1. (8 pts) Short answer. Put your answer in the blank. No explanation needed and NO PARTIAL CREDIT!

(a) Evaluate \( \int \frac{1}{x^2} \, dx \).
   \[ \text{Solution: } -\frac{1}{x} + C. \]

(b) Evaluate \( G'(x) \) for \( G(x) = \int_0^\pi \cos t \, dt \).
   \[ \text{Solution: } \cos x. \]

(c) Compute \( \int (x^4 + x^3) \, dx \).
   \[ \text{Solution: } \frac{1}{5}x^5 + \frac{1}{3}x^3 + C. \]

(d) Compute the sum \( \sum_{n=1}^{4} (n^2 + 1) \). Your answer should be in the form of an integer.
   \[ \text{Solution: } (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) = 2 + 5 + 10 + 17 = 34. \]

2. (3 pts) Identify the critical points and find the maximum value and the minimum value for \( f(x) = x^2 - 4x + 2 \) on the interval \([0, 3]\). Show your work.

   \[ \text{Solution: } \] Compute \( f'(x) = 2x - 4 \). \( f'(x) = 0 \) only when \( x = 2 \), which is in the interval \([0, 3]\). So we have three critical points: two endpoints \( x = 0, 3 \), and the stationary critical point \( x = 2 \). Compute \( f(0) = 2 \), \( f(2) = -2 \), \( f(3) = -1 \). So the minimum is at \( x = 2 \) with value \(-2\), and the maximum is at \( x = 0 \) with value 2.
3. (8 pts) The figure above is a graph of the derivative function \( y = f'(x) \). **BE SURE TO NOTE THE GRAPH ABOVE IS THE GRAPH OF \( y = f'(x) \), NOT THE GRAPH OF \( y = f(x) \).**

(a) (3 pts) Find all local minimum and maximum points of \( f(x) \) on the interval \([0, 5]\). (You should include the endpoints of the interval as possible local maxima and minima.)

**Solution:** Local minimum points: \( x = 0, 5 \). Local maximum point \( x = 4 \).

First the endpoints: \( f(x) \) is increasing once it leaves \( x = 0 \), and so \( x = 0 \) is a local minimum. On the other hand, \( f(x) \) is decreasing as it approaches \( x = 5 \), and so \( x = 5 \) is also a local minimum.

For the stationary critical points, the graph shows \( f'(x) = 0 \) when \( x = 2 \) and when \( x = 4 \). Use the First Derivative Test to show \( x = 2 \) is neither a local minimum nor a local maximum and \( x = 4 \) is a local maximum.

(b) (2 pts) Find all points of inflection of \( f(x) \) on the interval \([0, 5]\).

**Solution:** \( x = 2, 3 \) are the only inflection points. \( f \) goes from concave down to concave up at \( x = 2 \) (since \( f' \) changes from decreasing to increasing there), while \( f \) changes from concave up to concave down at \( x = 3 \) (since \( f' \) changes from increasing to decreasing there).

(c) (3 pts) On the axes provided below, sketch the graph of \( y = f(x) \), assuming that \( f(1) = 0 \). Be sure your graph reflects the information about the intervals on which \( f(x) \) is increasing, decreasing, concave up, and concave down.
4. (4 pts) A farmer is building a rectangular pen against the side of a long barn, as in the picture above. If the area of the pen is to be 800 ft², what is the minimum amount of fencing material (measured in ft) that the farmer must use? Show your work.

Solution: Let $x$ be the length of the sides of the pen perpendicular to the barn, and let $y$ be the length of the side of the pen parallel to the barn, as marked in the picture above. Then the total amount of fencing needed is $2x + y$. The area of the pen is fixed at $xy = 800$ ft². So we can solve to find $y = \frac{800}{x}$ and the amount of fencing needed is

$$f(x) = 2x + \frac{800}{x}.$$ 

The natural domain of $f$ is $x$ in $(0, \infty)$. Compute $f'(x) = 2 - \frac{800}{x^2}$. $f'(x) = 0$ when $2 = \frac{800}{x^2}$, or $x^2 = 400$, $x = 20$ (we only need to consider positive $x$). Thus $x = 20$ is the only critical point.
Compute \( f''(x) = \frac{1600}{x^3} \) and so \( f''(20) > 0 \). The second derivative test shows \( x = 20 \) is a local minimum, and it must be the global minimum since it’s the only critical point in the interval.

So the total amount of fencing material needed is

\[
2x + y = 2x + \frac{800}{x} = 2(20) + \frac{800}{20} = 80 \text{ ft}.
\]

5. (a) (3 pts) Find the general solution to the differential equation \( \frac{dy}{dx} = \frac{y^2}{x^3} \). Show your work.

**Solution:** Compute

\[
\frac{dy}{dx} = \frac{y^2}{x^3},
\]

\[
\frac{dy}{y^2} = \frac{dx}{x^3},
\]

\[
\int \frac{dy}{y^2} = \int \frac{dx}{x^3},
\]

\[
-\frac{1}{y} = -\frac{1}{2x^2} + C,
\]

\[
y = -\frac{1}{-\frac{1}{2x^2} + C} = \frac{2x^2}{1 - 2Cx^2}.
\]

(b) (3 pts) Find the particular solution to \( \frac{dy}{dx} = \frac{y^2}{x^3} \) which passes through the point \( (x, y) = (1, 2) \). Show your work.

**Solution:** Plug in \( (x, y) = (1, 2) \) to the general solution above to find

\[
2 = \frac{2 \cdot 1^2}{1 - 2C \cdot 1^2},
\]

\[
1 = 1 - 2C,
\]

\[
C = 0.
\]

So the particular solution is \( y = \frac{2x^2}{1 - 2 \cdot 0 \cdot x^2} = 2x^2 \).
6. (7 pts) Consider the function \( h(\theta) = \tan \theta - 4\theta \) for \( \theta \) in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\).

(a) (3 pts) Find all the critical points of \( h(\theta) \) in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\). Show your work.

**Solution:** First note \( h(\theta) \) is continuous on the given interval (the tangent function has vertical asymptotes at \( \theta = \pm \frac{\pi}{2} \)). Now compute the derivative \( h'(\theta) = \sec^2 \theta - 4 \). So \( h'(\theta) = 0 \) when

\[
0 = \sec^2 \theta - 4, \\
\sec^2 \theta = 4, \\
\cos^2 \theta = \frac{1}{4}, \\
\cos \theta = \pm \frac{1}{2}, \\
\theta = \pm \frac{\pi}{3}.
\]

So the only critical points in the interval are \( \theta = \pm \frac{\pi}{3} \).

(b) (2 pts) Classify each critical point from part (a) as a local minimum or a local maximum. Justify your answers.

**Solution:** Use the Second Derivative Test. Compute \( h''(\theta) = 2 \sec \theta(\sec \theta \tan \theta) = 2 \sec^2 \theta \tan \theta \). So \( h''\left(\frac{\pi}{3}\right) = 2 \sec^2 \frac{\pi}{3} \tan \frac{\pi}{3} = 2 \cdot 2^2 \cdot \sqrt{3} > 0 \), and so \( \frac{\pi}{3} \) is a local minimum. On the other hand, \( h''\left(-\frac{\pi}{3}\right) = 2 \sec^2 \left(-\frac{\pi}{3}\right) \tan\left(-\frac{\pi}{3}\right) = 2 \cdot 2^2 \cdot (-\sqrt{3}) < 0 \). So \(-\frac{\pi}{3}\) is a local maximum.

(c) (2 pts) Does \( h(\theta) \) have a global maximum point on the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\)? Why or why not?

**Solution:** \( h(\theta) \) does not have a global maximum point on this interval since

\[
\lim_{\theta \to \frac{\pi}{2}^-} h(\theta) = \lim_{\theta \to \frac{\pi}{2}^-} (\tan \theta - 4\theta) = \infty - 4\left(\frac{\pi}{2}\right) = \infty.
\]

So this infinite limit is larger than the value of \( h \) at the local max at \( \theta = -\frac{\pi}{3} \).