Calculus 1: Sample Questions, Exam 2

1. Short answer. Put your answer in the blank. **NO PARTIAL CREDIT!**

   (a) Evaluate $\int \cos x \, dx$. $+C$
   Solution: $\sin x + C$.

   (b) Evaluate $G'(x)$ for $G(x) = \int_0^x t^2 \, dt$.
   Solution: x^2 by the First Fundamental Theorem of Calculus.

   (c) Compute $\int (x^3 + x) \, dx$. $+C$
   Solution: $\frac{1}{4}x^4 + \frac{1}{2}x^2 + C$.

   (d) Compute the sum $\sum_{n=1}^{4} n^2$. Your answer should be in the form of *an integer*.
   Solution: $1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$.

2. Identify the critical points and find the maximum value and the minimum value for $f(x) = x^3 - 3x + 2$ on the interval $[0, 2]$. Show your work.

   Solution: Compute $f'(x) = 3x^2 - 3$. $f'(x) = 0$ when $3x^2 - 3 = 0$, $x^2 = 1$. So x is 1 or -1. Only the critical point $x = 1$ is in our interval $[0, 2]$.

   So we have 3 critical points: Endpoints $x = 0, 2$, and $x = 1$ as a stationary critical point. Now compute $f(0) = 2$, $f(1) = 0$, $f(2) = 4$. So the min is at $f(1) = 0$, and the max is at $f(2) = 4$.

3. Sketch the graph of a function $y = g(x)$ that has all the following properties:

(a) The domain of g is the open interval $(-3, 3)$.

(b) g is an odd function.

(c) $\lim_{x \to 3^-} g(x) = +\infty$.

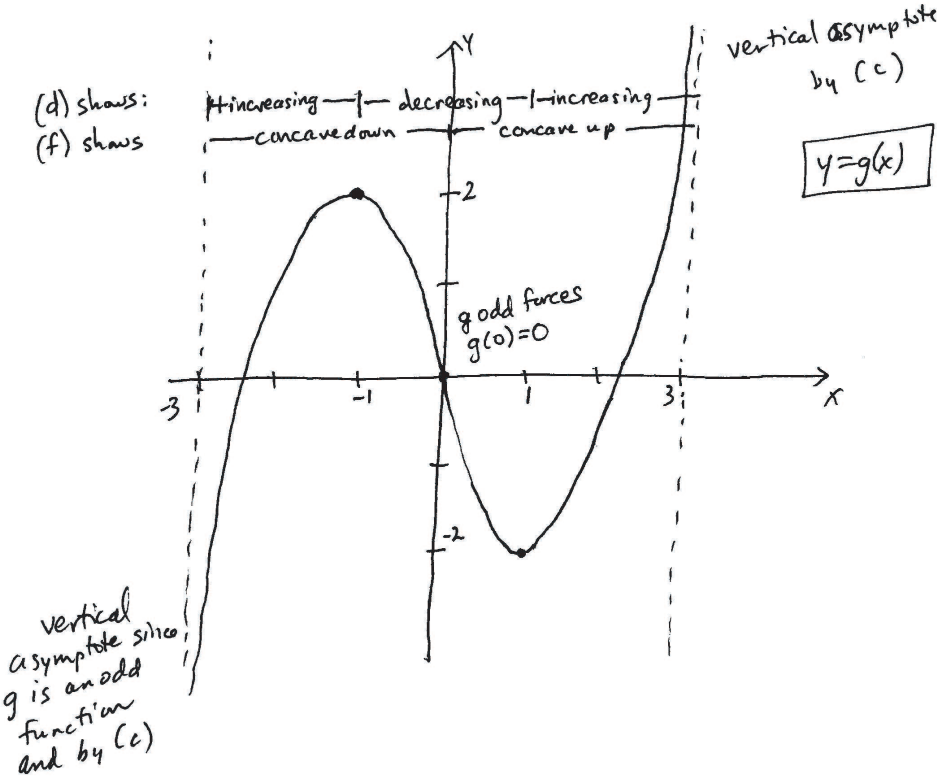
(d) $g'(x) < 0$ for x in $(-1, 1)$, while $g'(x) > 0$ for x in $(-3, -1)$ and for x in $(1, 3)$.

(e) $g'(-1) = g'(1) = 0$.

(f) $g''(x) < 0$ for x in $(-3, 0)$, while $g''(x) > 0$ for x in $(0, 3)$.

(g) $g''(0) = 0$.

(h) $g(1) = -2$, $g(-1) = 2$.



4. Show that for a rectangle of given perimeter 4, the one with the maximum area is a square. (Recall that for a rectangle with side lengths a and b, the perimeter is given by $2a + 2b$, and the area is given by ab.)

   Solution: The perimeter $P = 4 = 2a + 2b$, while the area $A = ab$. We want to maximize A. To make A a function of a single variable, solve for $4 = 2a + 2b$ to find $a = 2 - b$, and plug in to find

$$A = ab = (2 - b)b = 2b - b^2.$$

   To find critical points, compute

$$\frac{dA}{db} = 2 - 2b, \qquad \frac{d^2A}{db^2} = -2.$$

   So the only critical point is when $2 - 2b = 0$, which is when $b = 1$. So $A''(1) = -2 < 0$ and the second derivative test shows $b = 1$ is a local maximum. Since this is the only critical point, this must be the global maximum. For $b = 1$, then we find $a = 2 - b = 2 - 1 = 1$. So for the maximum area, the two sides $b = 1$ and $a = 1$, and so the rectangle is a square.

5. Find the particular solution to the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ which satisfies the condition that $y = 4$ when $x = 1$. Show your work.

   Solution: This equation is separable. Compute

$$\begin{aligned}
y^2\,dy &= x^2\,dx, \\
\int y^2\,dy &= \int x^2\,dx, \\
\frac{1}{3}y^3 &= \frac{1}{3}x^3 + C, \\
\frac{1}{3}(4)^3 &= \frac{1}{3}(1)^3 + C, \\
\frac{1}{3}(64) &= \frac{1}{3} + C, \\
21 &= C, \\
\frac{1}{3}y^3 &= \frac{1}{3}x^3 + 21, \\
y^3 &= x^3 + 63, \\
y &= \sqrt[3]{x^3 + 63}.
\end{aligned}$$

6. Consider $h(t) = 2t^3 - 3t^2 + 5$.

(a) Compute the critical points of $h(t)$. Show your work.

Solution: $h'(t) = 6t^2 - 6t$. $h'(t)$ always exists, so we just need to find $h'(t) = 0$.

$$\begin{aligned} 6t^2 - 6t &= 0, \\ 6t(t-1) &= 0. \end{aligned}$$

So $t = 0$ and $t = 1$ are the only critical points of $h(t)$.

(b) Compute the intervals where $h(t)$ is increasing, and where $h(t)$ is decreasing. Show your work.

Solution: The critical points $t = 0, 1$ split the real line into 3 intervals $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$. Pick representative points in each interval.

For $t = -1$ in $(-\infty, 0)$, $h'(-1) = 6(-1)^2 - 6(-1) = 12 > 0$. So $h'(t) > 0$ on the interval $(-\infty, 0)$, and h is increasing there.

For $t = 1/2$ in $(0, 1)$, $h'(1/2) = 6(1/2)^2 - 6(1/2) = -6/4 < 0$. So $h'(t) < 0$ on the interval $(0, 1)$, and h is decreasing there.

For $t = 2$ in $(1, \infty)$, $h'(2) = 6(2)^2 - 6(2) = 12 > 0$. So $h'(t) > 0$ on the interval $(0, \infty)$, and h is increasing there.

(c) Compute the intervals where $h(t)$ is concave up, and where $h(t)$ is concave down. Show your work.

Solution: Compute $h''(t) = D_t(6t^2 - 6t) = 12t - 6$. $h''(t) = 0$ only if $t = 1/2$. This splits the real line into two intervals $(-\infty, 1/2)$ and $(1/2, \infty)$. Pick representative points in each interval.

For $t = 0$ in $(-\infty, 1/2)$, $h''(0) = 12(0) - 6 = -6 < 0$. So $h''(t) < 0$ on the interval $(-\infty, 1/2)$, and h is concave down there.

For $t = 1$ in $(1/2, \infty)$, $h''(1) = 12(1) - 6 = 6 > 0$. So $h''(t) > 0$ on the interval $(1/2, \infty)$, and h is concave up there.