde{A}) = \bigoplus_{n \geq 0} \tilde{A}^{\otimes n} = C\mathbf{1} \oplus \tilde{A} \oplus \tilde{A}^{\otimes 2} \oplus \cdots$$

and

$$\mathbb{H}(\tilde{A}) = \tilde{A} \otimes \mathbb{H}^+(\tilde{A}) = (\tilde{A} \otimes C\mathbf{1}) \oplus \tilde{A}^{\otimes 2} \oplus \tilde{A}^{\otimes 3} \cdots .$$

Since $\mathbb{I}(\tilde{A})$ is an \tilde{A} -algebra, and hence a C -algebra, we have the structure map $\gamma : C \rightarrow \mathbb{I}(\tilde{A})$ given by $\gamma(c) = c\mathbf{1}_A \otimes \mathbf{1}$. By the universal property of the free C -Baxter algebra $\mathbb{I}(C)$, we have an induced homomorphism $\gamma : \mathbb{I}(C) \rightarrow \mathbb{I}(\tilde{A})$ of Baxter algebras. It is given by **[G-K1]**

$$\gamma(1 \otimes 1^{\otimes n}) = \mathbf{1}_A \otimes \mathbf{1}_A^{\otimes n}, \quad n \geq 0.$$

Let

$$\gamma^+ : \mathbb{I}^+(C) \rightarrow \mathbb{I}^+(\tilde{A}), \quad 1^{\otimes n} \mapsto \mathbf{1}_A^{\otimes n}, \quad n \geq 0.$$

We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{I}^+(C) & \xrightarrow{\gamma^+} & \mathbb{I}^+(\tilde{A}) \\ \downarrow & & \downarrow \\ \mathbb{I}(C) = C \otimes \mathbb{I}^+(C) & \xrightarrow{\gamma} & \mathbb{I}(\tilde{A}) = \tilde{A} \otimes \mathbb{I}^+(\tilde{A}) \end{array}$$

where the vertical arrows are the injective maps to the second tensor factors.

Theorem 3.3. *Let $C = \mathbf{k} \subseteq \mathbb{C}$ be a field. Let X be a Hoffman set and let the algebras A and \tilde{A} be the algebra and unitary algebra generated by X (as defined before Theorem 2.5). Let $\lambda \in \mathbf{k}$.*

- (1) *The algebra product of $\gamma^+(\mathbb{I}^+(\mathbf{k}))$ and $\mathbb{I}^+(A)$ in $\mathbb{I}^+(\tilde{A})$ has a Hopf algebra structure that expands the Hopf algebra structures on $\gamma^+(\mathbb{I}^+(\mathbf{k}))$ and $\mathbb{I}^+(A)$.*
- (2) *The algebra product of $\gamma(\mathbb{I}(\mathbf{k}))$ and $\mathbf{1}_A \otimes \mathbb{I}^+(A)$ in $\mathbb{I}(\tilde{A})$ has a Hopf algebra structure that expands the Hopf algebra structures on $\gamma(\mathbb{I}(\mathbf{k}))$ and $\mathbf{1}_A \otimes \mathbb{I}^+(A)$.*

See Theorem 3.7 for a characterization of the elements in these Hopf algebras.

Proof. Since the restriction of the isomorphism $\alpha : \mathbb{I}^+(\tilde{A}) \rightarrow \mathbf{1}_A \otimes \mathbb{I}^+(\tilde{A})$ in (10) restricts to isomorphisms $\gamma^+(\mathbb{I}^+(\mathbf{k})) \rightarrow \gamma(\mathbb{I}(\mathbf{k}))$ and $\mathbb{I}^+(A) \rightarrow \mathbf{1}_A \otimes \mathbb{I}^+(A)$, we only need to prove the first statement. Since the tensor product of two commutative, cocommutative Hopf algebras is a Hopf algebra **[Mo, L-N]**, assuming Proposition 3.4 which is stated and proved below, we see that $\gamma^+(\mathbb{I}^+(\mathbf{k})) \diamond^+ (\mathbf{1} \otimes \mathbb{I}^+(A))$ is a Hopf algebra for any $\lambda \in \mathbf{k}$ by Theorem 3.1 and Corollary 2.6. \square

Proposition 3.4. *For any weight $\lambda \in \mathbf{k}$, let \diamond^+ be the mixable shuffle product of weight λ . The two subalgebras $\gamma^+(\mathbb{I}^+(\mathbf{k}))$ and $\mathbb{I}^+(A)$ of $\mathbb{I}^+(\tilde{A})$ are linearly disjoint. Therefore, $\gamma^+(\mathbb{I}^+(\mathbf{k})) \diamond^+ \mathbb{I}^+(A)$ is isomorphic to the tensor product $\gamma(\mathbb{I}(\mathbf{k})) \otimes (\mathbf{1}_A \otimes \mathbb{I}^+(A))$.*

Proof. Let $\mathbf{k} \subseteq \mathbb{C}$, X, \tilde{X}, A and \tilde{A} be as in Theorem 3.3. Since X is locally finite, it is countable. So we can write $X = \{y_n \mid n \geq 1\}$. Also denote $y_0 = \mathbf{1}_A$, the unit of \tilde{A} . Thus $\tilde{X} = \{y_n \mid n \in \mathbb{N}\}$ and $\tilde{A} = \bigoplus_{n \geq 0} \mathbf{k}y_n$. For $r \geq 1$ and $I = (i_1, \dots, i_r) \in \mathbb{N}^r$, denote $y_{\otimes I} = y_{i_1} \otimes \dots \otimes y_{i_r}$. Then

$$A^{\otimes r} = \bigoplus_{I \in \mathbb{N}_{>0}^r} \mathbf{k}y_{\otimes I}, \quad \tilde{A}^{\otimes r} = \bigoplus_{I \in \mathbb{N}^r} \mathbf{k}y_{\otimes I}.$$

By convention, we define $\mathbb{N}^0 = \mathbb{N}_{>0}^0 = \{\emptyset\}$, and $y_{\otimes\emptyset} = \mathbf{1}$. Let $\mathcal{J} = \cup_{r \geq 0} \mathbb{N}_{>0}^r$ and $\tilde{\mathcal{J}} = \cup_{r \geq 0} \mathbb{N}^r$. We then have

$$\mathbb{I}^+(A) = \oplus_{I \in \mathcal{J}} \mathbf{k} y_{\otimes I}, \quad \mathbb{I}^+(\tilde{A}) = \oplus_{I \in \tilde{\mathcal{J}}} \mathbf{k} y_{\otimes I}.$$

Recall that

$$\gamma^+(\mathbb{I}^+(\mathbf{k})) = \oplus_{n \geq 0} \mathbf{k} \mathbf{1}_A^{\otimes n}.$$

So to prove that $\oplus_{n \geq 0} \mathbf{k} \mathbf{1}_A^{\otimes n}$ and $\mathbb{I}^+(A)$ are linearly disjoint under the product \diamond^+ , we only need to prove

Claim 3.5. *The set $\{\mathbf{1}_A^{\otimes n} \diamond^+ y_{\otimes I} \mid n \geq 0, I \in \mathcal{J}\}$ is linearly independent.*

Before proceeding further, we give a formula for the product $\mathbf{1}_A^{\otimes n} \diamond^+ y_{\otimes I}$ which express a mixable shuffle product as a sum of shuffle products in Eq. (2).

Lemma 3.6. *For any $m \geq 0$ and $I \in \mathcal{J}$, we have*

$$\mathbf{1}_A^{\otimes m} \diamond^+ y_{\otimes I} = \sum_{i=0}^m \lambda^i \binom{n}{i} \mathbf{1}_A^{\otimes(m-i)} \mathbb{I} y_{\otimes I}.$$

Proof. Define the length of $I \in \mathbb{N}^r$ to be $\ell(I) = r$. We will prove by induction on $w = m + \ell(I)$. When $w = 0$, we have $m = \ell(I) = 0$. Then $\mathbf{1}_A^{\otimes m}$ and $y_{\otimes I}$ are both $\mathbf{1}$, so the lemma is clear, as is if either $m = 0$ or $\ell(I) = 0$. Suppose it holds for all $\mathbf{1}_A^{\otimes m} \diamond^+ y_{\otimes I}$ with $m + \ell < w$. For given $\mathbf{1}_A^{\otimes m}$ and $y_{\otimes I} = y_{i_1} \otimes \cdots \otimes y_{i_r}$ with $m \geq 1$, $r \geq 1$ and $m + r = w$, let $y' = y_{i_2} \otimes \cdots \otimes y_{i_r}$ if $r > 1$ and $y' = \mathbf{1}$ if $r = 1$. Applying the recursive relation of \diamond^+ in Eq. (13), the induction hypothesis, the Pascal equality and the recursive relation of \mathbb{I} in Eq. (3), we have

$$\begin{aligned} \mathbf{1}_A^{\otimes m} \diamond^+ y_{\otimes I} &= \mathbf{1}_A \otimes (\mathbf{1}_A^{\otimes(m-1)} \diamond^+ y_{\otimes I}) + y_{i_1} \otimes (\mathbf{1}_A^{\otimes m} \diamond^+ y') + \lambda y_{i_1} \otimes (\mathbf{1}_A^{\otimes(m-1)} \diamond^+ y') \\ &= \mathbf{1}_A \otimes \left(\sum_{i=0}^{m-1} \lambda^i \binom{n}{i} \mathbf{1}_A^{\otimes(m-1-i)} \mathbb{I} y_{\otimes I} \right) + y_{i_1} \otimes \left(\sum_{i=0}^m \lambda^i \binom{n-1}{i} \mathbf{1}_A^{\otimes(m-i)} \mathbb{I} y' \right) \\ &\quad + \lambda y_{i_1} \otimes \left(\sum_{i=0}^{m-1} \lambda^i \binom{n-1}{i} \mathbf{1}_A^{\otimes(m-1-i)} \mathbb{I} y' \right) \\ &= \mathbf{1}_A \otimes \left(\sum_{i=0}^{m-1} \lambda^i \binom{n}{i} \mathbf{1}_A^{\otimes(m-1-i)} \mathbb{I} y_{\otimes I} \right) + y_{i_1} \otimes \left(\sum_{i=0}^m \lambda^i \binom{n-1}{i} \mathbf{1}_A^{\otimes(m-i)} \mathbb{I} y' \right) \\ &\quad + y_{i_1} \otimes \left(\sum_{i=1}^m \lambda^i \binom{n-1}{i-1} \mathbf{1}_A^{\otimes(m-i)} \mathbb{I} y' \right) \\ &= \mathbf{1}_A \otimes \left(\sum_{i=0}^{m-1} \lambda^i \binom{n}{i} \mathbf{1}_A^{\otimes(m-1-i)} \mathbb{I} y_{\otimes I} \right) + y_{i_1} \otimes \left(\sum_{i=0}^m \lambda^i \binom{n}{i} \mathbf{1}_A^{\otimes(m-i)} \mathbb{I} y' \right) \\ &\stackrel{(3)}{=} \sum_{i=0}^{m-1} \lambda^i \binom{n}{i} \mathbf{1}_A^{\otimes(m-i)} \mathbb{I} y_{\otimes I} + \lambda^m y \otimes \binom{n}{m} \mathbf{1} \mathbb{I} y'. \end{aligned}$$

Since $y_{i_1} \otimes (\mathbf{1} \amalg y') = y_{i_1} \otimes y' = y = \mathbf{1}_A^{\otimes 0} \amalg y$, we get exactly what we want. \square

We continue with the proof of Proposition 3.4. For $r \geq 1$, let $[r] = (1, \dots, r)$. For a sequence $I = (i_1, \dots, i_r) \in \mathbb{N}^r$, denote $SSupp(I)$ (called sequential support) for the subsequence (with ordering) of I of non-zero entries. For an all zero sequence $I = (0, \dots, 0)$ and the empty sequence \emptyset , we define $SSupp(I) = \emptyset$. We then get a map

$$SSupp : \tilde{\mathcal{J}} \rightarrow \tilde{\mathcal{J}}.$$

Clearly, $\mathcal{J} = \{I \in \tilde{\mathcal{J}} \mid SSupp(I) = I\}$. So

$$\tilde{\mathcal{J}} = \dot{\bigcup}_{I \in \mathcal{J}} SSupp^{-1}(I).$$

For each $I \in \mathcal{J}$, consider the subset $\mathcal{O}_I = \{y_{\otimes J} \mid J \in SSupp^{-1}(I)\}$. Then we have

$$\{y_{\otimes I} \mid I \in \tilde{\mathcal{J}}\} = \dot{\bigcup}_{I \in \mathcal{J}} \mathcal{O}_I.$$

So \mathcal{O}_I span linearly independent subspaces of $\amalg(\tilde{A})$.

By Lemma 3.6, $\mathbf{1}_A^{\otimes n} \diamond^+ y_{\otimes I}$, $n \geq 0$, is in the linear span of \mathcal{O}_I . Thus to prove Claim 3.5 and hence Proposition 3.4, we only need to prove that, for a fix $I \in \mathcal{J}$, the subset $\{\mathbf{1}_A^{\otimes n} \diamond^+ y_{\otimes I} \mid n \geq 0\}$ is linearly independent.

Suppose the contrary. Then there are integers $n_1 > n_2 > \dots > n_r \geq 0$ and $c_1 \neq 0$ in \mathbf{k} such that $\sum_{i=1}^r c_i \mathbf{1}_A^{\otimes n_i} \diamond^+ y_I = 0$. Express this sum as a linear combination in terms of the basis \mathcal{O}_I . By Lemma 3.6, the coefficient of $\mathbf{1}_A^{\otimes n_1} \otimes y_I$ is c_1 , so we must have $c_1 = 0$, a contradiction. \square

It is desirable to be able to characterize the elements in the Hopf algebra $\gamma^+(\amalg^+(\mathbf{k})) \diamond^+ \amalg^+(A)$. This is our last goal in this article. Recall that the **length** of $y_{\otimes I}$ with $I \in \mathbb{N}^r$, $r \geq 0$, is defined to be $\ell(y_{\otimes r}) = \ell(I) = r$. For a given $I \in \mathbb{N}^n$, the sum $\sum y_{\otimes J}$ over $J \in \mathbb{N}^n$ with $SSupp(J) = SSupp(I)$, is called the **one-shuffled element** of $y_{\otimes I}$, denoted by $O(y_{\otimes I})$. So $O(y_{\otimes I})$ is the subset of \mathcal{O}_I consisting of elements of length $\ell(I)$. For example, if $I = (2, 0, 1)$, then the corresponding one-shuffle element of $y_{\otimes I} = y_2 \otimes \mathbf{1}_A \otimes y_1$ is $O(y_{\otimes I}) = y_2 \otimes \mathbf{1}_A \otimes y_1 + \mathbf{1}_A \otimes y_2 \otimes y_1 + y_2 \otimes y_1 \otimes \mathbf{1}_A$. On the other hand, $O(y_{\otimes I})$ is itself if I is either an all zero sequence or an all non-zero sequence. It is so named because the sum can be obtained from shuffling the subsequence of $y_{\otimes I}$ of the $\mathbf{1}_A$ -entries with the subsequence of I of the non- $\mathbf{1}_A$ entries (from $SSupp(I)$). To put it in another way, define a relation \sim on $\tilde{\mathcal{J}}$ by $I_1 \sim I_2$ if $\ell(I_1) = \ell(I_2)$ and $SSupp(I_1) = SSupp(I_2)$. Then it is easy to check that \sim is an equivalence relation and a one-shuffled element is of the form $\sum y_{\otimes J}$ where the sum is taken over all J in an equivalence class.

We now give another version of Theorem 3.3.

Theorem 3.7. *Under the hypotheses of Theorem 3.3, the subspace of $\amalg^+(\tilde{A})$ spanned by one-shuffled elements form a Hopf algebra that contains the Hopf algebras $\gamma^+(\amalg^+(\mathbf{k}))$ and $\amalg^+(A)$.*

By Theorem 3.3, we only need to prove the following lemma.

Lemma 3.8. *The product of $\gamma^+(\mathbb{III}^+(\mathbf{k}))$ and $\mathbb{III}^+(A)$ in $\mathbb{III}^+(\tilde{A})$ is given by the subspace generated by one-shuffled elements.*

Proof. To prove the proposition, let U be the product of $\gamma^+(\mathbb{III}^+(\mathbf{k}))$ and $\mathbb{III}^+(A)$ in $\mathbb{III}^+(\tilde{A})$, and let V be the subspace of one-shuffled elements of $\mathbb{III}^+(\tilde{A})$. Then by Lemma 3.6 and the comments before the theorem, we have $U \subseteq V$. To prove $V \subseteq U$, we only need to show that, for each $k \geq 0$ and $I \in (\mathbb{N}^+)^n$, $n \geq 0$, the one-shuffled element $\mathbf{1}_A^{\otimes k} \mathbb{III} x_{\otimes I}$ is in U . When $n = 0$, $x_{\otimes I} = \mathbf{1}$. So $\mathbf{1}_A^{\otimes k} \mathbb{III} x_{\otimes I} = \mathbf{1}_A^{\otimes k}$ which is in $\gamma^+(\mathbb{III}^+(\mathbf{k}))$ and hence in U . When $n \geq 1$, we use induction on k . When $k = 0$, then $\mathbf{1}_A^{\otimes k} \mathbb{III} x_{\otimes I} = x_{\otimes I}$ which is in $\mathbb{III}^+(A)$, hence is in U . Assume that it is true for $\mathbf{1}_A^{\otimes k}$, $k < m$ and consider $\mathbf{1}_A^{\otimes m} \mathbb{III} x_{\otimes I}$. By Lemma 3.6, we have

$$\mathbf{1}_A^{\otimes m} \diamond^+ y_{\otimes I} = \sum_{i=0}^m \lambda^i \binom{n}{i} \mathbf{1}_A^{\otimes(m-i)} \mathbb{III} y_{\otimes I}.$$

The left hand side of the equation is in U and, by induction, every term on the right hand side except the first one (with $i = 0$) is also in U . Thus the first term, which is $\mathbf{1}_A^{\otimes m} \mathbb{III} y_{\otimes I}$, is also in U . This completes the induction. \square

Acknowledgements. We thank Zongzhu Lin for helpful discussions. The first author thanks the I.H.É.S. for its warm hospitality, and the Ev. Studienwerk for financial support.

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