Calculus 1: Sample Questions, Exam 1

1. (a) Use the definition of the derivative to compute the derivative $D_x \frac{1}{x}$. Show your work.

   Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}. \\ D_x \frac{1}{x} &= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{x(x+h)\left(\frac{1}{x+h}-\frac{1}{x}\right)}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{\frac{x(x+h)}{x+h}-\frac{x(x+h)}{x}}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{-h}{x(x+h)h} \\ &= \lim_{h\to 0} -\frac{1}{x(x+h)} \\ &= -\frac{1}{x^2} \end{aligned}$$

   (b) Compute the tangent line to the graph $y = \frac{1}{x}$ at the point $(x, y) = (2, \frac{1}{2})$. Put your answer in the form $y = mx + b$. Show your work.

   Solution: For $f(x) = \frac{1}{x}$, part (a) shows that $f'(x) = -\frac{1}{x^2}$. (You can also use the power rule $D_x x^n = nx^{n-1}$ with $n = -1$ to compute $f'(x) = -x^{-2} = -\frac{1}{x^2}$.)

   The slope of the tangent line at $(2, \frac{1}{2})$ is

$$f'(2) = -\frac{1}{2^2} = -\frac{1}{4}.$$

   So the slope $m = -\frac{1}{4}$. The point-slope formula of a line shows for $(x_0, y_0) = (2, \frac{1}{2})$

$$y - y_0 = m(x - x_0),$$

$$\begin{aligned} y-\frac{1}{2} &= -\frac{1}{4}(x-2) = -\frac{1}{4}x+\frac{1}{2}, \\ y &= -\frac{1}{4}x+1. \end{aligned}$$

2. Compute the derivative $f'(3)$ for $f(t) = t^2 - 3t + 4$. Show your work.

   Solution: $f'(t) = 2t - 3$ and so $f'(3) = 2(3) - 3 = 3$.

3. Let a be a constant, and let $y = \cos ax$. Compute the derivative y'. Show your work.

   Solution: $y = \cos(ax)$, and so the Chain Rule says

$$y' = -\sin(ax) \cdot D_x(ax) = -\sin(ax) \cdot a = -a \sin ax.$$

4. If $z = \sqrt{x^2+1}$, compute $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$. Show your work.

   Solution: By the Chain Rule,

$$\frac{dz}{dx} = \frac{1}{2\sqrt{x^2+1}} D_x(x^2+1) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

   On the other hand,

$$\begin{aligned} \frac{d^2z}{dx^2} &= \frac{d}{dx}\left(\frac{dz}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{x}{\sqrt{x^2+1}}\right) \\ &= \frac{d}{dx}\left(x(x^2+1)^{-\frac{1}{2}}\right) \\ &= D_x x \cdot (x^2+1)^{-\frac{1}{2}} + x \cdot D_x(x^2+1)^{-\frac{1}{2}} \\ &= 1 \cdot (x^2+1)^{-\frac{1}{2}} + x \cdot (-\frac{1}{2})(x^2+1)^{-\frac{3}{2}} \cdot D_x(x^2+1) \\ &= (x^2+1)^{-\frac{1}{2}} + x \cdot (-\frac{1}{2})(x^2+1)^{-\frac{3}{2}} \cdot (2x) \\ &= (x^2+1)^{-\frac{1}{2}} - x^2(x^2+1)^{-\frac{3}{2}} \\ &= \frac{1}{(x^2+1)^{\frac{1}{2}}} - \frac{x^2}{(x^2+1)^{\frac{3}{2}}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(x^2+1)}{(x^2+1)^{\frac{3}{2}}} - \frac{x^2}{(x^2+1)^{\frac{3}{2}}} \\
&= \frac{1}{(x^2+1)^{\frac{3}{2}}}.
\end{aligned}
$$

5. If $g(x)$ is a function satisfying $g(3) = 1$ and $g'(3) = 4$, and $f(x) = \frac{x^3}{g(x)}$, compute $f'(3)$. Show your work.

   Solution: Compute using the Quotient Rule

$$
\begin{aligned}
f'(x) &= \frac{g(x) \cdot 3x^2 - x^3 g'(x)}{g^2(x)}, \\
f'(3) &= \frac{g(3) \cdot 3(3^2) - 3^3 g'(3)}{g^2(3)} \\
&= \frac{1 \cdot 3(3^2) - 3^3 \cdot 4}{1^2} \\
&= \frac{27 - 108}{1} \\
&= -81
\end{aligned}
$$

6. Compute the limit $\lim_{y \to 2} \frac{y^2 - 4y + 4}{y^2 - 5y + 6}$. Show your work.

   Solution: Compute

$$
\begin{aligned}
\lim_{y \to 2} \frac{y^2 - 4y + 4}{y^2 - 5y + 6} &= \lim_{y \to 2} \frac{(y-2)^2}{(y-2)(y-3)} \\
&= \lim_{y \to 2} \frac{y-2}{y-3} \\
&= \frac{2-2}{2-3} \\
&= 0.
\end{aligned}
$$

7. Put the letter of the derivative of the function in the blank.

answer	$f(x)$		choices for $f'(x)$
		(a)	sin
(f)	$\cot x$	(b)	$\csc^2 x$
		(c)	$\cos x$
(k)	$\sqrt{x}$	(d)	$-\frac{1}{x^2}$
		(e)	$-\frac{1}{2}x^{-\frac{1}{2}}$
(g)	$\sec 2x$	(f)	$-\csc^2 x$
		(g)	$2\sec 2x \tan 2x$
(d)	x^{-1}	(h)	$2\sec^2 x \tan x$
		(i)	1^{-1}
(c)	$\sin x$	(j)	$2\tan^2 2x$
		(k)	$\frac{1}{2\sqrt{x}}$

Why are these true? The derivative of $\cot x$ is $-\csc^2 x$. The derivative

$$D_x\sqrt{x} = D_x x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

The derivative of $\sec 2x$ can be computed by the Chain Rule to be

$$\sec 2x \tan 2x \cdot D_x(2x) = 2\sec 2x \tan 2x.$$

The derivative

$$D_x x^{-1} = -1x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$

Finally the derivative of $\sin x$ is $\cos x$.

8. What is the slope of the tangent line to the graph of the relation

$$x^2y^2 + y^3 - y = \sqrt{x}$$

at the point $(x, y) = (1, 1)$? Show your work.

Solution: Differentiate both sides of the equation with respect to x to find

$$\begin{aligned} (x^2y^2 + y^3 - y)' &= (\sqrt{x})', \\ x^2(2yy') + 2xy^2 + 3y^2y' - y' &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

Plug in the values $(x, y) = (1, 1)$ and then solve for y':

$$\begin{aligned} 1^2(2 \cdot 1 \cdot y') + 2 \cdot 1 \cdot 1^2 + 3 \cdot 1^2 \cdot y' - y' &= \frac{1}{2\sqrt{1}}, \\ 2y' + 2 + 3y' - y' &= \frac{1}{2}, \\ 4y' &= -\frac{3}{2}, \\ y' &= -\frac{3}{8} \end{aligned}$$

is the slope of the graph of the relation at the point $(1, 1)$.

9. A spherical balloon is being inflated at a rate of π in^3/sec. How fast is the radius of the balloon changing when the volume of the balloon is $\frac{4}{3}\pi$ in^3? Show your work. (Recall the formula for the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.)

Solution: We know $V = \frac{4}{3}\pi r^3$. So $\frac{dV}{dr} = 4\pi r^2$. We also are given that $\frac{dV}{dt} = \pi$. When $V = \frac{4}{3}\pi$ in^3, we can solve to find

$$\begin{aligned} \frac{4}{3}\pi r^3 &= V = \frac{4}{3}\pi, \\ r^3 &= 1, \\ r &= 1. \end{aligned}$$

So when $V = \frac{4}{3}\pi$ in^3, $r = 1$ in. We're asked to find the rate of change $\frac{dr}{dt}$. Compute using the Chain Rule

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt}, \\ \frac{dr}{dt} &= \left(\frac{dV}{dt}\right) / \left(\frac{dV}{dr}\right) \\ &= (\pi)/(4\pi r^2) \\ &= \frac{1}{4r^2} \\ &= \frac{1}{4} \end{aligned}$$

since $r = 1$ as we computed above. So the rate of change of the radius $\frac{dr}{dt}$ is $\frac{1}{4}$ in/sec.