

The Bloch-Kato conjecture for adjoint motives of modular forms

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ABSTRACT. The Tamagawa number conjecture of Bloch and Kato describes the behavior at integers of the L -function associated to a motive over \mathbf{Q} . Let f be a newform of weight $k \geq 2$, level N with coefficients in a number field K . Let M be the motive associated to f and let A be the adjoint motive of M . Let λ be a finite prime of K . We verify the λ -part of the Bloch-Kato conjecture for $L(A, 0)$ and $L(A, 1)$ when $\lambda \nmid Nk!$ and the mod λ representation associated to f is absolutely irreducible when restricted to the Galois group over $\mathbf{Q} \left(\sqrt{(-1)^{(\ell-1)/2\ell}} \right)$ where $\lambda \mid \ell$.

This is a summary of results on the Tamagawa number conjecture of Bloch and Kato [B-K] for adjoint motives of modular forms of weight $k \geq 2$. The conjecture relates the value at 0 of the associated L -function to arithmetic invariants of the motive. We prove in [D-F-G] that it holds up to powers of certain “bad primes.” The strategy for achieving this is essentially due to Wiles [Wi], as completed with Taylor in [T-W]. The Taylor-Wiles construction yields a formula relating the size of a certain module measuring congruences between modular forms to that of a certain Galois cohomology group. This was carried out in [Wi] and [T-W] in the context of modular forms of weight 2, where it was used to prove results in the direction of the Fontaine-Mazur conjecture [F-M]. While it was no surprise that the method could be generalized to higher weight modular forms and that the resulting formula would be related to the Bloch-Kato conjecture, there remained many technical details to verify in order to accomplish this. In particular, the very formulation of the conjecture relies on a comparison isomorphism between the ℓ -adic and de Rham realizations of the motive provided by theorems of Faltings [Fa] or Tsuji [Ts], and verification of the conjecture requires the careful application of such a theorem. We also need to generalize results on congruences between modular forms to higher weight, and to compute certain local Tamagawa numbers.

Suppose f is a newform of weight $k \geq 2$, level N and character ψ^{-1} with coefficients in the ring of integers \mathcal{O} of a number field K . For $* = B, \text{ dr}$ or λ (a prime of K), we let M_* denote the corresponding realization of the motive attached to f (see [De2]). This is a two-dimensional K_* -subspace (where $K_B = K_{\text{dr}} = K$) of

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the corresponding cohomology group of the modular curve $X_1(N)$ with coefficients in a sheaf depending on k . We have certain additional structure on each M_* (an involution, filtration or $G_{\mathbf{Q}}$ -action) and comparison isomorphisms relating certain pairs of realizations. Let S be a set of primes containing those dividing $Nk!$, and let $\mathcal{O}_B = \mathcal{O}_{\text{dr}} = \mathcal{O}_S$ denote the S -integers in K . For $* = B, \text{dr}$ or λ for $\lambda \notin S$, we also have natural \mathcal{O}_* -lattices \mathcal{M}_* in M_* and integral versions of the comparison isomorphisms involving the \mathcal{M}_λ .

Let D_* denote the $*$ -realization of the $(1-k)$ -Tate twist of the motive associated to ψ . Poincaré duality for $X_1(N)$ gives rise to a perfect alternating pairing on M_* with values in D_* , i.e., an isomorphism

$$\mu_* : \wedge_{K_*}^2 M_* \rightarrow D_*,$$

respecting the various additional structures and comparison isomorphisms. We also have natural \mathcal{O}_* -lattices \mathcal{D}_* in D_* , but the map $\wedge_{\mathcal{O}_*}^2 \mathcal{M}_* \rightarrow \mathcal{D}_*$ need not be an isomorphism; the image is $\eta \mathcal{D}_*$ for a certain ideal η of \mathcal{O}_S . More generally, if Σ is a finite set of primes contained in S , we analogously define \mathcal{M}_*^Σ , μ_*^Σ and η^Σ using level structure which is, roughly speaking, minimal outside Σ .

We let A_* denote the set of trace-zero elements in $\text{End}_{K_*} M_*$. So A_* is three-dimensional over K_* and we have comparison isomorphisms, including

$$I_\infty : A_{\text{dr}} \otimes \mathbf{C} \rightarrow A_B \otimes \mathbf{C},$$

induced by those for the M_* . Replacing f by a twist does not change this data, so we always assume f has minimal conductor among its twists. The L -function associated to A_λ is independent of λ ; we denote it $L(A, s)$ and regard it as taking values in $K \otimes \mathbf{C}$. We know also that $L(A, s)$ extends to an entire function on the complex plane and satisfies the functional equation (see [G-J], [Sc]),

$$\Lambda(A, s) = \epsilon(A, s) \Lambda(A, 1 - s),$$

where

$$\Lambda(A, s) = L(A, s) \Gamma_{\mathbf{R}}(s + 1) \Gamma_{\mathbf{C}}(s + k - 1)$$

and $\epsilon(A, s)$ is as defined by Deligne [De1]. Moreover $L(A, 0)$ and $L(A, 1)$ are invertible elements of $\mathbf{R} \otimes K$, and $\epsilon(A, s) = \pm C^{1-2s}$ for some positive integer C dividing N .

We show that the conjecture of Bloch and Kato [B-K] correctly predicts the values of $L(A, 0)$ and $L(A, 1)$ up to an element of \mathcal{O}_S^\times , where S is a certain finite set of primes containing those dividing $Nk!$. We use the formulation of the conjecture given by Fontaine and Perrin-Riou [F-P], which is easily generalized to the setting of K -coefficients.

The *fundamental line* for A is the K -line defined by

$$\begin{aligned} \Delta &= \text{Hom}_K(A_B^+, A_{\text{dr}}/\text{Fil}^0 A_{\text{dr}}) \\ &\cong \text{Hom}_K(\text{Fil}^{k-1} M_{\text{dr}}, M_{\text{dr}}/\text{Fil}^{k-1} M_{\text{dr}}). \end{aligned}$$

The *Deligne period* for A is the natural basis c^+ for $\Delta \otimes \mathbf{R}$ gotten from I_∞ . One finds that $c^+ L(A, 0)^{-1}$ is an element $\delta \in \Delta$ characterized by

$$(0.1) \quad \mu_{\text{dr}}(f \wedge \delta(f)) = \frac{i^{k-2-\eta}(k-2)! \epsilon(M) \prod_{p \in P} (1 + p^{-1})}{2\epsilon(D)\epsilon(A)} \cdot G_{\text{dr}},$$

where $\eta \in \{0, 1\}$ has the same parity as k , P is a certain exceptional set of primes dividing N and G_{dr} (for Gauss sum) is the usual basis for D_{dr} . This is proved by

relating the Deligne period to the Petersson product, applying a method of Rankin and Shimura to relate this to $L(A, 1)$ (see [St], [Hi], [Sc]) and using the functional equation.

To state the λ -part of the Bloch-Kato conjecture, one chooses a Galois-stable \mathcal{O}_λ -lattice \mathcal{A}_λ in $A_\lambda \cong A_B \otimes_K K_\lambda$ and an \mathcal{O}_λ -lattice ω in $(A_{\text{dr}}/\text{Fil}^0 A_{\text{dr}}) \otimes_K K_\lambda$. The conjecture, which turns out to be independent of these choices, states that

- $H_f^1(G_{\mathbf{Q}}, A_\lambda/\mathcal{A}_\lambda)$ is finite;
- δ generates the \mathcal{O}_λ -submodule

$$\frac{\text{Fitt}_{\mathcal{O}_\lambda} H^0(G_{\mathbf{Q}}, A_\lambda/\mathcal{A}_\lambda) \cdot \text{Fitt}_{\mathcal{O}_\lambda} H^0(G_{\mathbf{Q}}, B_\lambda/\mathcal{B}_\lambda)}{\text{Tam}_\omega^0(\mathcal{A}_\lambda) \cdot \text{Fitt}_{\mathcal{O}_\lambda} H_f^1(G_{\mathbf{Q}}, A_\lambda/\mathcal{A}_\lambda)} \cdot \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{A}_\lambda^+, \omega)$$

of $\Delta \otimes_K K_\lambda$, where $\mathcal{B}_\lambda = \text{Hom}_{\mathbf{Z}_\ell}(\mathcal{A}_\lambda, \mathbf{Z}_\ell(1))$, $B_\lambda = \mathcal{B}_\lambda \otimes \mathbf{Q}$ and $\text{Tam}_\omega^0(\mathcal{A}_\lambda)$ is the Tamagawa ideal of \mathcal{A}_λ relative to ω .

The two main problems are to compute the local contribution at ℓ to the Tamagawa ideal, $\text{Tam}_{\ell, \omega}(\mathcal{A}_\lambda)$, and the length of $H_f^1(G_{\mathbf{Q}}, A_\lambda/\mathcal{A}_\lambda)$. We restrict our attention to primes λ not in S , and let \mathcal{A}_* denote the set of elements of $\text{End}_{\mathcal{O}_*}(\mathcal{M}_*)$ of trace zero. One can then use existing machinery to show that $\text{Tam}_{\ell, \omega}(\mathcal{A}_\lambda) = \mathcal{O}_\lambda$, where $\omega = (A_{\text{dr}}/\text{Fil}^0 A_{\text{dr}}) \otimes_{\mathcal{O}_{\text{dr}}} \mathcal{O}_\lambda$. The proof requires only that λ not divide $Nk!$ and uses the integral version of Faltings' comparison theorem [Fa]. For primes between k and $2k$, the argument is slightly delicate since the filtration length of A_{dr} is greater than $\ell - 2$.

The computation of the H_f^1 relies on the methods of Wiles [Wi] and Taylor-Wiles [T-W], as modified in [Di1] and [Di2]. Now we have to impose another hypothesis on λ :

- (2) $\bar{\rho} : G_{\mathbf{Q}} \rightarrow \text{Aut}_{\mathcal{O}_\lambda/\lambda}(\mathcal{M}_\lambda/\lambda\mathcal{M}_\lambda)$ has absolutely irreducible restriction to G_L , where $L = \mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$.

We give an axiomatic formulation of the method of the following nature: Suppose

$$\mathcal{R} = \{\rho : G_{\mathbf{Q}} \rightarrow \text{Aut}_{K_\rho} V_\rho\}$$

is a set of liftings of $\bar{\rho}$, where for each ρ in \mathcal{R} , K_ρ is a finite extension of K_λ in \bar{K}_λ and the restriction of ρ to $G_{\mathbf{Q}_\ell}$ is crystalline of Hodge-Tate type $(0, k-1)$ (see [Fo]). For any finite set of primes Σ not containing ℓ , let \mathcal{R}^Σ denote the set of $\rho \in \mathcal{R}$ such that ρ is minimally ramified outside Σ . Suppose that for each $\rho \in \mathcal{R}^\Sigma$ we are given an isomorphism

$$\mu_\rho^\Sigma : \wedge_{K_\rho}^2 V_\rho \rightarrow D_\lambda \otimes_{K_\lambda} K_\rho.$$

We assume that for fixed ρ and varying Σ , the ratios of the μ_ρ^Σ are determined by Euler factors. We assume also that $\bigoplus_{\rho \in \mathcal{R}} (V_\rho \otimes_{K_\rho} \bar{K}_\lambda)$ contains a lattice \mathcal{V} having a certain self-duality property with respect to the pairings. We show that if the numbers $\#\mathcal{R}^\Sigma$ satisfy a certain numerical criterion, then

- every lifting of $\bar{\rho}$ which is crystalline of Hodge-Tate type $(0, k-1)$ is isomorphic to an element of \mathcal{R} ;
- if $\rho \in \mathcal{R}^\Sigma$ and $K_\rho = K_\lambda$, then

$$\text{Fitt}_{\mathcal{O}_\lambda} H_\Sigma^1(G_{\mathbf{Q}}, \mathcal{W}_\rho \otimes_{\mathcal{O}_\lambda} (K_\lambda/\mathcal{O}_\lambda)) = \text{Fitt}_{\mathcal{O}_\lambda} \mathcal{D}_\lambda / \mu_\rho^\Sigma (\wedge_{\mathcal{O}_\lambda}^2 \mathcal{V}_\rho),$$

where $\mathcal{V}_\rho = \mathcal{V} \cap V_\rho$ and \mathcal{W}_ρ is the set of elements of $\text{End}_{\mathcal{O}_\lambda} \mathcal{V}_\rho$ of trace zero.

Generalizing results and methods of Ribet, Wiles and others ([**Ri**], [**D-T**], [**Wi**], [**T-W**], [**Di1**]), we verify that these hypotheses are satisfied by the set \mathcal{R} of liftings of $\bar{\rho}$ associated to newforms of weight k , level prime to ℓ and character ψ^{-1} . We thus obtain the following result in the direction of the Fontaine-Mazur conjecture [**F-M**]:

THEOREM 0.1. *Suppose $\rho : G_{\mathbf{Q}} \rightarrow \mathrm{Aut}_{K_{\lambda}} V_{\lambda}$ is continuous, two-dimensional, unramified outside a finite set of primes and has crystalline restriction to $G_{\mathbf{Q}_{\ell}}$ of Hodge-Tate type $(0, d)$ with $0 < d < \ell - 1$. If $\bar{\rho}$ is modular and has absolutely irreducible restriction to G_L , then ρ is modular.*

We also find that $\mathrm{Fitt}_{\mathcal{O}_{\lambda}} H_{\Sigma}^1(G_{\mathbf{Q}}, A_{\lambda}/\mathcal{A}_{\lambda}) = \eta^{\Sigma} \mathcal{O}_{\lambda}$. Using the modified Poitou-Tate sequence (see [**F12**]), we can then compute the length of $H_f^1(G_{\mathbf{Q}}, A_{\lambda}/\mathcal{A}_{\lambda})$, which is dual to $H_f^1(G_{\mathbf{Q}}, B_{\lambda}/\mathcal{B}_{\lambda})$ [**F11**]. Note that (2) implies that $H^0(G_{\mathbf{Q}}, A_{\lambda}/\mathcal{A}_{\lambda})$ and $H^0(G_{\mathbf{Q}}, B_{\lambda}/\mathcal{B}_{\lambda})$ both vanish. Applying equation (1) and its analogue for $L(A, 1)$, we conclude:

THEOREM 0.2. *Let f be a newform of weight $k \geq 2$ and level N with coefficients in a number field K . Let S denote the set of primes λ such that λ divides $Nk!$ or (2) fails. For λ not in S , the λ -part of the Bloch-Kato conjecture holds for $L(A, 0)$ and $L(A, 1)$.*

Finally we remark that the set S is finite. In fact, if λ does not divide $Nk!$ and (2) fails, then either $\ell = 2k - 1$ or $\mathcal{M}_{\lambda}/\lambda\mathcal{M}_{\lambda}$ defines a reducible representation of $G_{\mathbf{Q}}$, and reducibility for infinitely many λ would violate the Ramanujan conjecture.

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