Calculus I, Sections 1-5, Exam 1, Version B

Solution

1. (15 pts) Compute the following derivatives, and simplify your answers. Show your work if you want to be eligible for partial credit.

   (a) (3 pts) $D_x(x^3 - 2x + 7)$.

   Solution: $D_x(x^3 - 2x + 7) = 3x^2 - 2$.

   (b) (3 pts) $D_x[\cos x(\sin x - 2)]$.

   Solution:

$$\begin{aligned} D_x[\cos x(\sin x - 2)] &= \cos x\, D_x(\sin x - 2) + (\sin x - 2) D_x \cos x \\ &= \cos x(\cos x) + (\sin x - 2)(-\sin x) \\ &= \cos^2 x - \sin^2 x + 2\sin x. \end{aligned}$$

   (c) (3 pts) $D_x\sqrt{x^4 - 3x}$.

   Solution:

$$\begin{aligned} D_x\sqrt{x^4 - 3x} &= \frac{1}{2\sqrt{x^4 - 3x}} D_x(x^4 - 3x) \\ &= \frac{4x^3 - 3}{2\sqrt{x^4 - 3x}}. \end{aligned}$$

   (d) (3 pts) $D_x \csc x$.

   Solution: $D_x \csc x = -\csc x \cot x$.

   (e) (3 pts) $D_x \dfrac{x^4 - 2x^3 + 1}{3x - 9}$.

   Solution:

$$\begin{aligned} D_x \frac{x^4 - 2x^3 + 1}{3x - 9} &= \frac{(3x - 9)D_x(x^4 - 2x^3 + 1) - (x^4 - 2x^3 + 1)D_x(3x - 9)}{(3x - 9)^2} \\ &= \frac{(3x - 9)(4x^3 - 6x^2) - (x^4 - 2x^3 + 1)(3)}{(3x - 9)^2} \\ &= \frac{3x^4 - 16x^3 + 18x^2 - 1}{3(x - 3)^2}. \end{aligned}$$

2. (a) (3 pts) Give an example of a function $p(x)$ whose derivative at 0 is equal to 3. In other words, $p'(0) = 3$. Verify your answer is correct.

   Solution: For example, $p(x) = 3x$, $p'(x) = 3$, $p'(0) = 3$.

   (b) (3 pts) Give an example of a function $g(x)$ which is not differentiable at $x = 0$. In other words, $g'(0)$ should not exist. Give a reason for your answer.

   Solution: For example, $g(x) = |x|$. $g'(0)$ does not exist since the graph of g has a corner at $x = 0$.

3. (4 pts) Use the definition of the derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ to compute the derivative of $f(x) = 4x^2 - 2x + 1$. Show your work.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{4(x+h)^2 - 2(x+h) + 1 - (4x^2 - 2x + 1)}{h} \\ &= \lim_{h\to 0} \frac{8xh + 4h^2 - 2h}{h} \\ &= \lim_{h\to 0}(8x + 4h - 2) \\ &= 8x - 4(0) - 2 \\ &= 8x - 2. \end{aligned}$$

4. (3 pts) Let $h(x)$ be a function which satisfies $h'(4) = 5$. If $k(x) = h(x^2)$, compute $k'(2)$. Show your work.

Solution: The Chain Rule says $k'(x) = h'(x^2)D_x(x^2) = h'(x^2) \cdot 2x$. Then plug in $k'(2) = h'(2^2) \cdot 2(2) = h'(4) \cdot 4 = 5 \cdot 4 = 20$.

5. (3 pts) Compute the limit $\lim_{y\to 2} \frac{y^2 - 2y}{y^2 + 5y - 14}$. Show your work.

Solution:

$$\begin{aligned} \lim_{y\to 2} \frac{y^2 - 2y}{y^2 + 5y - 14} &= \lim_{y\to 2} \frac{y(y-2)}{(y-2)(y+7)} \\ &= \lim_{y\to 2} \frac{y}{y+7} = \frac{2}{2+7} = \frac{2}{9}. \end{aligned}$$

6. (4 pts) Compute the tangent line to the graph of the relation $y^4 + 3xy = 6 + y$ at the point $(x, y) = (2, 1)$. Put your answer in the form $y = mx + b$. Show your work.

Solution: Use Implicit Differentiation to compute y' when $(x, y) = (2, 1)$:

$$\begin{aligned} (y^4 + 3xy)' &= (6 + y)' \\ 4y^3y' + 3y + 3xy' &= y' \\ 4(1)^3y' + 3(1) + 3(2)y' &= y' \\ 4y' + 3 + 6y' &= y' \\ y' &= -\frac{1}{3}. \end{aligned}$$

For the tangent line, the slope $m = y' = -\frac{1}{3}$. So use the point slope formula for $(x_0, y_0) = (2, 1)$ to show

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= -\frac{1}{3}(x-2) = -\frac{1}{3}x + \frac{2}{3} \\ y &= -\frac{1}{3}x + \frac{5}{3}. \end{aligned}$$

7. (6 pts) At time t (measured in hours), the position $s(t)$ of a car along a road is given by $s(t) = 4t^3 - t$ mi.

   (a) (2 pts) Compute the velocity of the car at time t.
   Solution: $v(t) = s'(t) = (12t^2 - 1)$ mi/hr.

   (b) (2 pts) Compute the acceleration of the car at time t.
   Solution: $a(t) = v'(t) = 24t$ mi/hr^2.

   (c) (2 pts) At what time is the acceleration equal to 9 mi/hr^2?
   Solution: $a(t) = 24t = 9$ when $t = 9/24 = 3/8$ hr.