

Calculus 1: Sample Questions, Final Exam, Solutions

1. Short answer. Put your answer in the blank. **NO PARTIAL CREDIT!**

- (a) Evaluate $\int_{e^2}^{e^3} \frac{1}{x} dx$. Your answer should be in the form of *an integer*.

Solution: $\int_{e^2}^{e^3} \frac{1}{x} dx = \ln|x| \Big|_{e^2}^{e^3} = \ln|e^3| - \ln|e^2| = \ln(e^3) - \ln(e^2) = 3 - 2 = 1.$

- (b) Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta$. Your answer should be in the form of *an integer*.

Solution: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta = \sin \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2}) = 1 - (-1) = 2.$

- (c) Compute $e^{\ln 3 + \ln 2}$. Your answer should be in the form of *an integer*.

Solution: $e^{\ln 3 + \ln 2} = e^{\ln 3} e^{\ln 2} = 3 \cdot 2 = 6.$

- (d) Compute $\int_5^5 \frac{x^3}{e^x + 9} dx$. Your answer should be in the form of *an integer*.

Solution: $\int_5^5 \frac{x^3}{e^x + 9} dx = 0$ since it is a definite integral over an interval of zero length. (This integral is probably impossible to do otherwise.)

- (e) Evaluate $\int 3x^2 \sin(x^3 + 1) dx$. + C

Solution: Calculate for a substitution $u = x^3 + 1$, $du = 3x^2 dx$, and so

$$\int 3x^2 \sin(x^3 + 1) dx = \int \sin u du = -\cos u + C = -\cos(x^3 + 1) + C.$$

- (f) Evaluate the derivative $D_x e^x$.

Solution: e^x .

- (g) Compute the derivative $D_x \ln(x^2 + 1)$.

Solution: Use the Chain Rule. $D_x \ln(x^2 + 1) =$

$$\frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}.$$

- (h) Evaluate $\int_{-1}^1 \frac{y^5}{y^2 + 1} dy$. Your answer should be in the form of *an integer*.

Solution: 0. This is because it is an integral of an odd function over a symmetric interval $[-1, 1]$.

To see the integrand is an odd function, compute

$$\frac{(-y)^5}{(-y)^2 + 1} = \frac{-y^5}{y^2 + 1} = -\frac{y^5}{y^2 + 1}.$$

You can also do this the long way. Use long division of polynomials to show $\frac{y^5}{y^2 + 1} = y^3 - y + \frac{y}{y^2 + 1}$ and integrate

$$\begin{aligned} \int_{-1}^1 \frac{y^5}{y^2 + 1} dy &= \int_{-1}^1 \left(y^3 - y + \frac{y}{y^2 + 1} \right) dy \\ &= \int_{-1}^1 (y^3 - y) dy + \int_{-1}^1 \frac{y}{y^2 + 1} dy \\ &= \left(\frac{1}{4}y^4 - \frac{1}{2}y^2 \right) \Big|_{-1}^1 + \frac{1}{2} \ln(y^2 + 1) \Big|_{-1}^1 \\ &= \left(\frac{1}{4} - \frac{1}{2} \right) - \left(\frac{1}{4} - \frac{1}{2} \right) + \frac{1}{2} \ln 2 - \frac{1}{2} \ln 2 = 0. \end{aligned}$$

(This computation uses the fact that $\int \frac{y}{y^2 + 1} dy = \frac{1}{2} \ln(y^2 + 1)$, which follows by using the substitution $u = y^2 + 1$.)

- (i) Evaluate $\int_2^{10} 3\sqrt{z-1} dz$. Your answer should be in the form of *an integer*.

Solution: Make the substitution $u = z - 1$, $du = dz$, $u(2) = 1$, $u(10) = 9$ to compute

$$\int_2^{10} 3\sqrt{z-1} dz = \int_1^9 3\sqrt{u} du = 3 \left(\frac{2}{3} u^{\frac{3}{2}} \right) \Big|_1^9 = 2(9^{\frac{3}{2}} - 1^{\frac{3}{2}}) = 2(27 - 1) = 52.$$

2. Let f be a continuous function on the interval $[0, 2]$ which satisfies $\int_0^2 f(x) dx = 5$. Given this information, compute the integral $\int_0^1 f(2y) dy$. Show your work and justify your answer.

Solution: Use the substitution $x = 2y$, $dx = 2 dy$, $dy = \frac{1}{2} dx$, $x(0) = 2(0) = 0$, $x(1) = 2(1) = 2$ to find

$$\int_0^1 f(2y) dy = \int_0^2 f(x) \frac{1}{2} dx = \frac{1}{2} \int_0^2 f(x) dx = \frac{1}{2}(5) = \frac{5}{2}.$$

3. A population of bacteria undergoes exponential growth. If at noon, there are 1000 bacteria, and there are 2000 by 2pm, when does the number of bacteria reach 8000? Show your work and simplify your answer.

Solution: The doubling time is 2 hours, and so to reach 8000 bacteria from 2000, it must double twice (since $8000/2000 = 4 = 2^2$). This means that $2 \cdot 2 = 4$ more hours past 2pm are required, and so the population will be 8000 at 6pm.

Alternately, the general formula for exponential growth is $y = y_0 e^{kt}$, where t is the time in hours past noon and $y_0 = 1000$ is the initial size. We need to solve for the constant k : At $t = 2$, we know the population is $y = 2000$, so we find

$$2000 = y = y(2) = y_0 e^{kt} = 1000 e^{2k}$$

and so $e^{2k} = 2$, $2k = \ln 2$, $k = \frac{1}{2} \ln 2$. So plug this back in to find the general formula

$$y = y_0 e^{kt} = 1000 e^{t(\frac{1}{2} \ln 2)} = 1000 e^{(\ln 2) \frac{t}{2}} = 1000 (e^{\ln 2})^{\frac{t}{2}} = 1000 \cdot 2^{\frac{t}{2}}.$$

Therefore, $y = 8000$ when $8000 = 1000 \cdot 2^{\frac{t}{2}}$, and $2^3 = 8 = 2^{\frac{t}{2}}$, $3 = \frac{t}{2}$ and so $t = 6$ pm.

4. Find the particular solution to the differential equation $\frac{dy}{dx} = xy + x$ which satisfies $y = 3$ when $x = 0$. Show your work.

Solution: This equation is separable:

$$\begin{aligned}\frac{dy}{dx} &= xy + x = x(y + 1), \\ \frac{dy}{y + 1} &= x dx, \\ \int \frac{dy}{y + 1} &= \int x dx, \\ \ln|y + 1| &= \frac{1}{2}x^2 + C \quad (\text{substitute } u = y + 1)\end{aligned}$$

Now the point we are interested in is $y = 3$, which means that $y + 1 > 0$ for our solution. So we can drop the absolute value and find

$$\begin{aligned}\ln(y + 1) &= \frac{1}{2}x^2 + C, \\ y + 1 &= e^{\frac{1}{2}x^2 + C}, \\ y &= e^{\frac{1}{2}x^2 + C} - 1.\end{aligned}$$

For the particular solution, we should plug in $y = 3$ and $x = 0$ to compute C .

$$\begin{aligned}3 &= e^{\frac{1}{2}(0^2) + C} - 1, \\ 4 &= e^C, \\ C &= \ln 4, \\ y &= e^{\frac{1}{2}x^2 + \ln 4} - 1 \\ &= 4 \cdot e^{\frac{1}{2}x^2} - 1.\end{aligned}$$

5. Consider the following functions. Circle the one(s) which are concave up on an open interval containing $x = 0$. No explanation necessary.

- $\ln x$
- x^2
- $\cos x$
- $\frac{1}{x^2 - 1}$
- $\tan x$

Solution: Only x^2 is concave up near $x = 0$:

$\ln x$ is not even defined at $x = 0$.

If $f(x) = x^2$, then $f''(x) = 2 > 0$ and so f is concave up always.

If $g(x) = \cos x$, $g''(x) = -\cos x$ and $g''(0) = -\cos 0 = -1 < 0$, so g is concave down near $x = 0$.

If $h(x) = \frac{1}{x^2 - 1} = (x^2 - 1)^{-1}$, $h'(x) = -2x(x^2 - 1)^{-2}$ and $h''(x) = -2(x^2 - 1)^{-2} - 2x(-2)(x^2 - 1)^{-3}(2x)$ and $h''(0) = -2 < 0$. So h is concave down near $x = 0$.

Finally $k(x) = \tan x$ has an inflection point at $x = 0$. Compute $k'(x) = \sec^2 x$, $k''(x) = 2 \sec x(\sec x \tan x) = 2 \sec^2 x \tan x$, and $k''(0) = 2 \sec^2 0 \tan 0 = 2(1)^2(0) = 0$. We can also see that $k''(x) > 0$ for $0 < x < \frac{\pi}{2}$ (concave up there) and $k''(x) < 0$ for $-\frac{\pi}{2} < x < 0$ (concave down there), and so $x = 0$ is an inflection point.

6. Consider the function $g(x) = \frac{1}{2}x + \cos x$ for $0 < x < 2\pi$.

- (a) Find all the critical points of $g(x)$ for $0 < x < 2\pi$. Show your work. Hint: there are two of them.

Solution: $g'(x) = \frac{1}{2} - \sin x = 0$ when $\sin x = \frac{1}{2}$. This is possible only in the first and second quadrants (this is where the sine function is positive); these correspond to $0 < x < \frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$. To find the specific value, note that $\sin \frac{\pi}{6} = \frac{1}{2}$ for $\frac{\pi}{6}$ in the first quadrant. The corresponding solution in the second quadrant is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$. So the only 2 critical points are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

- (b) Classify each of the critical points you found in part (a) as a local maximum or a local minimum (or neither). Justify your answers.

Solution: Apply the second derivative test: $g''(x) = -\cos x$. $g''(\frac{\pi}{6}) = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2} < 0$. (You can tell the sign of the solution just by knowing that $\frac{\pi}{6}$ is in the first quadrant, and thus its cosine must be positive.) This means $x = \frac{\pi}{6}$ is a local maximum.

Similarly $g''(\frac{5\pi}{6}) = -\cos(\frac{5\pi}{6}) = -(-\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2} > 0$ and so $x = \frac{5\pi}{6}$ is a local minimum.

7. Compute the derivative $\frac{d}{dx} \left[\frac{x^2 + 1}{x^3(x - 1)^2} \right]$. Show your work.

Solution: Use logarithmic differentiation.

$$\begin{aligned}y &= \frac{x^2 + 1}{x^3(x-1)^2}, \\ \ln y &= \ln \left[\frac{x^2 + 1}{x^3(x-1)^2} \right] = \ln(x^2 + 1) - 3 \ln x - 2 \ln(x-1), \\ \frac{d}{dx}(\ln y) &= \frac{d}{dx}[\ln(x^2 + 1) - 3 \ln x - 2 \ln(x-1)], \\ \frac{1}{y} \frac{dy}{dx} &= \frac{2x}{x^2 + 1} - \frac{3}{x} - \frac{2}{x-1}, \\ \frac{dy}{dx} &= y \left[\frac{2x}{x^2 + 1} - \frac{3}{x} - \frac{2}{x-1} \right] \\ &= \frac{x^2 + 1}{x^3(x-1)^2} \left[\frac{2x}{x^2 + 1} - \frac{3}{x} - \frac{2}{x-1} \right] \\ &= \frac{2}{x^2(x-1)^2} - \frac{3(x^2 + 1)}{x^4(x-1)^2} - \frac{2(x^2 + 1)}{x^3(x-1)^3} \\ &= \frac{2x^2(x-1) - 3(x^2 + 1)(x-1) - 2(x^2 + 1)x}{x^4(x-1)^3} \\ &= \frac{-3x^3 + x^2 - 5x + 3}{x^4(x-1)^3}\end{aligned}$$

8. Consider the function $h(x) = \frac{e^x + e^{-x}}{2}$ for $-\infty < x < \infty$.

(a) Find the interval(s) on which $h(x)$ is increasing. Show your work.

Solution: Compute $h'(x) = \frac{1}{2}[e^x + e^{-x}(-1)] = \frac{1}{2}(e^x - e^{-x})$. So $h'(x) = 0$ if

$$\begin{aligned}\frac{1}{2}(e^x - e^{-x}) &= 0, \\ e^x &= e^{-x}, \\ e^{2x} &= 1, \\ \ln(e^{2x}) &= \ln 1 = 0, \\ 2x &= 0, \\ x &= 0.\end{aligned}$$

So this splits the real line up into two intervals $(-\infty, 0)$ and $(0, \infty)$. Check for each interval $h'(-1) = \frac{1}{2}(e^{-1} - e) < 0$ (use your calculator

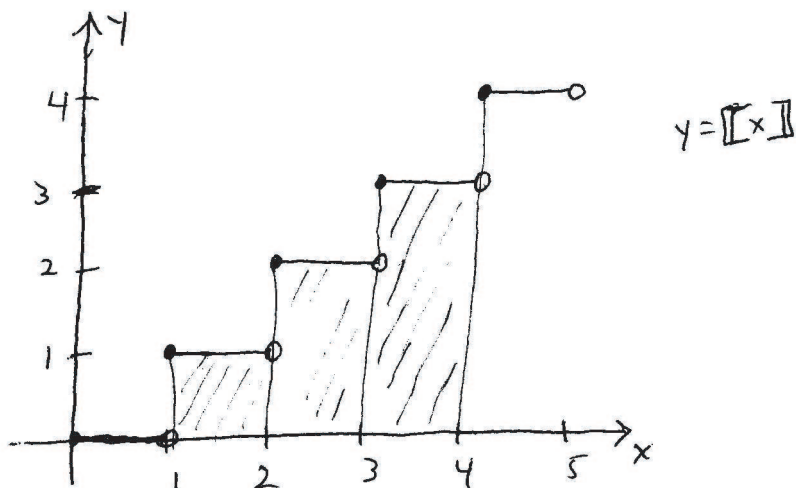
or the fact that $e > 1$). So $h'(x) < 0$ on the interval $(-\infty, 0)$. Similarly, plug in $h'(1) = \frac{1}{2}(e - e^{-1}) > 0$ and so $h'(x) > 0$ on the interval $(0, \infty)$. Thus h is increasing on the interval $(0, \infty)$.

(b) Show that $h(x)$ is always concave up.

Solution: Compute $h''(x) = \frac{1}{2}(e^x + e^{-x}) = h(x)$, which is clearly always positive. Therefore h is always concave up.

9. Recall that $\llbracket x \rrbracket$ represents the greatest integer function ($\llbracket x \rrbracket$ is the greatest integer $\leq x$). Compute $\int_0^4 \llbracket x \rrbracket dx$. Show your work. (Hint: Draw the graph.)

Solution: See the picture below. The integral is equal to the area under the graph, which is $0 + 1 + 2 + 3 = 6$.



10. (a) If $f(x) = \sin x$, what is $f'(x)$?

Solution: $\cos x$.

(b) For $f(x) = \sin x$, write down the formula for $f'(0)$ using the definition of the derivative.

Solution:

$$\begin{aligned}f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(0+h) - \sin 0}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin h}{h}.\end{aligned}$$

- (c) Use parts (a) and (b) to compute the limit $\lim_{h \rightarrow 0} \frac{\sin h}{h}$. Your answer should be in the form of *an integer*. Justify your answer.

Solution: $\lim_{h \rightarrow 0} \frac{\sin h}{h} = f'(0) = \cos 0 = 1.$

11. Consider the function $p(r) = r^3 + 6r^2 + 9r - 4$.

- (a) Find all the critical points of $p(r)$ for r in the interval $[-2, 2]$. Show your work.

Solution: The critical points in this interval are $r = -2, 2, -1$. First of all, the endpoints $r = -2, 2$ are critical points.

Compute $p'(r) = 3r^2 + 12r + 9 = 0$ if $3(r+3)(r+1) = 0$ if $r = -1, -3$. Only $r = -1$ is in our interval, so this is the only stationary critical point.

- (b) For which r does $p(r)$ attain its minimum and maximum on the interval $[-2, 2]$? Show your work.

Solution: Compute

$$\begin{aligned}p(-2) &= (-2)^3 + 6(-2)^2 + 9(-2) - 4 = -8 + 24 - 18 - 4 = -6, \\p(-1) &= (-1)^3 + 6(-1)^2 + 9(-1) - 4 = -1 + 6 - 9 - 4 = -8, \\p(2) &= 2^3 + 6(2^2) + 9(2) - 4 = 8 + 24 + 18 - 4 = 46.\end{aligned}$$

So $r = -1$ is the minimum and $r = 2$ is the maximum.

12. Compute the limit $\lim_{u \rightarrow 3} \frac{u^2 - 9}{u^2 - 4u + 3}$. Show your work.

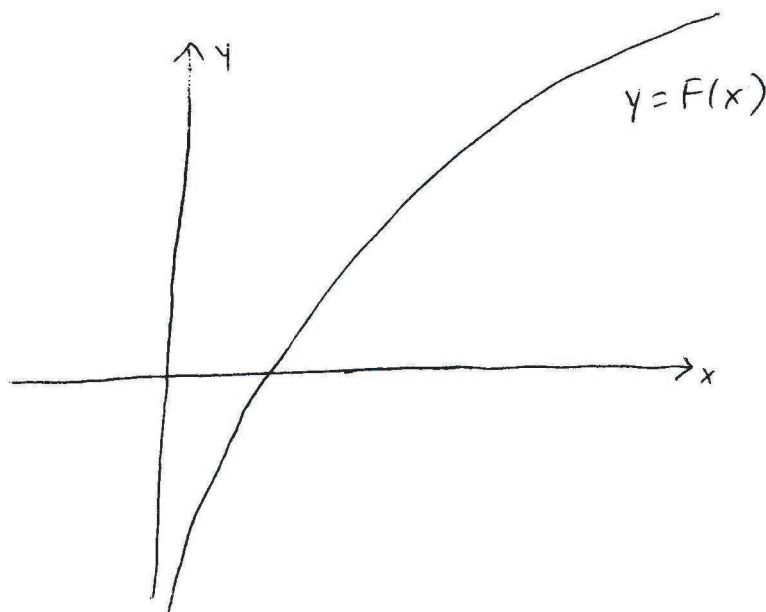
Solution: First of all, if we plug in $u = 3$, we get $\frac{0}{0}$, and so we must do some work:

$$\lim_{u \rightarrow 3} \frac{u^2 - 9}{u^2 - 4u + 3} = \lim_{u \rightarrow 3} \frac{(u-3)(u+3)}{(u-3)(u-1)} = \lim_{u \rightarrow 3} \frac{u+3}{u-1} = \frac{3+3}{3-1} = 3.$$

13. (a) Sketch the graph of a function $y = F(x)$ which satisfies all the following properties:

- $F(x)$ has domain $(0, \infty)$.
- $F(x)$ has a vertical asymptote at $x = 0$.
- $F(x)$ is increasing on the interval $(0, \infty)$.
- $F(x)$ is concave down on the interval $(0, \infty)$.

Solution:



(b) Give a formula for a function $F(x)$ which satisfies all the properties listed in part (a). Justify your answer.

Solution: $F(x) = \ln x$ works: It has the correct domain. $\lim_{x \rightarrow 0^+} \ln x = -\infty$ and so there is a vertical asymptote at $x = 0$. $F'(x) = \frac{1}{x} > 0$ for $x > 0$ and so F is increasing on $(0, \infty)$. Moreover, $F''(x) = -\frac{1}{x^2} < 0$ for $x > 0$ and so F is concave down on $(0, \infty)$.

There are other possible answers. $F(x) = -\frac{1}{\sqrt{x}}$ also works.