

Calculus 1, Sections 1-5, Exam 2, Version A
Monday, November 17, 2008

1. (8 pts) Short answer. Put your answer in the blank. No explanation needed and **NO PARTIAL CREDIT!**

(a) Evaluate $\int \sqrt{x} dx$. + C
Solution: $\frac{2}{3}x^{\frac{3}{2}} + C$.

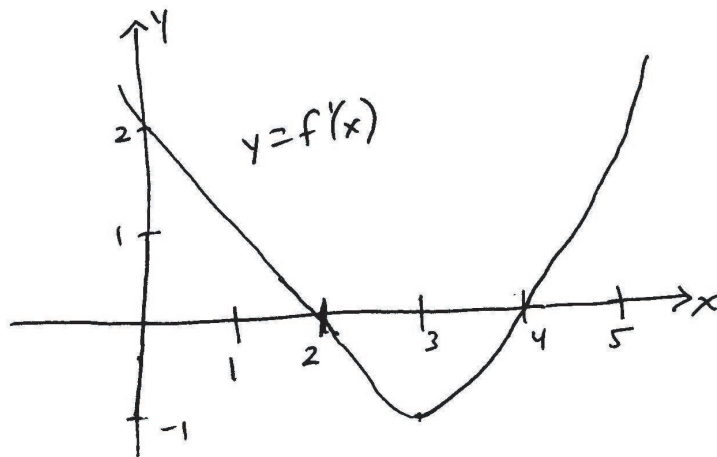
(b) Evaluate $G'(x)$ for $G(x) = \int_3^x t^3 dt$.
Solution: x^3 .

(c) Compute $\int (x^3 - x^2) dx$. + C
Solution: $\frac{1}{4}x^4 - \frac{1}{3}x^3 + C$.

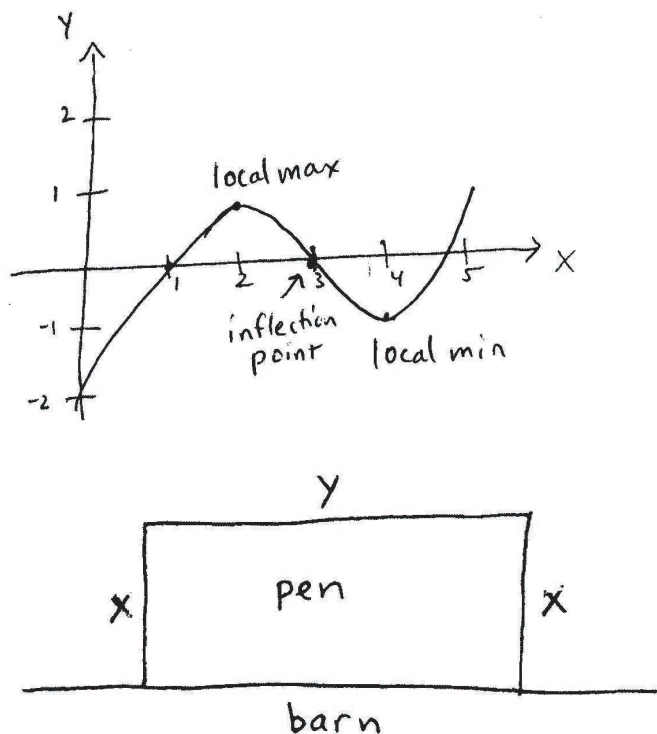
(d) Compute the sum $\sum_{n=1}^3 (n^2 + 1)$. Your answer should be in the form of *an integer*.
Solution: $(1^2 + 1) + (2^2 + 1) + (3^2 + 1) = 2 + 5 + 9 = 16$.

2. (3 pts) Identify the critical points and find the maximum value and the minimum value for $f(x) = x^2 - 3x + 2$ on the interval $[0, 2]$. Show your work.

Solution: Compute $f'(x) = 2x - 3$. $f'(x) = 0$ only when $x = \frac{3}{2}$, which is in the interval $[0, 2]$. So we have three critical points: two endpoints $x = 0, 2$, and the stationary critical point $x = \frac{3}{2}$. Compute $f(0) = 2$, $f(2) = 0$, $f(\frac{3}{2}) = -\frac{1}{4}$. So the minimum is at $x = \frac{3}{2}$ with value $-\frac{1}{4}$, and the maximum is at $x = 0$ with value 2.



3. (8 pts) The figure above is a graph of *the derivative function* $y = f'(x)$. **BE SURE TO NOTE THE GRAPH ABOVE IS THE GRAPH OF $y = f'(x)$, NOT THE GRAPH OF $y = f(x)$.**
- (a) (3 pts) Find all local minimum and maximum points of $f(x)$ on the interval $[0, 5]$. (You should include the endpoints of the interval as possible local maxima and minima.)
- Solution:** First the endpoints: $f(x)$ is increasing once it leaves $x = 0$, and so $x = 0$ is a local minimum. On the other hand, $f(x)$ is increasing as it approaches $x = 5$, and so $x = 5$ is a local maximum.
- For the stationary critical points, the graph shows $f'(x) = 0$ when $x = 2$ and when $x = 4$. Use the First Derivative Test to show $x = 2$ is a local maximum and $x = 4$ is a local minimum.
- (b) (2 pts) Find all points of inflection of $f(x)$ on the interval $[0, 5]$.
- Solution:** $x = 3$ is the only inflection point. f goes from concave down to concave up there (since f' changes from decreasing to increasing there).
- (c) (3 pts) On the axes provided below, sketch the graph of $y = f(x)$, assuming that $f(1) = 0$. Be sure your graph reflects the information about the intervals on which $f(x)$ is increasing, decreasing, concave up, and concave down.



4. (4 pts) A farmer is building a rectangular pen against the side of a long barn, as in the picture above. If the area of the pen is to be 1800 ft^2 , what is the minimum amount of fencing material (measured in ft) that the farmer must use? Show your work.

Solution: Let x be the length of the sides of the pen perpendicular to the barn, and let y be the length of the side of the pen parallel to the barn, as marked in the picture above. Then the total amount of fencing needed is $2x + y$. The area of the pen is fixed at $xy = 1800 \text{ ft}^2$.

So we can solve to find $y = \frac{1800}{x}$ and the amount of fencing needed is

$$f(x) = 2x + \frac{1800}{x}.$$

The natural domain of f is x in $(0, \infty)$. Compute $f'(x) = 2 - \frac{1800}{x^2}$. $f'(x) = 0$ when $2 = \frac{1800}{x^2}$, or $x^2 = 900$, $x = 30$ (we only need to consider positive x). Thus $x = 30$ is the only critical point.

Compute $f''(x) = \frac{3600}{x^3}$ and so $f''(30) > 0$. The second derivative test shows $x = 30$ is a local minimum, and it must be the global minimum since it's the only critical point in the interval.

So the total amount of fencing material needed is

$$2x + y = 2x + \frac{1800}{x} = 2(30) + \frac{1800}{30} = 120 \text{ ft.}$$

5. (a) (3 pts) Find the general solution to the differential equation $\frac{dy}{dx} = \frac{y^2}{x^2}$. Show your work.

Solution: Compute

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2}{x^2}, \\ \frac{dy}{y^2} &= \frac{dx}{x^2}, \\ \int \frac{dy}{y^2} &= \int \frac{dx}{x^2}, \\ -\frac{1}{y} &= -\frac{1}{x} + C, \\ y &= \frac{1}{-\frac{1}{x} + C} \\ &= \frac{x}{1 - Cx}. \end{aligned}$$

- (b) (3 pts) Find the particular solution to $\frac{dy}{dx} = \frac{y^2}{x^2}$ which passes through the point $(x, y) = (1, 2)$. Show your work.

Solution: Plug in $(x, y) = (1, 2)$ to the general solution above to find

$$\begin{aligned} 2 &= \frac{1}{1 - C \cdot 1}, \\ \frac{1}{2} &= 1 - C, \\ C &= \frac{1}{2}. \end{aligned}$$

So the particular solution is $y = \frac{x}{1 - \frac{1}{2}x}$.

6. (7 pts) Consider the function $h(\theta) = \tan \theta - 2\theta$ for θ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

- (a) (3 pts) Find all the critical points of $h(\theta)$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Show your work.

Solution: First note $h(\theta)$ is continuous on the given interval (the tangent function has vertical asymptotes at $\theta = \pm\frac{\pi}{2}$.) Now compute the derivative $h'(\theta) = \sec^2 \theta - 2$. So $h'(\theta) = 0$ when

$$\begin{aligned}0 &= \sec^2 \theta - 2, \\ \sec^2 \theta &= 2, \\ \frac{1}{\cos^2 \theta} &= 2, \\ \cos^2 \theta &= \frac{1}{2}, \\ \cos \theta &= \pm \frac{\sqrt{2}}{2}, \\ \theta &= \pm \frac{\pi}{4}.\end{aligned}$$

So the only critical points in the interval are $\theta = \frac{\pi}{4}$, $\theta = -\frac{\pi}{4}$.

- (b) (2 pts) Classify each critical point from part (a) as a local minimum or a local maximum. Justify your answers.

Solution: Use the Second Derivative Test. Compute $h''(\theta) = 2 \sec \theta (\sec \theta \tan \theta) = \sec^2 \theta \tan \theta$. So $h''(\frac{\pi}{4}) = \sec^2 \frac{\pi}{4} \tan \frac{\pi}{4} = (\sqrt{2})^2 \cdot 1 = 2 > 0$, and so $\frac{\pi}{4}$ is a local minimum. On the other hand, $h''(-\frac{\pi}{4}) = \sec^2(-\frac{\pi}{4}) \tan(-\frac{\pi}{4}) = (\sqrt{2})^2 \cdot (-1) = -2 < 0$. So $-\frac{\pi}{4}$ is a local maximum.

- (c) (2 pts) Does $h(\theta)$ have a global maximum point on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$? Why or why not?

Solution: $h(\theta)$ does not have a global maximum point on this interval since

$$\lim_{\theta \rightarrow \frac{\pi}{2}^-} h(\theta) = \lim_{\theta \rightarrow \frac{\pi}{2}^-} (\tan \theta - 2\theta) = \infty - 2(\frac{\pi}{2}) = \infty.$$

So this infinite limit is larger than the value of h at local max at $\theta = -\frac{\pi}{4}$.